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Existence of periodic solutions to fractional $p(z)$-Laplacian parabolic problems

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Abstract. We consider a class of nonlinear parabolic initial boundary value problems having the fractional $p(z)$-Laplacian operator. By combining variable exponent fractional Sobolev spaces with topological degree theory, we establish the existence of a time-periodic non-trivial weak solution.

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1. Introduction and motivation

The main objective of this investigation is to analyse the existence of a time-periodic non-trivial weak solution for a nonlinear parabolic equation containing a fractional $p(z)$-Laplacian operator. The model for this investigation can be described as follows

$$\begin{cases} \dfrac{\partial w}{\partial t} + (-\triangle)^{\sigma}_{p(z)} w = \xi(z,t) & \text{in} \quad Q := \Omega \times (0,T), \\ u(z,0) = u(z,T) & \text{on} \quad \Omega \\ w(z,t) = 0 & \text{on} \quad \partial Q := \big(\mathbb{R}^N \backslash \Omega\big) \times (0,T), \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$ $(N \geq 2)$ is a bounded open set with smooth boundary $\partial\Omega$, $T > 0$ is the period, $\sigma \in (0,1)$, $\xi \in \mathcal{V}^* := L^{(p^-)'}\big(0,T;\mathcal{W}^*\big)$, with $\mathcal{V} := L^{p^-}\big(0,T;\mathcal{W}\big)$, and let $p \in \mathcal{C}(\mathbb{R}^N \times \mathbb{R}^N)$ satisfying

$$1 < p^- = \min_{(z,y)\in\overline{\Omega}\times\overline{\Omega}} p(z,y) \leq p(z,y) \leq p^+ = \max_{(z,y)\in\overline{\Omega}\times\overline{\Omega}} p(z,y) < +\infty, \tag{1.2}$$

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p is symmetric i.e.

$$p(z,y)=p(y,z), \quad \text{for all } (z,y)\in\overline{\Omega}\times\overline{\Omega},$$

we denote by

$$\widetilde{p}(z)=p(z,z), \quad \text{for every } z\in\overline{\Omega}. \tag{1.3}$$

Here, the main operator $(-\Delta)^{\sigma}_{p(z)}$ is the fractional $p(z)$-Laplacian which is non-local operator described on smooth functions by

$$(-\Delta)^{\sigma}_{p(z)}w(z)=p.v.\int_{\mathbb{R}^N\setminus B_\varepsilon(z)}\frac{|w(z)-w(y)|^{p(z,y)-2}(w(z)-w(y))}{|z-y|^{N+\sigma p(z,y)}}dy,$$

where $z\in\mathbb{R}^N$, $p.v.$ is a commonly used abbreviation in the principal value sense, $B_\varepsilon(z):=\{y\in\mathbb{R}^N \ : \ |z-y|<\varepsilon\}$. As far as we know, the introduction of this operator can be credited to Kaufmann et al. [26]. In their work, the authors extended the Sobolev spaces with variable exponents to the fractional case and demonstrated a compact embedding theorem. Additionally, they applied this development to establish the existence and uniqueness of weak solutions for the following fractional $p(x)$-Laplacian problem

$$\begin{cases} (-\Delta)^{s}_{p(z)}u+|u|^{q(z)-2}u=f(z) & \text{in } \Omega \\ u=0 & \text{on } \partial\Omega \end{cases}$$

where $f\in L^{a(z)}(\Omega)$ for some $a(x)>1$.

Recently, considerable attention has been paid to the study of fractional $p(z)$-Laplacian and non-local differential problems. The importance of studying equations similar to (1.1) goes beyond mathematical interests and finds applications in various fields of modern applied science, including phase continuum mechanics, fluid dynamics, image processing, game theory, transition phenomena and population dynamics. These problems arise as a natural consequence of the stochastic stabilization of Lévy processes, as evidenced by the works of [7, 28, 32] and other relevant references.

Of particular note is the seminal work of Caffarelli and Silvestre [18], who introduced the concept of the σ-harmonic extension to describe the fractional Laplacian operator. This development has led to significant progress in the understanding of elliptic problems associated with the fractional Laplacian. Notable advances in this context can be found in references such as [23, 36] and related sources. Furthermore, for hyperbolic problems, important contributions have been made by [14, 33], while the Camassa-Holm system has been treated by [30]. Taken together, these studies emphasise the relevance and wide-ranging applications of such non-local operators in various scientific fields.

The study of parabolic equations involving fractional Laplacian operators has attracted considerable interest in recent years, mainly due to their prevalence in various phenomena in physics, ecology, biology, geophysics, finance and other fields characterized by non-Brownian scaling.

