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Hyers-Ulam stability of some positive linear operators

Jaspreet Kaur and Meenu Goyal

Abstract. The present article deals with the Hyers-Ulam stability of positive linear operators in approximation theory. We discuss the HU-stability of Bernstein-Schurer type operators, Bernstein-Durrmeyer operators and find the HU-stability constant for these operators. Also, we show that the beta operators with Jacobi weights are HU-unstable.

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1. Introduction

In a conference at the University of Wisconsin, Madison, Ulam asked a question regarding the stability of an equation in a metric group. The question posed by Ulam was whether,
"Given a metric group $(G, ., \rho)$, a number $\epsilon > 0$, and a mapping $f : G \to G$ that satisfies the inequality

$$\rho(f(xy), f(x)f(y)) < \epsilon \text{ for all } x, y \in G,$$

does there exists a homomorphism a of G and a constant $k > 0$ (dependent only on G) such that

$$\rho(a(x), f(x)) \leq k\epsilon \text{ for all } x \in G?"$$

This question is concerned with finding an exact solution close to every approximate solution. If the answer to this question is positive, then the equation $a(xy) = a(x)a(y)$ is called HU-stable, indicating the existence of a unique exact solution close to the

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approximate solution.
In 1941, Hyers [8] provided a proof for a specific equation of the form

$$f(x+y) = f(x) + f(y)$$

in Banach spaces, known as the Cauchy functional equation. This equation is fundamental in Mathematics. Further developments in this field can be found in the references: [3, 4, 9, 10, 12, 15, 16, 21]. For unbounded Cauchy difference equations, Aoki [1] and Rassias [19] introduced another type of stability for functional equations, where the parameter ϵ is replaced by a function depending on x and y.

The Hyers-Ulam stability of linear operators was first observed in the papers by Miura et al. [2, 6, 7, 13], who provided characterizations of HU-stability and its constants for linear operators.

To the best of our knowledge, the HU-stability of positive linear operators in approximation theory was first investigated by Popa and Raşa [17], who examined the HU-stability of both discrete and integral operators. They established the general result that every positive linear operator with finite-dimensional range is HU-stable. Additionally, they determined the HU-stability constant for Bernstein operators and showed that Szász-Mirakyan and beta operators are unstable. In another article [18], the authors obtained stability constants for more general operators and improved the constant for Bernstein operators.

In 2015, Mursaleen and Ansari [14] found the best constant in terms of HU-stability for Kantorovich-Stancu and King's operators. They also demonstrated the unstability of Szász-Mirakyan type operators.

Positive linear operators have many applications in various areas of Mathematics, including functional analysis, approximation theory, and numerical analysis. Hyers-Ulam stability helps us to see the change in behavior of positive linear operators under perturbations. This implies that the operators T has a stable behavior with respect to small perturbations in the function it operates on.

Motivated by the applications of positive linear operators and behavior of their solution with Hyers-Ulam stabiility, in the present article, we determine the stability and the best constant for Bernstein-Stancu type operators and Bernstein-Durrmeyer operators. We also establish the unstability of beta operators with Jacobi weights.
The paper is organized as: Section 1 includes introduction that provides an overview of the problem and the motivation behind studying HU-stability of operators in approximation theory. In section 2, we provide basic definitions and results useful in the subsequent sections. In next section, we discuss the HU-stability of two specific types of operators: Bernstein-Schurer type operators and Bernstein-Durrmeyer operators and determine the HU-stability constants for these operators, which quantify how close the approximate solutions are to the exact solutions. Section 4 investigates the unstability of beta operators with Jacobi weights.

2. Basic definitions and results

Definition 2.1. [20] Let X and Y are two normed spaces and $L : X \to Y$ is a mapping. We say that L has the Hyers-Ulam stability property or L is HU-stable if there exists

a constant K such that
(i) for any $g \in L(X)$, $\epsilon > 0$ and $f \in X$ with $\| Lf - g \| \leq \epsilon$, there exists a $f_0 \in X$ such that $Lf_0 = g$ and $\| f - f_0 \| \leq K\epsilon$.
The condition expresses the Hyers-Ulam stabiliy of the equation

$$Lf = g,$$

where $g \in R(L)$ is given and $f \in X$ is unknown. The number K is called Hyers-Ulam stability (HUS) constant of L, and the infimum of all HUS contants is denoted by K_L, which, in general, is not a HUS constant.

