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An optimal quadrature formula exact to the exponential function by the phi function method

Abdullo Hayotov, Samandar Babaev,
Alibek Abduakhadov and Javlon Davronov

Abstract. The numerical integration of definite integrals is essential in fundamental and applied sciences. The accuracy of approximate integral calculations is contingent upon the initial data and specific requirements, leading to the imposition of diverse conditions on the resultant computations. Classical methods for the numerical analysis of definite integrals are known, such as the quadrature formulas of Gregory, Newton-Cotes, Euler, Gauss, Markov, etc. Since the middle of the last century, the theory of constructing optimal formulas for numerical integration based on variational methods began to develop. It should be noted that there are optimal quadrature formulas in the sense of Nikolsky and Sard.
In this paper, we study the problem of constructing an optimal quadrature formula in the sense of Sard. When constructing a quadrature formula, the method of φ-functions is used. The error of the formula is estimated from above using the integral of the square of the function φ from a specific Hilbert space. Next, such a φ function is selected, and the integral of the square in this interval takes the smallest value. The coefficients of the optimal quadrature formula are calculated using the resulting φ function. The optimal quadrature formula in this work is exact on the functions $e^{\sigma x}$ and $e^{-\sigma x}$, where σ is a nonzero real parameter.

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1. Introduction

Many problems of science and technology lead to integral and differential equations or their systems. Solutions to such equations are often expressed in terms of definite integrals. In most cases, these integrals cannot be calculated accurately. Therefore, it is necessary to calculate the approximate value of such integrals with the highest possible accuracy and at a low cost.

Given the known geometric value, the problem of finding the numerical value of integrals is often called quadrature and cubature, respectively. Various quadrature and cubature methods allow the calculation of the integral using a finite number of values of the integrated function. These methods are universal and can be used where other calculation methods fail.

Many researchers have constructed various quadrature formulas based on certain ideas and taking into account the properties of the integrand. Thus, the well-known quadrature formulas of Gregory, Newton-Cotes, Simpson, Euler, Gauss, Chebyshev, Markov and others appeared, still used in practice.

Currently, in the theory of constructing quadrature and cubature formulas, there are the following main approaches: *algebraic, probabilistic, numerical theoretic and functional.*

1. *In the algebraic approach*, it is necessary to choose the nodes and coefficients of quadrature and cubature formulas so that these formulas are accurate for all functions of a particular set F. Taking into account the properties of the integrand. Usually, the set F is taken to be algebraic and trigonometric polynomials whose degrees do not exceed a certain number of m or bounded rational functions.
2. *The probabilistic approach* to constructing cubature formulas is based on the Monte Carlo method.
3. *Number-theoretic approach* to constructing cubature formulas is based on methods of number theory.
4. *For functional approach* to constructing quadrature and cubature formulas, in the functional system, it is considered that the integrands belong to some Banach space, and the difference between the integral and the combination of values of the integrand that approximates it is considered some linear continuous functional. This functional is called the error functional of the cubature formula, and the error of the formula is estimated through the norms of the error functional. By minimizing the norms of the error functional according to the parameters of quadrature and cubature formulas, optimal formulas for numerical integration of various meanings are obtained.

Since the research in this work relates to the latter approach, we will provide an overview of the results in this area.

The construction of quadrature formulas and the study of their error estimates, based on the methods of functional analysis, were first given in the scientific works of A. Sard [20, 21] (minimizing the norm of the error functional can be achieved by adjusting the coefficients at fixed nodes) and S. M. Nikolsky [18] (minimization of the error functional by coefficients and nodes), and the emergence of the theory of cubature formulas is associated with the scientific research of S.L. Sobolev [29].

The works of S.M. Nikolsky, N.P. Korneichuk, N.E. Lushpay, T.N. Busarova, B. Boyanov, V.P. Motorny, A.A. Ligun, A.A. Zhensykbaev, K.I. Oskolkov, M.A. Chakhkiev, T.A. Grankina are devoted to the problems of minimizing the norm of the error functional over coefficients and over nodes in various spaces in the one-dimensional case. For example, detailed results and a complete bibliography are given in the creation of S.M. Nikolsky [19].