An essential and highly recommended work in this area is the book by Bisci et al. [15], which provides a comprehensive and in-depth introduction to the study of fractional problems. Building on this foundation, several previous studies have focused on the study of specific instances of the problem (1.1). In particular, we will now review some of the key results of previous research on parabolic problems (1.1) with initial conditions w_0.

In [16], Boudjeriou consider a non-local diffusion equation involving the fractional $p(z)$-Laplacian with nonlinearities of the variable exponent type. By employing the subdifferential approach, the author ensured the existence of local solutions. Thereafter, there obtained the existence of global solutions and the explosion of solutions in finite time via the potential well theory and the Nehari manifold. there then studed the asymptotic stability of global solutions when time goes to infinity in certain Lebesgue spaces with variable exponents. The case of $p = p(z)$ and $s \to 1^-$, problem (1.1), with an initial data $w_0 \in L^2$, was studied by Hammou [25] by applying the theory of topological degrees. In this direction, we also refer to [24, 29] and references therein for the interested reader.

Concurrently, there was comprehensive research in the literature on periodic solutions. In the book by Lions [29], a qualitative investigation of periodic solutions to the problem (1.1) was conducted when $p(z, y) = p$ and $\sigma = 1$ is an integer. The author explored the existence, regularity, and uniqueness of weak periodic solutions to (1.1) under the condition that $f \in L^{p'}\big(0, T; W^{-1,p'}(\Omega)\big)$, where p' is the conjugate exponent of p.

In a more recent study, Pucci, Xiang, and Zhang [35] employed standard techniques to demonstrate the existence of periodic solutions for a similar initial-boundary value problem involving fractional p-Laplacian parabolic equations (1.1), but with an additional Kirchhoff term. For further details, interested readers can also refer to [37] and [39]. In [31], by means of the sub-differential approach, Mazăn et al. established the existence and uniqueness of strong solutions for the following diffusion problems implying a nonlocal fractional p-Laplacian operator

$$\frac{\partial w}{\partial t} + (-\Delta)_p^\sigma w = 0, \qquad \text{in } \Omega,\ t > 0, \tag{1.4}$$

with $\Omega \subseteq \mathbb{R}^N$, $N \geq 1$, $\sigma \in (0, 1)$ and $p \in (1, \infty)$. In addition, the authors also prove that when $p \neq 2$ and s tends to 1^-, after inserting a normalizing constant, the equation (1.4) is simplified to the following evolution equation $w_t - \Delta_p w = 0$ implying the p-Laplacian. For all $2 < p \leq 2^*_\sigma = 2N/(N - 2\sigma)$ and $\sigma \in (0, 1), N > 2\sigma$, Fu and Pucci [22] tackled the following problem

$$\begin{cases} \dfrac{\partial w(z,t)}{\partial t} + (-\Delta)^\sigma w(z,t) = |w(z,t)|^{p-2} w(z,t), & z \in \Omega,\ t > 0, \\ u(z, 0) = w_0(z), & z \in \Omega, \\ w(z,t) = 0, & z \in \mathbb{R}^N \backslash \Omega,\ t > 0. \end{cases}$$

Basing on the potential well theory, they showed the existence of global weak solutions to the considered problem. Thereafter, they obtained the vacuum isolating and blow-up of strong solutions.

Taking inspiration from previous research, we employ topological degree theory to establish the existence of periodic weak solutions for the nonlinear parabolic problems (1.1), which involve the fractional $p(z)$-Laplacian operator. To the best of our knowledge, these problems have not been addressed in earlier studies. We transform this fractional parabolic problem into a new problem governed by an operator equation of the form $\mathcal{N}w + \Phi w = \xi$, where $\mathcal{N}$ is a densely defined monotone maximal linear operator, and Φ is a demicontinuous bounded map of type (S_+) with respect to the domain of $\mathcal{N}$.

The topological degree theory for perturbations of linear maximal monotone mappings and its application to a class of parabolic problems were proposed by Berkovits and Mustonen in 1991, as described in [13]. This method has been extensively employed by various authors to study nonlinear parabolic problems and has proven to be a highly effective tool. For more details, interested readers can refer to the works [4, 5, 8, 9]. For additional background information and applications of this theory, readers can consult the articles [3, 1, 2, 6, 12, 20].

The organization of this paper is as follows: Section 2 presents essential preliminary results and related lemmas that will be utilized in subsequent sections. In Section 3, we provide the proof of the main results of this paper.

2. Preliminary results

In this section, we initiate by introducing the necessary functional framework to explore the problem (1.1). Additionally, we provide essential explanations and characteristics of topological degree theory that are pertinent to our objective.