For any bounded linear operator L with kernel $N(L)$ and the range space $R(L)$, we can consider a one-to-one operator $\tilde{L}$ from the quotient space $X/N(L)$ into Y defined as:

$$\tilde{L}(f + N(L)) = Lf, \ f \in X.$$

The inverse of this operator is $\tilde{L}^{-1} : R(L) \to X/N(L)$.

Theorem 2.2. [20] *Let X and Y be Banach spaces and $L : X \to Y$ be a bounded linear operator. Then the following statements are equivalent:*
(I) *L is HU-stable;*
(II) *$R(L)$ is closed;*
(III) *$\tilde{L}^{-1}$ is bounded.*
Moreover, if any of the above conditions are satisfied, then $K_L = \| \tilde{L}^{-1} \|$.

Remark 2.3. If $L : X \to Y$ is bounded linear operator, then (i) in Definition 2.1 is equivalent to:
for any $f \in X$ with $\| Lf \| \leq 1$ there exists an $f_0 \in N(L)$ such that

$$\| f - f_0 \| \leq K. \tag{2.1}$$

It is clear from Remark 2.3 that, to study the HU-stability of a bounded linear operator $L : X \to Y$, we need to show either the existence of a constant K for (2.1) or the boundedness of the operators $\tilde{L}^{-1}$.
Let $g \in \Pi_n$, where Π_n is the set of all polynomials of degree at most n with real coefficients. Then g has a unique Lorentz representation of the form

$$g(x) = \sum_{k=0}^{n} c_k x^k (1-x)^{n-k}, \tag{2.2}$$

where $c_k \in \mathbb{R}, \ k = 0, 1, \cdots, n$.
Let T_n denotes the usual nth degree Chebyshev polynomial of the first kind. Then the following representation [11] holds:

$$T_n(2x-1) = \sum_{k=0}^{n} d_{n,k} \, x^k (1-x)^{n-k} (-1)^{n-k}, \tag{2.3}$$

where

$$d_{n,k} := \sum_{j=0}^{\min\{k,n-k\}} \binom{n}{2j} \binom{n-2j}{k-j} 4^j, \ k = 0, 1, \cdots, n.$$

It is proved in [17] that

$$d_{n,k} = \binom{2n}{2k}, \; k = 0, 1, \cdots, n. \tag{2.4}$$

Therefore,

$$T_n(2x-1) = \sum_{k=0}^{n} \binom{2n}{2k} x^k (1-x)^{n-k} (-1)^{n-k}.$$

Theorem 2.4. [11] *Let $g(x)$ has the representation* (2.2) *and $0 \leq k \leq n$. Then*

$$\mid c_k \mid \leq d_{n,k}. \parallel g \parallel_\infty,$$

where equality holds if and only if gis a constant multiple of $T_n(2x-1)$.

3. HU-stability of Bernstein-Schurer type Operators and Bernstein-Durrmeyer Operators

3.1. Bernstein-Schurer type operators

For any integer $n \geq 1$. Let Π_n denote the space of all polynomials of degree $\leq n$, which is a subspace of $C[0,1]$, a space consisting all continuous functions on $[0,1]$. Consider $C[0,1+p]$ be the linear space of all continuous functions $f : [0,1+p] \to \mathbb{R}$ having supremum norm. Let $0 \leq a \leq b$, the Bernstein-Schurer type operators $S_{n,p} : C[0,1+p] \to \Pi_{n+p}$ are defined by

$$S_{n,p}(f;x) = \sum_{k=0}^{n+p} \binom{n+p}{k} x^k (1-x)^{n+p-k} f\left(\frac{k+a}{n+b}\right).$$

These operators are HU-stable being finite dimensional operators. Here, we find the HUS constant for Bernstein-Schurer type operators.
The kernel of $S_{n,p}$ is given as:

$$N(S_{n,p}) = \left\{ f \in C[0,1+p]; f\left(\frac{k+a}{n+b}\right) = 0, 0 \leq k \leq n+p \right\}.$$

$N(S_{n,p})$ is closed subspace of $C[0,1+p]$ and $R(S_{n,p}) = \Pi_{n+p}$.