Note that there is a spline method, a method of φ-functions and a Sobolev method for constructing optimal formulas obtained by minimizing the norm of the error functional over coefficients at fixed nodes. A.Sard [20, 21], L.F.Meyers [17], G.Coman [6, 7], I.J.Schoenberg [22, 23, 24, 25], S.D.Silliman [25], P.Köhler [15], based on the spline method, and A.Ghizzetti and A.Ossicini [9], F.Lanzara [16], T.Catinaş and G.Coman [5] , using the method of φ-functions, constructed optimal quadrature formulas in the space $L_2^{(m)}$. In constructing optimal cubature formulas using the Sobolev method, the results of S.L. Sobolev [30] on finding the coefficients of optimal quadrature formulas generalized the studies mentioned earlier in which the spline method was applied. The algorithm proposed by S.L. Sobolev in the $L_2^{(m)}$ space was implemented in scientific research by Z.Z.Zhamalov, F.Ya.Zagirova, Kh.M. Shadimetov, A.R. Hayotov and others. Recent results on optimal formulas obtained using the Sobolev method can be found, for example, in the works [1, 27].

Note that the results of this work are closely related to the results of the works [26, 2, 3, 8, 14, 10, 28, 4, 11, 12], which are devoted to the construction of optimal quadrature formulas using the Sobolev method. In particular, our results generalize the results of recent work [13].

2. Statement of the problem

In this work, we study the construction of an optimal quadrature formula using the method of φ-functions. In this regard, consider quadrature formulas of the form

$$\int_a^b f(x)dx = \sum_{k=0}^{n} A_k f(x_k) + R_n(f), \tag{2.1}$$

where A_k and x_k are the coefficients and the nodes of the quadrature formula. Let the nodes of the formula be located on the segment $[a, b]$ as follows

$$a = x_0 < x_2 < \ldots < x_n = b, \tag{2.2}$$

and $R_n(f)$ is the error of formula (2.1).

Suppose that the integrand function $f(x)$ is from the space $W_{2,\sigma}^{(1,0)}(a, b)$, where $W_{2,\sigma}^{(1,0)}(a, b)$ is the Hilbert space of absolutely continuous functions that are quadratically integrable with the first-order derivative on the interval $[a, b]$. The scalar product of any two functions $f(x)$ and $g(x)$ from this space is defined by the following formula

$$\langle f(x), g(x)\rangle = \int_a^b (f'(x) + \sigma f(x))(g'(x) + \sigma g(x))dx, \tag{2.3}$$

where $\sigma \in \mathbb{R}$ and $\sigma \neq 0$. This space is provided with the corresponding norm

$$\|f(x)\|_{W_{2,\sigma}^{(1,0)}} = \left(\int_a^b (f'(x) + \sigma f(x))^2 dx \right)^{1/2}. \quad (2.4)$$

One of the important problems in the theory of quadrature formulas is the problem of the optimality of quadrature formulas relative to the error of this formula. In this paper, we will consider the problem of optimality of a formula in the sense of Sard. We use the one-to-one correspondence between quadrature formulas and φ - functions in this.

For convenience, we introduce the multi-index notations

$$A = (A_0, A_1, \dots, A_n) \text{ and } X = (x_0, x_1, \dots, x_n). \quad (2.5)$$

Definition 2.1. The quadrature formula (2.1) is called *optimal in the sense of Nikolsky* in space $W_{2,\sigma}^{(1,0)}$, if the value

$$F_n(W_{2,\sigma}^{(1,0)}, A, X) = \sup_{f \in W_{2,\sigma}^{(1,0)}} |R_n(f)| \quad (2.6)$$

reaches its smallest value relative to A and X, where A and X are defined in (2.5).

Definition 2.2. The quadrature formula (2.1) is called *optimal in the sense of Sard* in the space $W_{2,\sigma}^{(1,0)}$ if the quantity

$$F_n(W_{2,\sigma}^{(1,0)}, A) = \sup_{f \in W_{2,\sigma}^{(1,0)}} |R_n(f)| \quad (2.7)$$

reaches its smallest value relative to A for fixed X, where A and X are defined in (2.5).

In this work, we solve the problem of constructing an optimal quadrature formula of the form (2.1) in the sense of Sard in the space $W_{2,\sigma}^{(1,0)}(a,b)$, i.e., let us find such coefficients of the formula (2.1) that give the smallest value to the quantity (2.7) for fixed X. In this case, we use the method of φ-functions.