Let Ω be a bounded open subset of $\mathbb{R}^N, N \geq 2$, we denote

$$C_+(\overline{\Omega}) = \left\{p(\cdot) : \Omega \to \mathbb{R} \text{ such that } p^- \leq p(z) \leq p^+ < +\infty\right\},$$

where

$$p^- := \operatorname*{ess\,inf}_{z\in\overline{\Omega}} p(z); \quad p^+ := \operatorname*{ess\,sup}_{z\in\overline{\Omega}} p(z).$$

We define the Lebesgue space with variable exponents $L^{p(z)}(\Omega)$, as follows

$$L^{p(z)}(\Omega) = \{w : \Omega \to \mathbb{R}, \text{ measurable} : \int_\Omega |w|^{p(z)} dz < \infty\},$$

endowed with the norm

$$\|w\|_{p(z)} = \inf\left\{\lambda > 0 \mid \varrho_{p(\cdot)}\left(\frac{z}{\lambda}\right) \leq 1\right\}$$

where

$$\varrho_{p(\cdot)}(w) = \int_\Omega |w(z)|^{p(z)} dz, \qquad \text{for all } w \in L^{p(z)}(\Omega)$$

$\left(L^{p(z)}(\Omega), \|\cdot\|_{p(z)}\right)$ is a Banach space, separable and reflexive. Its dual space is $L^{p'(z)}(\Omega)$, where $\frac{1}{p(z)} + \frac{1}{p'(z)} = 1$ for all $z \in \Omega$. We have also the following result

Proposition 2.1. [21] *For any $w \in L^{p(z)}(\Omega)$ we have*

1. $\|w\|_{p(z)} < 1 \,(= 1; > 1) \Leftrightarrow \varrho_{p(\cdot)}(w) < 1 \,(= 1; > 1)$,
2. $\|w\|_{p(z)} \geq 1 \Rightarrow \|w\|_{p(z)}^{p^-} \leq \varrho_{p(\cdot)}(w) \leq \|w\|_{p(z)}^{p^+}$,
3. $\|w\|_{p(z)} \leq 1 \Rightarrow \|w\|_{p(z)}^{p^+} \leq \varrho_{p(\cdot)}(w) \leq \|w\|_{p(z)}^{p^-}$.

From this statement, we can infer the following inequalities

$$\|w\|_{p(z)} \leq \varrho_{p(\cdot)}(w) + 1 \quad \text{and} \quad \varrho_{p(\cdot)}(w) \leq \|w\|_{p(z)}^{p^-} + \|w\|_{p(z)}^{p^+}$$

If p, $q \in \mathcal{C}_+(\overline{\Omega})$ such that $p(z) \leq q(z)$ for any $z \in \overline{\Omega}$, then there exists the continuous embedding $L^{q(z)}(\Omega) \hookrightarrow L^{p(z)}(\Omega)$.

Following this, we will provide an introduction to fractional Sobolev spaces with variable exponents, as introduced in the references [11, 26].

Suppose σ is a constant real number, where $0 < \sigma < 1$, and let p and q be two continuous functions mapping from the closed set $\overline{\Omega}$ to the interval $(0, \infty)$. Additionally, assuming that the conditions (1.2) and (1.3) hold, we proceed to define the fractional Sobolev space with a variable exponent using the Gagliardo approach in the following manner

$$\mathcal{W} = W^{\sigma,q(z),p(z,y)}(\Omega) = \Big\{ w \in L^{q(z)}(\Omega) : \int_{\Omega\times\Omega} \frac{|w(z)-w(y)|^{p(z,y)}}{\lambda^{p(z,y)}|z-y|^{N+\sigma p(z,y)}} dzdy < +\infty \text{ for some } \lambda > 0 \Big\}$$

We endow the space $\mathcal{W}$ with the norm given by

$$\|w\|_{\mathcal{W}} = \|w\|_{q(z)} + [w]_{\sigma,p(z,y)},$$

where $[\,\cdot\,]_{\sigma,p(z,y)}$ is a Gagliardo seminorm with variable exponent, which is defined as follows

$$[w]_{\sigma,p(z,y)} = \inf \Big\{ \lambda > 0 : \int_{\Omega\times\Omega} \frac{|w(z)-w(y)|^{p(z,y)}}{\lambda^{p(z,y)}|z-y|^{N+\sigma p(z,y)}} dzdy \leq 1 \Big\}.$$

The space $\big(\mathcal{W}, \|\cdot\|_{\mathcal{W}}\big)$ is a Banach space (as referenced in [19]), and it possesses the properties of separability and reflexivity (as mentioned in [11, Lemma 3.1]).

We define $\mathcal{W}_0$ as a subspace of $\mathcal{W}$, obtained by taking the closure of $\mathcal{C}_0^\infty(\Omega)$ with respect to the norm $\|\cdot\|_{\mathcal{W}}$. This construction is based on [10, Theorem 2.1 and Remark 2.1].

$$\|\cdot\|_{\mathcal{W}_0} := [\,\cdot\,]_{\sigma,p(z,y)}$$

is a norm on $\mathcal{W}_0$ which is equivalent to the norm $\|\cdot\|_{\mathcal{W}}$, and we have the compact embedding $\mathcal{W}_0 \hookrightarrow\hookrightarrow L^{q(z)}(\Omega)$. Consequently, the space $\big(\mathcal{W}_0, \|\cdot\|_{\mathcal{W}_0}\big)$ is a Banach space that is also separable and reflexive.