Thus, $S_{n,p} : \dfrac{C[0,1+p]}{N(S_{n,p})} \to \Pi_{n+p}$ is bijective. Hence, $\tilde{S}_{n,p}^{-1} : \Pi_{n+p} \to \dfrac{C[0,1+p]}{N(S_{n,p})}$ exists and bijective.
Now, to find the HUS constant, we need to find the $\|\tilde{S}_{n,p}^{-1}\|$.
Let $g \in \Pi_{n+p}$with $\|g\| \leq 1$ has its Lorentz representation as

$$g(x) = \sum_{k=0}^{n+p} c_k(g) x^k (1-x)^{n+p-k}, \quad x \in [0,1].$$

Consider a piecewise function

$$f_g(x)=\begin{cases} c_0(g), & x\in\left[0,\dfrac{a}{n+b}\right)\\ \dfrac{c_k(g)}{\binom{n+p}{k}}, & x\in\left[\dfrac{k+a}{n+b},\dfrac{k+a+1}{n+b}\right)\ 0\le k\le n-1\\ c_{n+p}(g), & x\in\left[\dfrac{n+a}{n+b},1\right]. \end{cases} \tag{3.1}$$

Clearly, $S_{n,p}(f_g;x)=g(x)$ that is $\tilde{S}^{-1}_{n,p}(g(x))=f_g+N(S_{n,p})$.
Thus,

$$\begin{aligned} \|\tilde{S}^{-1}_{n,p}\| &= \sup_{\|g\|\le 1}\|\tilde{S}^{-1}_{n,p}(g)\|=\sup_{\|g\|\le 1}\inf_{h\in N(S_{n,p})}\|f_g+h\|\\ &= \sup_{\|g\|\le 1}\|f_g\|=\sup_{\|g\|\le 1}\max_{0\le k\le n+p}\frac{|c_k(g)|}{\binom{n+p}{k}}\\ &\le \sup_{\|g\|\le 1}\max_{0\le k\le n+p}\frac{d_{n+p,k}\,\|g\|}{\binom{n+p}{k}} \quad \text{[Using Theorem 2.4]}\\ &\le \max_{0\le k\le n+p}\frac{d_{n+p,k}}{\binom{n+p}{k}}. \end{aligned} \tag{3.2}$$

Now, let $q(x)=T_n(2x-1),\ x\in[0,1]$ be Chebyshev poynomials. Then $\|q\|=1$ and from Theorem 2.4 $|c_k(q)|=d_{n+p,k}$. So,

$$\|\tilde{S}^{-1}_{n,p}\|\ge\max_{0\le k\le n+p}\frac{|c_k(q)|}{\binom{n+p}{k}}=\max_{0\le k\le n+p}\frac{d_{n+p,k}}{\binom{n+p}{k}}. \tag{3.3}$$

Combining (3.2) and (3.3), we get

$$\|\tilde{S}^{-1}_{n,p}\|=\max_{0\le k\le n+p}\frac{d_{n+p,k}}{\binom{n+p}{k}}=\max_{0\le k\le n+p}\frac{\binom{2n+2p}{2k}}{\binom{n+p}{k}} \qquad \text{[By (2.4)]}$$

Let $a_k=\dfrac{\binom{2n+2p}{2k}}{\binom{n+p}{k}}$, $0\le k\le n+p$. Then, $\dfrac{a_{k+1}}{a_k}=\dfrac{2n+2p-2k-1}{2k+1}$, $0\le k\le n+p-1$.
The inequality $\dfrac{a_{k+1}}{a_k}\ge 1$ is satisfied if and only if $k\le\left[\frac{n+p-1}{2}\right]$, where $[x]$ denotes the greatest integer function. So, maximum value of $a_k,\ 0\le k\le n+p$ will be at $\left[\frac{n+p-1}{2}\right]+1$.

i.e. $\displaystyle\max_{0\le k\le n+p}a_k=a_{\left[\frac{n+p-1}{2}\right]+1}=\begin{cases} a_{\left[\frac{n+p}{2}\right]}, & \text{if } n+p \text{ is even}\\ a_{\left[\frac{n+p}{2}\right]+1}=a_{\left[\frac{n+p}{2}\right]}, & \text{if } n+p \text{ is odd.}\end{cases}$

Hence, $\displaystyle\max_{0\le k\le n+p}a_k=a_{\left[\frac{n+p}{2}\right]}$.

Finally, using (3.3), $\|\tilde{S}_{n,p}^{-1}\| = \dfrac{\dbinom{2(n+p)}{2\left[\frac{n+p}{2}\right]}}{\dbinom{n+p}{\left[\frac{n+p}{2}\right]}}$.

When $p = 0$, it will reduce to HUS constant for Bernstein-Stancu operators. Also, when $p = a = b = 0$, it will reduce the HUS constant for Bernstein operators.