Next, the rest of this work is organized as follows. Section 3 describes the method of φ-functions for constructing quadrature formulas of the form (2.1) in the space $W_{2,\sigma}^{(1,0)}$ and provides the relationship between the coefficients and φ-functions. Section 4 is devoted to obtaining φ-functions that give the smallest error value of a quadrature formula of the form (2.1). Using the obtained φ-functions, the coefficients of the optimal quadrature formula of the form (2.1) are calculated.

3. Method of φ - functions for constructing quadrature formulas in the space $W_{2,\sigma}^{(1,0)}$

In this section, we explain the method of φ - functions for constructing optimal quadrature formulas of the form (2.1) in the sense of Sard in the space $W_{2,\sigma}^{(1,0)}$. For more details on the φ - function method see, for instance, [5, 16, 9].

Let functions $f(x)$ be from the space $W_{2,\sigma}^{(1,0)}(a,b)$ and for a given positive integer n the nodes of the quadrature formula under consideration are located as in (2.5). Then for each subinterval $[x_{k-1},x_k]$, $k=1,2,\dots,n$, consider the functions φ_k, $k=1,2,\dots,n$ having the following property

$$\varphi_k'(x)-\sigma\varphi_k(x)=1,\quad k=1,2,\dots,n. \tag{3.1}$$

Then the function φ is defined as follows

$$\varphi|_{[x_{k-1},x_k]}=\varphi_k(x),\quad k=1,2,\dots,n. \tag{3.2}$$

That is, the restriction of the function φ to the interval $[x_{k-1},x_k]$ is equal to φ_k.

Let us introduce the following notation

$$I(f):=\int_a^b f(x)dx, \tag{3.3}$$

$$Q_n(f):=\sum_{k=0}^{n} A_k f(x_k). \tag{3.4}$$

Now, using the property of additivity of a definite integral, taking into account the equalities (3.1), from (3.3), we have (see, e.g., [5, 16, 9])

$$\begin{aligned}
I(f):&=\int_a^b f(x)dx=\sum_{k=1}^{n}\int_{x_{k-1}}^{x_k}(\varphi_k'(x)-\sigma\varphi_k(x))f(x)dx\\
&=\sum_{k=1}^{n}\Big(\int_{x_{k-1}}^{x_k}\varphi_k'(x)f(x)dx-\sigma\int_{x_{k-1}}^{x_k}\varphi_k(x)f(x)dx\Big)\\
&=\sum_{k=1}^{n}\Big(\varphi_k(x)f(x)\Big|_{x_{k-1}}^{x_k}-\int_{x_{k-1}}^{x_k}\varphi_k(x)f'(x)dx-\sigma\int_{x_{k-1}}^{x_k}\varphi_k(x)f(x)dx\Big)\\
&=\sum_{k=1}^{n}\Big(\varphi_k(x)f(x)\Big|_{x_{k-1}}^{x_k}-\int_{x_{k-1}}^{x_k}\varphi_k(x)(f'(x)+\sigma f(x))\Big)dx\\
&=\sum_{k=1}^{n}\Big(\varphi_k(x_k)f(x_k)-\varphi_k(x_{k-1})f(x_{k-1})\Big)\\
&-\sum_{k=1}^{n}\int_{x_{k-1}}^{x_k}\varphi_k(x)\Big(f'(x)+\sigma f(x)\Big)dx\\
&=\sum_{k=1}^{n}\varphi_k(x_k)f(x_k)-\sum_{k=1}^{n}\varphi_k(x_{k-1})f(x_{k-1})\\
&-\sum_{k=1}^{n}\int_{x_{k-1}}^{x_k}\varphi_k(x)\Big(f'(x)+\sigma f(x)\Big)dx
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{k=1}^{n} \varphi_k(x_k) f(x_k) - \sum_{k=0}^{n-1} \varphi_{k+1}(x_k) f(x_k) \\
&- \sum_{k=1}^{n} \int_{x_{k-1}}^{x_k} \varphi_k(x) \Big(f'(x) + \sigma f(x) \Big) dx \\
&= \varphi_n(x_n) f(x_n) + \sum_{k=1}^{n-1} \varphi_k(x_k) f(x_k) - \sum_{k=1}^{n-1} \varphi_{k+1}(x_k) f(x_k) \\
&- \varphi_1(x_0) f(x_0) - \sum_{k=1}^{n} \int_{x_{k-1}}^{x_k} \varphi_k(x) \Big(f'(x) + \sigma f(x) \Big) dx.
\end{aligned}
$$