We define the modular $\varrho_{p(\cdot,\cdot)} : \mathcal{W}_0 \to \mathbb{R}$ by

$$\varrho_{p(\cdot,\cdot)}(w) = \int_{\Omega\times\Omega} \frac{|w(z)-w(y)|^{p(z,y)}}{|z-y|^{N+\sigma p(z,y)}} dzdy$$

The modular ϱ_p checks the following results

Proposition 2.2. [27] *For any $w \in \mathcal{W}_0$ we have*

(i) $\|w\|_{\mathcal{W}_0} \geq 1 \Rightarrow \|w\|_{\mathcal{W}_0}^{p^-} \leq \varrho_{p(\cdot,\cdot)}(w) \leq \|w\|_{\mathcal{W}_0}^{p^+}$,

(ii) $\|w\|_{\mathcal{W}_0} \leq 1 \Rightarrow \|w\|_{\mathcal{W}_0}^{p^+} \leq \varrho_{p(\cdot,\cdot)}(w) \leq \|w\|_{\mathcal{W}_0}^{p^-}$.

Afterwards, we present the devised approach to address the problem (1.1). For a given time interval $0 < T < \infty$, we consider the functional space

$$\mathcal{V} := L^{p^-}(0, T; \mathcal{W}_0),$$

that is a separable and reflexive Banach space with the norm

$$|w|_{\mathcal{V}} = \Big(\int_0^T \|w\|_{\mathcal{W}}^{p^-} dt \Big)^{\frac{1}{p^-}}.$$

In view of [38], we can clearly establish that the norm $|w|_{\mathcal{V}}$ is equivalent to the following standard norm

$$\|w\| := \|w\|_{\mathcal{V}} = \Big(\int_0^T \|w\|_{\mathcal{W}_0}^{p^-} dt \Big)^{\frac{1}{p^-}}.$$

For reader's convenience, we start by recalling some results and properties from the Berkovits and Mustonen degree theory for demicontinuous operators of generalized (S_+) type in real separable reflexive Banach $\mathcal{Z}$.
In what follows, We respectively denote by $\mathcal{Z}^*$ the topological dual of the Banach space $\mathcal{Z}$ with continuous dual pairing $\langle \cdot , \cdot \rangle$ and $\rightharpoonup$ represents the weak convergence. Given a nonempty subset Ω of $\mathcal{Z}$.
Let $\mathcal{A}$ from $\mathcal{Z}$ to $2^{\mathcal{Z}^*}$ be a multi-values mapping. We designate by $Gr(\mathcal{A})$ the graph of $\mathcal{A}$, i.e.

$$Gr(\mathcal{A}) = \big\{ (w, v) \in \mathcal{Z} \times \mathcal{Z}^* \ : \ v \in \mathcal{A}(w) \big\}.$$

Definition 2.3. The multi-values mapping $\mathcal{A}$ is called

1. monotone, if for each pair of elements $(\eta_1, \theta_1), (\eta_2, \theta_2)$ in $Gr(\mathcal{A})$, we have the inequality
$$\langle \theta_1 - \theta_2, \eta_1 - \eta_2 \rangle \geq 0.$$
2. maximal monotone, if it is monotone and maximal in the sense of graph inclusion among monotone multi-values mappings from $\mathcal{Z}$ to $2^{\mathcal{Z}^*}$. The last clause has an analogous variant in that, for each $(\eta_0, \theta_0) \in \mathcal{Z} \times \mathcal{Z}^*$ for which $\langle \theta_0 - \theta, \eta_0 - \eta \rangle \geq 0$, for all $(\eta, \theta) \in Gr(\mathcal{A})$, we have $(\eta_0, \theta_0) \in Gr(\mathcal{A})$.

Let $\mathcal{Y}$ be another real Banach space.

Definition 2.4. A mapping $\Phi : D(\Phi) \subset \mathcal{Z} \to \mathcal{Y}$ is said to be

1. demicontinuous, if for each sequence $(w_n) \subset \Omega$, $w_n \to w$ implies $\Phi(w_n) \rightharpoonup \Phi(w)$.
2. of type (S_+), if for any sequence $(w_n) \subset D(\Phi)$ such that $w_n \rightharpoonup w$ and $\limsup_{n\to\infty} \langle \Phi w_n, w_n - w \rangle \leq 0$, we have $w_n \to w$.