3.2. Bernstein-Durrmeyer operators

Durrmeyer [5] in 1967 defined Bernstein-Durrmeyer operators $D_n : C[0,1] \to C[0,1]$ as:

$$D_n(f;x) = \sum_{k=0}^{n} p_{n,k}(x) \int_0^1 p_{n,k}(t)\, f(t)\, dt, \quad x \in [0,1],\ n \geq 1. \tag{3.4}$$

As the range of the operators (3.4) is Π_n, which is finite dimensional. Hence, the operators are HU-stable. Now, we will find the HUS constant for these operators. Therefore, we will check that boundedness of its inverse operators. The kernel of $D_n(.;x)$ is:

$$N(D_n) = \left\{ f \in C[0,1];\ \int_0^1 p_{n,k}(t)\, f(t)\, dt = 0 \right\}.$$

$N(D_n)$ is closed subspace of $C[0,1]$ and $R(D_n) = \Pi_n$.

Hence, $\tilde{D}_n : \dfrac{C[0,1]}{N(D_n)} \to \Pi_n$ is bijective. So, $\tilde{D}_n^{-1}$ exists and bijective, where

$$\tilde{D}_n^{-1} : \Pi_n \to \frac{C[0,1]}{N(D_n)}.$$

Let Lorentz representation of $g(x) = \sum_{k=0}^{n} x^k (1-x)^{n-k} c_k(g)$ such that $g \in \Pi_n$ and $\|g\| \leq 1$.

Define a function $f_g \in C[0,1]$ as: $f_g(x) = \dfrac{c_k(g)}{\binom{n}{k}},\ 0 \leq k \leq n$.

Clearly, $D_n(f_g;x) = g(x)$, therefore $\tilde{D}_n^{-1}(g(x)) = f_g + N(D_n)$.

$$\begin{aligned}
\|\tilde{D}_n^{-1}\| &= \sup_{\|g\|\leq 1} \|\tilde{D}_n^{-1}(g)\| = \sup_{\|g\|\leq 1} \inf_{h \in N(D_n)} \|f_g + h\| \\
&= \sup_{\|g\|\leq 1} \|f_g\| \leq \sup_{\|g\|\leq 1} \max_{0\leq k\leq n} \frac{|\ c_k(g)\ |}{\binom{n}{k}} \\
&\leq \sup_{\|g\|\leq 1} \max_{0\leq k\leq n} \frac{d_{n,k}\|g\|}{\binom{n}{k}} \leq \max_{0\leq k\leq n} \frac{d_{n,k}}{\binom{n}{k}} \quad \text{[Using Theorem 2.4]}.
\end{aligned} \tag{3.5}$$

Now, choose $q(x) = T_n(2x-1),\ x \in [0,1]$. Clearly, $\|q\| = 1$ and $|c_k(q)| = d_{n,k}$.

$$\|\tilde{D}_n^{-1}\| \geq \max_{0\leq k\leq n} \frac{|\ c_k(q)\ |}{\binom{n}{k}} = \max_{0\leq k\leq n} \frac{d_{n,k}}{\binom{n}{k}}. \tag{3.6}$$

Using (3.5) and (3.6), we get:

$$\|\tilde{D}_n^{-1}\| = \max_{0\le k\le n} \frac{d_{n,k}}{\binom{n}{k}} = \max_{0\le k\le n} \frac{\binom{2n}{2k}}{\binom{n}{k}} \quad \text{[By (2.4)]}. \tag{3.7}$$

Consider $a_k = \frac{\binom{2n}{2k}}{\binom{n}{k}}$ and $a_{k+1} = \frac{\binom{2n}{2k+2}}{\binom{n}{k+1}}$. By simple calculations, we get

$$\frac{a_{k+1}}{a_k} = \frac{2n-2k-1}{2k+1}.$$

For $k \le \left[\frac{n-1}{2}\right]$, we have $a_{k+1} \ge a_k$.
Therefore,

$$\begin{aligned} \max_{0\le k\le n} a_k &= a_{\left[\frac{n-1}{2}\right]+1} \\ &= \begin{cases} a_{\left[\frac{n}{2}\right]}, & \text{if } n \text{ is even} \\ a_{\left[\frac{n}{2}\right]+1} = a_{\left[\frac{n}{2}\right]}, & \text{if } n \text{ is odd.} \end{cases} \end{aligned} \tag{3.8}$$

Thus, $\max\limits_{0\le k\le n} a_k = a_{\left[\frac{n}{2}\right]}$, and by (3.7)

$$\|\tilde{D}_n^{-1}\| = \frac{\binom{2n}{2\left[\frac{n}{2}\right]}}{\binom{n}{\left[\frac{n}{2}\right]}},$$

which is the HUS constant for Bernstein-Durrmeyer operators.