From here we have

$$
\begin{aligned}
I(f) := \; & -\varphi_1(x_0) f(x_0) + \sum_{k=1}^{n-1} \Big(\varphi_k(x_k) - \varphi_{k+1}(x_k) \Big) f(x_k) \\
& + \varphi_n(x_n) f(x_n) - \sum_{k=1}^{n} \int_{x_{k-1}}^{x_k} \varphi_k(x) \Big(f'(x) + \sigma f(x) \Big) dx \\
= \; & A_0 f(x_0) + \sum_{k=1}^{n-1} A_k f(x_k) + A_n f(x_n) + R_n[f]. \qquad (3.5)
\end{aligned}
$$

From (3.5) we get

$$
\begin{aligned}
A_0 &= -\varphi_1(x_0), \\
A_k &= \varphi_k(x_k) - \varphi_{k+1}(x_k), \quad k = 1, 2, \ldots, n-1, \qquad (3.6) \\
A_n &= \varphi_n(x_n)
\end{aligned}
$$

and the error of the formula has the form

$$
\begin{aligned}
R_n[f] &= -\sum_{k=1}^{n} \int_{x_{k-1}}^{x_k} \varphi_k(x) \Big(f'(x) + \sigma f(x) \Big) dx \\
&= -\int_a^b \varphi(x) \Big(f'(x) + \sigma f(x) \Big). \qquad (3.7)
\end{aligned}
$$

Remark 3.1. Knowing the function φ from (3.6) we can find the coefficients A_k, $k = 0, 1, \ldots, n$. This method of constructing a quadrature formula is called the method of φ - functions (see, [5, 16, 9]).

Remark 3.2. From the expression (3.7) it is clear that the quadrature formula (2.1) is exact on functions that are a solution to the equation

$$
f'(x) + \sigma f(x) = 0. \qquad (3.8)
$$

Further, in the next section, we are engaged in calculating the coefficients of the optimal quadrature formula of the form (2.1) in the space $W_{2,\sigma}^{(1,0)}(a, b)$.

4. The optimality problem for a quadrature formula

In this section, we will discuss the problem of optimality of a quadrature formula of the form (2.1) in the space $W_{2,\sigma}^{(1,0)}(a,b)$.

Using the Cauchy-Schwartz inequality for the absolute value of the error (3.7) of the quadrature formula (2.1) we have the following

$$\begin{aligned} |R_n(f)| &\leq \|f'(x)+\sigma f(x)\|_{L_2(a,b)} \left(\int_a^b \varphi^2(x)dx\right)^{1/2} \\ &= \|f(x)\|_{W_{2,\sigma}^{(1,0)}} \|\varphi(x)\|_{L_2(a,b)}. \end{aligned} \tag{4.1}$$

It should be noted that here the task of constructing an optimal quadrature formula of the form (2.1) in the sense of Sard in the space $W_{2,\sigma}^{(1,0)}(a,b)$ is the task of finding the coefficients $A=(A_0,A_1,\dots,A_n)$ (for fixed nodes $X=(x_0,x_1,\dots,x_n)$ satisfying the condition (2.2)) giving the smallest value to the quantity

$$F_n(A)=\int_a^b \varphi^2(x)dx. \tag{4.2}$$

In turn, this problem is equivalent to finding functions $\varphi_k(x)$, $k=1,2,\dots,n$, satisfying the equation (3.1) and giving the smallest value to the quantity (4.2) on each interval $[x_{k-1},x_k]$, $k=1,2,\dots,n$.

Next, for the beginning we will find the functions $\varphi_k(x)$, $k=1,2,\dots,n$ that give the smallest value to the quantity (4.2) and then using the formulas (3.6) we will calculate coefficients A_k, $k=0,1,\dots,n$ of the optimal quadrature formula (2.1).