Let $\mathcal{N} : \mathcal{D}(\mathcal{N}) \subset \mathcal{Z} \to \mathcal{Z}^*$ be a linear maximal monotone map such that $\mathcal{D}(\mathcal{N})$ is dense in $\mathcal{Z}$.
In the following, for each open and bounded subset $\mathcal{O}$ on $\mathcal{Z}$, we consider classes of operators :

$$\mathcal{F}_{\mathcal{O}}(\Omega) := \big\{ \mathcal{N} + \Phi : \overline{\mathcal{O}} \cap \mathcal{D}(\mathcal{N}) \to \mathcal{Z}^* \mid \Phi \quad \text{is bounded, demicontinuous} \\ \text{and of type } (S_+) \text{ with respect to } \mathcal{D}(\mathcal{N}) \text{ from } \mathcal{O} \text{ to } \mathcal{Z}^* \big\},$$

$$\mathcal{H}_{\mathcal{O}} := \big\{\mathcal{N} + \Phi(t) \,:\, \overline{\mathcal{O}} \cap D(L) \to \mathcal{Z}^* \mid \Phi(t) \text{ is a bounded homotopy of type } (S_+) \text{ with respect to } \mathcal{D}(\mathcal{N}) \text{ from } \overline{\mathcal{O}} \text{ to } \mathcal{Z}^*\big\}.$$

Remark 2.5. [13] Remark that the class $\mathcal{H}_{\mathcal{O}}$ contains all affine homotopy

$$\mathcal{N} + (1-t)\Phi_1 + t\Phi_2 \quad \text{with} \quad (\mathcal{N} + \Phi_i) \in \mathcal{F}_{\mathcal{O}}, \ \ i = 1, 2.$$

The following theorem provides the notion of the Berkovits and Mustonen topological degree for a class of demicontinuous operators satisfying the condition (S_+), which is the main key to the existence proof, for more details see [13].

Theorem 2.6. *Let $\mathcal{N}$ be a linear maximal monotone densely defined map from $\mathcal{D}(\mathcal{N}) \subset \mathcal{Z}$ to $\mathcal{Z}^*$ and*

$$\mathcal{M} = \Big\{(F, G, \psi) \,:\, F \in \mathcal{F}_{\mathcal{O}},\ \mathcal{O} \text{ an open bounded subset in } \mathcal{Z}, \ \psi \notin F\big(\partial\mathcal{O} \cap \mathcal{D}(\mathcal{N})\big)\Big\}.$$

There is a unique degree function $d \,:\, \mathcal{M} \longrightarrow \mathbb{Z}$ which satisfies the following properties :

1. *(Normalization) $\mathcal{N} + \mathcal{J}$ is a normalising map, where $\mathcal{J}$ is the duality mapping of $\mathcal{Z}$ into $\mathcal{Z}^*$, that is, $d(\mathcal{N} + \mathcal{J}, \mathcal{O}, \psi) = 1$, when $\psi \in (\mathcal{N} + \mathcal{J})(\mathcal{O} \cap \mathcal{D}(\mathcal{N}))$.*
2. *(Additivity) Let $\Phi \in \mathcal{F}_{\mathcal{O}}$. If $\mathcal{O}_1$ and $\mathcal{O}_2$ are two disjoint open subsets of $\mathcal{O}$ such that $\psi \notin \Phi\big((\overline{\mathcal{O}} \backslash (\mathcal{O}_1 \cup \mathcal{O}_2)) \cap D(L)\big)$ then we have*
$$d(\Phi, \mathcal{O}, \psi) = d(\Phi, \mathcal{O}_1, \psi) + d(\Phi, \mathcal{O}_2, \psi).$$
3. *(Homotopy invariance) If $\Phi(t) \in \mathcal{H}_{\mathcal{O}}$ and $\psi(t) \notin \Phi(t)(\partial\mathcal{O} \cap \mathcal{D}(\mathcal{N}))$ for every $t \in [0,1]$, where $\psi(t)$ is a continuous curve in $\mathcal{Z}^*$, then*
$$d(\Phi(t), \mathcal{O}, \psi(t)) = \text{ constant}, \qquad \text{for all } t \in [0,1].$$
4. *(Existence) if $d(\Phi, \mathcal{O}, \psi) \neq 0$, then the equation $\Phi w = \psi$ has a solution in $\mathcal{O} \cap \mathcal{D}(\mathcal{N})$.*