4. Unstability of Beta Operators with Jacobi Weights

For any $\alpha, \beta \ge -1$, the operators are defined as:

$$B_n^{\alpha,\beta}(f;x) = \frac{\displaystyle\int_0^1 t^{nx+\alpha}(1-t)^{n-nx+\beta} f(t)\,dt}{B(nx+\alpha+1, n-nx+\beta+1)}, \tag{4.1}$$

where $B(m,n)$ is the beta function. For $\alpha = \beta = 0$, these operators reduce to beta operators by Lupaş.

Theorem 4.1. *For each $n \ge 1$, the beta operators with Jacobi weights are HU-unstable.*

Proof. To define the inverse of the operators (4.1), firstly, we prove that the operators $B_n^{\alpha,\beta}(.;x)$ are injective.
Consider $B_n^{\alpha,\beta} f = 0$, for some $f \in C[0,1]$.
Thus,

$$\int_0^1 t^{nx+\alpha}(1-t)^{n-nx+\beta} f(t)\,dt = 0.$$

Now, by changing the variable $\dfrac{t}{1-t} = u$, we get

$$\int_0^\infty \frac{u^{nx+\alpha}}{(1+u)^{n+\alpha+\beta+2}} f\left(\frac{u}{1+u}\right) du = 0.$$

As $f \in C[0,1]$, therefore g defined as:

$$g(u) = \frac{1}{(1+u)^{n+\alpha+\beta+2}} f\left(\frac{u}{1+u}\right), \quad u \in [0,\infty).$$

is also continuous function on $[0,\infty)$.
Now, we have $\int_0^\infty u^{nx+\alpha}\, g(u)\, du = 0, \quad x \in [0,1]$,
Using Mellin transformation, we get:

$$M[g](nx+\alpha+1) = 0, \quad x \in [0,1].$$

Put $nx+\alpha+1 = s$, we have: $M[g](s) = 0 \quad \forall s \in [\alpha+1, n+\alpha+1]$, which gives $g(u) = 0$ a.e. on $[0,\infty)$.
But $g \in C[0,\infty)$, which implies $g(u) = 0$ on $[0,\infty)$. Therefore, $f(t) = 0$ on $[0,1]$. Hence, $B_n^{\alpha,\beta}(.;x)$ are injective.
Now, consider the inverse operators

$$(B_n^{\alpha,\beta})^{-1} : R(B_n^{\alpha,\beta}) \to C[0,1].$$

Denote $e_j(x) = x^j, j = 0, 1, \cdots, \quad x \in [0,1]$.
Clearly, $B_n^{\alpha,\beta}(e_0;x) = 1$ and

$$B_n^{\alpha,\beta}(e_j;x) = \frac{(nx+\alpha+1)(nx+\alpha+2)\cdots(nx+\alpha+j)}{(n+\alpha+\beta+2)(n+\alpha+\beta+3)\cdots(n+\alpha+\beta+j+1)}.$$

The eigenvalues of

$$B_n^{\alpha,\beta}(f;x) = \frac{n^j}{(n+\alpha+\beta+2)(n+\alpha+\beta+3)\cdots(n+\alpha+\beta+j+1)}.$$

Thus, eigenvalues of $(B_n^{\alpha,\beta})^{-1}$ are

$$\frac{(n+\alpha+\beta+2)(n+\alpha+\beta+3)\cdots(n+\alpha+\beta+j+1)}{n^j}.$$

Since, $\lim_{j\to\infty} \dfrac{(n+\alpha+\beta+2)(n+\alpha+\beta+3)\cdots(n+\alpha+\beta+j+1)}{n^j} = \infty$.
We can say that $(B_n^{\alpha,\beta})^{-1}$ is unbounded, so the operators $B_n^{\alpha,\beta}(.;x)$ are HU-unstable.
□

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Jaspreet Kaur
Department of Mathematics,
Thapar Institute of Engineering and Technology, Patiala,
Patiala 147004, India
and
Department of Mathematics,
GLA University, Mathura 281406, India
e-mail: jazzbagri3@gmail.com

Meenu Goyal
Department of Mathematics,
Thapar Institute of Engineering and Technology, Patiala,
Patiala-147004, India
e-mail: meenu_rani@thapar.edu