4.1. Finding functions φ_k

Now we are engaged in finding the functions φ_k on each interval $[x_{k-1},x_k]$ for $k=1,2,\dots,n$, which are the solution to the equation

$$y'-\sigma y=1. \tag{4.3}$$

We will seek a solution to this equation in the form of the product $y=uy_1$ of the functions $u(x)$ and $y_1(x)$, where y_1 is the solution to the corresponding homogeneous equation

$$y'-\sigma y=0. \tag{4.4}$$

It is easy to check that one of the solutions to the equation (4.4) has the form

$$y_1(x)=e^{\sigma x}. \tag{4.5}$$

Now we can solve equation (4.3). By assumption, the solution to equation (4.3) has the form

$$y(x)=uy_1, \tag{4.6}$$

where $y_1(x)$ is defined by equality (4.5). Then we just need to find the function $u(x)$. To find it, we first calculate the first-order derivative of the unknown function $y(x)$ in (4.6). Then we have

$$y'=u'y_1+uy_1'. \tag{4.7}$$

Substituting (4.6) into (4.3), taking into account (4.5), we get

$$u'e^{\sigma x} + u\sigma e^{\sigma x} - u\sigma e^{\sigma x} = 1.$$

From here

$$u' = e^{-\sigma x}.$$

Integrating both sides of the last equality we have

$$u(x) = -\frac{1}{\sigma}e^{-\sigma x} + C.$$

This means, taking into account the last equality and (4.5), for the solution of equation (4.3) we obtain

$$y = -\frac{1}{\sigma} + Ce^{\sigma x}. \tag{4.8}$$

Next, on each interval $[x_{k-1}, x_k]$, $k = 1, 2, \dots, n$ we take functions $\varphi_k(x)$ in the form (4.8), i.e.

$$\varphi_k(x) = -\frac{1}{\sigma} + C_1^{(k)} e^{\sigma x}, \quad x \in [x_{k-1}, x_k],\ k = 1, 2, \dots, n. \tag{4.9}$$

From here we conclude that to find the functions $\varphi_k(x)$ we need to find such coefficients $C_1^{(k)}$, $k = 1, 2, \dots, n$, which give the smallest values to the quantity (4.2) on each of the intervals $[x_{k-1}, x_k]$ for $k = 1, 2, \dots, n$. Next, we find $C_1^{(k)}$ such that the integral of the square of the function $\varphi_k(x)$ defined by equality (4.9) on the interval $[x_{k-1}, x_k]$ takes the smallest value. In this regard, consider the following functions

$$\mathcal{F}_k(C_1^{(k)}) = \int\limits_{x_{k-1}}^{x_k} \varphi_k^2(x)dx, \quad k = 1, 2, \dots, n.$$

Then from here, taking into account (4.9), we have

$$\begin{aligned}
\mathcal{F}_k(C_1^{(k)}) &= \int\limits_{x_{k-1}}^{x_k} \left(-\frac{1}{\sigma} + C_1^{(k)} e^{\sigma x}\right)^2 dx \\
&= \int\limits_{x_{k-1}}^{x_k} \frac{1}{\sigma^2} dx - 2C_1^{(k)}\frac{1}{\sigma} \int\limits_{x_{k-1}}^{x_k} e^{\sigma x} dx \\
&\quad +(C_1^{(k)})^2 \int\limits_{x_{k-1}}^{x_k} e^{2\sigma x} dx,\ k = 1, 2, \dots, n.
\end{aligned}$$

Then calculating the first order derivatives of the functions $\mathcal{F}_k(C_1^{(k)})$ with respect to $C_1^{(k)}$ and equating them to zero, we have

$$2C_1^{(k)} \int\limits_{x_{k-1}}^{x_k} e^{2\sigma x} dx - \frac{2}{\sigma} \int\limits_{x_{k-1}}^{x_k} e^{\sigma x} dx = 0,\ k = 1, 2, \dots, n.$$

From the last equalities we get the following

$$C_1^{(k)} = \frac{\frac{1}{\sigma}\int\limits_{x_{k-1}}^{x_k} e^{\sigma x}dx}{\int\limits_{x_{k-1}}^{x_k} e^{2\sigma x}dx} = \frac{2}{\sigma\left(e^{\sigma x_k}+e^{\sigma x_{k-1}}\right)},\ k=1,2,\dots,n. \tag{4.10}$$