Lemma 2.7. *Let $\mathcal{N} + \Phi \in \mathcal{F}_{\mathcal{Z}}$ and $\psi \in \mathcal{Z}^*$. Suppose that there is $R > 0$ such as*

$$\langle \mathcal{N}w + \Phi w - \psi, w\rangle > 0, \tag{2.1}$$

for each $w \in \partial B_R(0) \cap \mathcal{D}(\mathcal{N})$. Hence

$$(\mathcal{N} + \Phi)(\mathcal{D}(\mathcal{N})) = \mathcal{Z}^*. \tag{2.2}$$

Proof. Let $\varepsilon > 0,\ \theta \in [0,1]$ and

$$\mathcal{S}_\varepsilon(\theta, w) = \mathcal{N}w + (1-\theta)\mathcal{J}w + \theta(\Phi w + \varepsilon\mathcal{J}w - \psi).$$

As $0 \in \mathcal{N}(0)$ and applying the boundary condition (2.1), we have

$$\begin{aligned}
\langle \mathcal{S}_\varepsilon(\theta, w), w\rangle &= \big\langle \theta(\mathcal{N}w + \Phi w - \psi, w\big\rangle + \big\langle (1-\theta)\mathcal{N}w + (1-\theta+\varepsilon)\mathcal{J}w, w\big\rangle \\
&\geq \big\langle (1-\theta)\mathcal{N}w + (1-\theta+\varepsilon)\mathcal{J}w, w\big\rangle \\
&= (1-\theta)\big\langle \mathcal{N}w, w\big\rangle + (1-\theta+\varepsilon)\big\langle \mathcal{J}w, w\big\rangle \\
&\geq (1-\theta+\varepsilon)\|w\|^2 = (1-\theta+\varepsilon)R^2 > 0.
\end{aligned}$$

Which means that $0 \notin \mathcal{S}_\varepsilon(\theta, w)$. As $\mathcal{J}$ and $\Phi+\varepsilon\mathcal{J}$ are bounded, continuous and of type (S_+), $\{\mathcal{S}_\varepsilon(\theta,\cdot)\}_{\theta\in[0,1]}$ is an admissible homotopy. Hence, by using the normalisation and invariance under homotopy, we get

$$d(\mathcal{S}_\varepsilon(\theta,\cdot), B_R(0), 0) = d(\mathcal{N}+\mathcal{J}, B_R(0), 0) = 1.$$

As a result, there is $w_\varepsilon \in \mathcal{D}(\mathcal{N})$ such that $0 \in \mathcal{S}_\varepsilon(\theta,\cdot)$.
If we take $\theta = 1$ and when $\varepsilon \to 0^+$, then we have $\psi = \mathcal{N}w + \Phi w$ for certain $w \in \mathcal{D}(\mathcal{N})$. As $\psi \in \mathcal{Z}^*$ is of any kind, we deduce that $(\mathcal{N}+\Phi)(\mathcal{D}(\mathcal{N})) = \mathcal{Z}^*$. □

3. Main result

To demonstrate the existence of a weak periodic solution for (1.1), we employ compactness methods. Initially, we transform this nonlinear parabolic problem into a new one governed by an operator equation of the type $\mathcal{N}w + \Phi w = \xi$. Subsequently, we employ the theory of topological degrees to further investigate the problem.

In this context, we take into consideration the mapping $\Phi\colon \mathcal{V} \longrightarrow \mathcal{V}^*$, where

$$\langle \Phi w, \varphi\rangle = \int_0^T \int_{\Omega\times\Omega} |w(z,t)-w(y,t)|^{p(z,y)-2}(w(z,t)-w(y,t)) \times (\varphi(z,t)-\varphi(y,t))\mathcal{L}(z,y)dzdydt, \quad (3.1)$$

for all $v \in \mathcal{V}$, with $\mathcal{L}(z,y) = |z-y|^{-N-\sigma p(z,y)}$.
The central outcome of this investigation is encapsulated in the subsequent theorem.

Theorem 3.1. *Assuming that $\xi \in \mathcal{V}^*$ and $u(z,0) = u(z,T) \in L^2(\Omega)$ are satisfied, the problem (1.1) admits at least one weak periodic solution $u \in \mathcal{D}(\mathcal{N})$ in the following sense*

$$-\int_Q u\varphi_t dzdt + \langle \Phi w, \varphi\rangle = \int_Q h\varphi dzdt, \quad (3.2)$$

for each $\varphi \in \mathcal{V}$.

To prove Theorem 3.1, we initially relied on the subsequent technical lemma

Lemma 3.2. *For $0 < \sigma < 1$ and $2 < p^- \le p(z,y) < +\infty$, the operator Φ defined in (3.1) possesses the following properties*

(i) *It is bounded and demicontinuous.*
(ii) *It is strictly monotone.*
(iii) *It is of type (S_+).*