It is easy to check that this value of $C_1^{(k)}$ gives the smallest value to the function $\mathcal{F}_k(C_1^{(k)})$ on the interval $[x_{k-1},x_k]$. Then, taking into account (4.10), from (4.9) we have

$$\varphi_k(x) = -\frac{1}{\sigma} + \frac{2e^{\sigma x}}{\sigma\left(e^{\sigma x_k}+e^{\sigma x_{k-1}}\right)},\quad x\in[x_{k-1},x_k],\ k=1,2,\dots,n. \tag{4.11}$$

4.2. Calculation of coefficients of the optimal quadrature formula

Now, using (4.11), from the formulas (3.6) we calculate the coefficients A_k, $k=0,1,\dots,n$ of the optimal quadrature formula of the form (2.1).

First, let's calculate A_0. From (3.6), taking into account $\varphi_1(x)$, we have

$$\begin{aligned} A_0 &= -\varphi_1(x_0) = -\left(-\frac{1}{\sigma}+\frac{2e^{\sigma x_0}}{\sigma\left(e^{\sigma x_1}+e^{\sigma x_0}\right)}\right)\\ &= \frac{e^{\sigma x_1}-e^{\sigma x_0}}{\sigma(e^{\sigma x_1}+e^{\sigma x_0})}. \end{aligned} \tag{4.12}$$

Now let's calculate the coefficients A_k, $k=1,2,\dots,n-1$. From (3.6), using $\varphi_k(x)$ for $k=1,2,\dots,n-1$, we have

$$\begin{aligned} A_k &= \varphi_k(x_k)-\varphi_{k+1}(x_k)\\ &= \left(-\frac{1}{\sigma}+\frac{2e^{\sigma x_k}}{\sigma\left(e^{\sigma x_k}+e^{\sigma x_{k-1}}\right)}\right)-\left(-\frac{1}{\sigma}+\frac{2e^{\sigma x_k}}{\sigma\left(e^{\sigma x_{k+1}}+e^{\sigma x_k}\right)}\right)\\ &= \frac{2e^{\sigma x_k}\left(e^{\sigma x_{k+1}}-e^{\sigma x_{k-1}}\right)}{\sigma(e^{\sigma x_{k+1}}+e^{\sigma x_k})(e^{\sigma x_k}+e^{\sigma x_{k-1}})}. \end{aligned} \tag{4.13}$$

Finally, let's calculate the last coefficient A_n. Then, from (3.6), taking into account (4.11), we obtain

$$\begin{aligned} A_n &= -\varphi_n(x_n) = -\left(-\frac{1}{\sigma}+\frac{2e^{\sigma x_n}}{\sigma\left(e^{\sigma x_n}+e^{\sigma x_{n-1}}\right)}\right)\\ &= \frac{e^{\sigma x_n}-e^{\sigma x_{n-1}}}{\sigma(e^{\sigma x_n}+e^{\sigma x_{n-1}})}. \end{aligned} \tag{4.14}$$

Thus, summing up the results of (4.12), (4.13) and (4.14), we obtain the following main theorem of this work.

Theorem 4.1. *In the space $W_{2,\sigma}^{(1,0)}(a,b)$ for each fixed positive integer n, there is a unique quadrature formula that is optimal in the sense of Sard of the form*

$$\int\limits_a^b f(x)dx = \sum_{k=0}^{n} A_k f(x_k) + R_n(f)$$

with coefficients

$$\begin{aligned} A_0 &= \frac{e^{\sigma x_1} - e^{\sigma x_0}}{\sigma(e^{\sigma x_1} + e^{\sigma x_0})}, \\ A_k &= \frac{2e^{\sigma x_k}(e^{\sigma x_{k+1}} - e^{\sigma x_{k-1}})}{\sigma(e^{\sigma x_{k+1}} + e^{\sigma x_k})(e^{\sigma x_k} + e^{\sigma x_{k-1}})}, \quad k = 1, 2, \ldots, n-1, \\ A_n &= \frac{e^{\sigma x_n} - e^{\sigma x_{n-1}}}{\sigma(e^{\sigma x_n} + e^{\sigma x_{n-1}})}, \end{aligned}$$

for fixed nodes x_k, $k = 0, 1, \ldots, n$ satisfying the inequality $a = x_0 < x_1 < \ldots < x_k = b$.