Proof. (i) As in [23], the operator given by

$$\langle \mathcal{B}u, \varphi\rangle = \int_{\Omega\times\Omega} |w(z,t)-w(y,t)|^{p(z,y)-2}(w(z,t)-w(y,t)) \times (\varphi(z,t)-\varphi(y,t))\mathcal{L}(z,y)dzdy, \quad \forall w,\varphi \in \mathcal{W}_0$$

is well defined, bounded, continuous.
Furthermore, the form $\mathcal{B}$ gives rise to a Nemytskii operator that inherits the aforementioned properties, implying that the nonlinear operator Φ is bounded and demicontinuous.
(ii) Thanks to Perera et al. [34, Lemma 6.3], It is sufficient to show that

$$\langle \Phi w, \varphi \rangle \leq \|w\|^{p^{\pm}-1}\|\varphi\| \qquad \text{for all } \; w, \varphi \in \mathcal{V}$$

Additionally, the equality holds if and only if $\delta w = \gamma\varphi$ for some $\delta, \gamma \geq 0$, with both not being zero simultaneously.
Applying Hölder's inequality, we obtain (without loss of generality, we can assume that $\beta(z,y) = p(z,y) - 1$)

$$\langle \Phi w, \varphi \rangle \leq \int_0^T \int_{\Omega\times\Omega} |w(z,t) - w(y,t)|^{p(z,y)-1}|\varphi(z,t) - \varphi(y,t)| \\ \times \mathcal{L}(z,y)dzdydt \leq \|w\|^{p^{\pm}-1}\|\varphi\|,$$

where $p^{\pm} = \begin{cases} p^{+} & \text{if } \|w\| \geq 1 \\ p^{-} & \text{if } \|w\| < 1 \end{cases}$

The equivalence becomes apparent when $\delta w = \gamma\varphi$ for any $\delta, \gamma \geq 0$, with both not being zero simultaneously. Conversely, if $\langle \Phi w, v \rangle = \|w\|^{p^{\pm}-1}\|\varphi\|$, equality occurs in both inequalities. Consequently, the equality in the second inequality results in

$$\delta|w(z,t) - w(y,t)| = \gamma|\varphi(z,t) - \varphi(y,t)| \quad \text{ a.e. in } \mathbb{R}^N \times \mathbb{R}^N \times (0,T)$$

for each $\delta, \gamma \geq 0$, not both null. Therefore, the equality in the first inequality implies

$$\delta(w(z,t) - w(y,t)) = \gamma(\varphi(z,t) - \varphi(y,t)) \quad \text{ a.e. in } \mathbb{R}^N \times \mathbb{R}^N \times (0,T).$$

Since w and φ disappear a.e. in $\mathbb{R}^N\backslash\Omega \times (0,T)$, it results that $\delta w = \gamma\varphi$ a.e. in Q.
(iii) We still need to demonstrate that the operator Φ is of type (S_+).
Let $(w_n)_n$ be a sequence in $D(\Phi)$ such that

$$\begin{cases} w_n \rightharpoonup w & \text{in } \mathcal{V} \\ \limsup\limits_{n\to\infty} \langle \Phi w_n, w_n - w \rangle \leq 0. \end{cases}$$

We want to demonstrate that $w_n \to w$ in $\mathcal{V}$.
From the weak convergence $w_n \rightharpoonup w$, $\limsup\limits_{n\to\infty} \langle \Phi w_n - \Phi w, w_n - w \rangle \leq 0$ and (ii), we infer

$$\lim_{n\to+\infty} \langle \Phi w_n, w_n - w \rangle = \lim_{n\to+\infty} \langle \Phi w_n - \Phi w, w_n - w \rangle = 0. \tag{3.3}$$

According to the compact embedding $\mathcal{W}_0 \hookrightarrow\hookrightarrow L^{p(z)}(\Omega)$ and [29, Theorem 5.1] informs us that $\mathcal{V} \hookrightarrow\hookrightarrow L^{p^-}(Q)$. Consequently, there is a subsequence still referred to as (w_n), such that

$$w_n(z,t) \to w(z,t), \text{ a.e. } (z,t) \in Q. \tag{3.4}$$

Thus, we have from the Fatou lemma and (3.4)

$$\liminf_{n\to+\infty}\int_0^T\int_{\Omega\times\Omega}|w_n(z)-w_n(y)|^{p(z,y)}\mathcal{L}(z,y)dzdydt \geq \int_0^T\int_{\Omega\times\Omega}|w(z)-w(y)|^{p(z,y)}\mathcal{L}(z,y)dzdydt, \quad (3.5)$$

On the other hand, By using the Young inequality, there is a positive constant C such that

$$\begin{aligned}\langle\Phi w_n, w_n-w\rangle &= \int_0^T\int_{\Omega\times\Omega}|w_n(z,t)-w_n(y,t)|^{p(z,y)}\mathcal{L}(z,y)dzdydt\\ &-\int_0^T\int_{\Omega\times\Omega}|w_n(z,t)-w_n(y,t)|^{p(z,y)-2}(w_n(z,t)-w_n(y,t))\\ &\qquad\times(w(z,t)-w(y,t))\mathcal{L}(z,y)dzdydt \qquad (3.6)\\ &\geq \int_0^T\int_{\Omega\times\Omega}|w_n(z,t)-w_n(y,t)|^{p(z,y)}\mathcal{L}(z,y)dzdydt\\ &-\int_0^T\int_{\Omega\times\Omega}|w_n(z,t)-w_n(y,t)|^{p(z,y)-1}|w(z,t)-w(y,t)|\mathcal{L}(z,y)dzdydt\\ &\geq C\int_0^T\int_{\Omega\times\Omega}|w_n(z,t)-w_n(y,t)|^{p(z,y)}\mathcal{L}(z,y)dzdydt\\ &-C\int_0^T\int_{\Omega\times\Omega}|w(z,t)-w(y,t)|^{p(z,y)}\mathcal{L}(z,y)dzdydt,\end{aligned}$$

by (3.5), (3.3) and (3.6), we drive

$$\lim_{n\to+\infty}\int_0^T\int_{\Omega\times\Omega}|w_n(z,t)-w_n(y,t)|^{p(z,y)}\mathcal{L}(z,y)dzdydt = \int_0^T\int_{\Omega\times\Omega}|w(z,t)-w(y,t)|^{p}\mathcal{L}(z,y)dzdydt. \quad (3.7)$$

Combining (3.4), (3.7) with the Brezis-Lieb lemma [17], our conclusion has been in place. □

We are now ready to prove Theorem 3.1.

Proof of Theorem 3.1. To demonstrate the existence of a weak solution to (1.1), we aim to utilize the topological degree methods. To accomplish this objective, we introduce

$$\mathcal{D}(\mathcal{N}) = \{\varphi\in\mathcal{V} \,:\, \varphi'\in\mathcal{V}^*,\ \varphi(0)=0\},$$

By exploiting the density property of $\mathcal{C}_c^\infty(Q_T)$ within $\mathcal{V}$ and considering that $\mathcal{C}_c^\infty(Q_T)\subset\mathcal{D}(\mathcal{N})$, we can conclude that $\mathcal{D}(\mathcal{N})$ densely exists in $\mathcal{V}$. Now, let us examine the operator $\mathcal{N} : \mathcal{D}(\mathcal{N}) \subset \mathcal{V} \longrightarrow \mathcal{V}^*$ defined as follows

$$\langle \mathcal{N}w, \varphi\rangle = -\int_Q w\varphi dzdt, \qquad \text{for all } w \in \mathcal{D}(\mathcal{N}),\ \varphi \in \mathcal{V}.$$

Thereby, the operator $\mathcal{N}$ is generated by $\partial/\partial t$ by making of the relation

$$\langle \mathcal{N}w, \varphi\rangle = \int_0^T \Big\langle \frac{\partial w(t)}{\partial t}, \varphi(t)\Big\rangle dt, \text{ for each } w \in \mathcal{D}(\mathcal{N}),\ \varphi \in \mathcal{V}.$$

Thanks to the outcome presented in [29, Lemma 1.1, p. 313], it can be deduced that $\mathcal{L}$ qualifies as a maximal monotone operator. For further details, one may refer to the comprehensive information provided in [40].
On a separate note, the fact that $\mathcal{N}$ is a monotone operator (i.e., $\langle \mathcal{N}w, w\rangle \geq 0$ for all $u \in \mathcal{D}(\mathcal{N})$) ensures that

$$\begin{aligned}\langle \mathcal{N}w + \Phi w, w\rangle &\geq \langle \Phi w, w\rangle \\ &= \int_0^T \int_{\Omega\times\Omega} |w(z,t) - w(y,t)|^{p(z,y)-2}(w(z,t) - w(y,t))^2 \mathcal{L}(z,y) dzdydt \\ &= \int_0^T \int_{\Omega\times\Omega} |w(z,t) - w(y,t)|^{p(z,y)} \mathcal{L}(z,y) dzdydt \\ &\geq \min\{\|w\|^{p^-}, \|w\|^{p^+}\} \qquad (3.8)\end{aligned}$$

for all $w \in \mathcal{V}$.
From (3.8) the right hand side goes to infinity as $\|w\| \to \infty$, as for each $\xi \in \mathcal{V}^*$ there exists $R = R(h)$ for which

$$\langle \mathcal{N}w + \Phi w - \xi, w\rangle > 0 \qquad \text{for all } w \in B_R(0) \cap \mathcal{D}(\mathcal{N}).$$

By relying on the principles established in Lemma 2.7, it follows that there exists an element $w \in \mathcal{D}(\mathcal{N})$ that serves as a solution to the operator equation $\mathcal{N}w + \Phi w = \xi$. Consequently, this result indicates the existence of a weak periodic solution to the problem (1.1). □

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