Remark 4.2. It should be noted that for $[a, b] = [0, 1]$ and $x_k = kh$, where $k = 0, 1, \ldots, N$, $h = 1/N$ from Theorem 1 we obtain the result of the work [13].

5. The norm of φ–function

According to inequality (4.1), we need to calculate the norm of the function φ to get an upper bound of the absolute value of the error (3.7)

$$\|\varphi\|^2_{L_2(a,b)} = \int_0^1 \varphi^2(x)\mathrm{d}x = \sum_{k=1}^{N} \int_{x_{k-1}}^{x_k} \varphi_k^2(x)\mathrm{d}x. \tag{5.1}$$

From the expression of the function φ_k in equation (4.11), we can calculate the following

$$\begin{aligned} \varphi_k^2(x) &= \left(\frac{-1}{\sigma} + \frac{2e^{\sigma x}}{\sigma(e^{\sigma x_k} + e^{\sigma x_{k-1}})}\right)^2 \\ &= \frac{1}{\sigma^2} - \frac{4e^{\sigma x}}{\sigma^2(e^{\sigma x_k} + e^{\sigma x_{k-1}})} + \frac{4e^{2\sigma x}}{\sigma^2(e^{\sigma x_k} + e^{\sigma x_{k-1}})^2}. \end{aligned}$$

Substituting the last expression into equation (5.1), we get the following

$$\begin{aligned} \int_{x_{k-1}}^{x_k} \varphi_k^2(x)dx &= \int_{x_{k-1}}^{x_k} \left(\frac{1}{\sigma^2} - \frac{4e^{\sigma x}}{\sigma^2(e^{\sigma x_k} + e^{\sigma x_{k-1}})} + \frac{4e^{2\sigma x}}{\sigma^2(e^{\sigma x_k} + e^{\sigma x_{k-1}})^2}\right) dx \\ &= \int_{x_{k-1}}^{x_k} \frac{1}{\sigma^2} dx - \int_{x_{k-1}}^{x_k} \frac{4e^{\sigma x}}{\sigma^2(e^{\sigma x_k} + e^{\sigma x_{k-1}})} dx \\ &\quad + \int_{x_{k-1}}^{x_k} \frac{4e^{2\sigma x}}{\sigma^2\left(e^{\sigma x_k} + e^{\sigma x_{k-1}}\right)^2} dx \\ &= \frac{x_k - x_{k-1}}{\sigma^2} - \frac{4\left(e^{\sigma x_k} - e^{\sigma x_{k-1}}\right)}{\sigma^3\left(e^{\sigma x_k} + e^{\sigma x_{k-1}}\right)} + \frac{2\left(e^{2\sigma x_k} - e^{2\sigma x_{k-1}}\right)}{\sigma^3\left(e^{\sigma x_k} + e^{\sigma x_{k-1}}\right)^2} \\ &= \frac{x_k - x_{k-1}}{\sigma^2} - \frac{2\left(e^{\sigma x_k} - e^{\sigma x_{k-1}}\right)}{\sigma^3\left(e^{\sigma x_k} + e^{\sigma x_{k-1}}\right)}. \end{aligned}$$

Thus, putting the obtained expression into equation (5.1), we get the following result

$$\begin{aligned}\|\varphi\|^2_{L_2(a,b)} &= \sum_{k=1}^{N}\int_{x_{k-1}}^{x_k}\varphi_k^2(x)dx = \sum_{k=1}^{N}\left(\frac{x_k-x_{k-1}}{\sigma^2}-\frac{2\left(e^{\sigma x_k}-e^{\sigma x_{k-1}}\right)}{\sigma^3\left(e^{\sigma x_k}+e^{\sigma x_{k-1}}\right)}\right)\\ &= \frac{x_n-x_0}{\sigma^2}-\frac{2}{\sigma^3}\sum_{k=1}^{N}\frac{e^{\sigma x_k}-e^{\sigma x_{k-1}}}{e^{\sigma x_k}+e^{\sigma x_{k-1}}}.\end{aligned}$$

We have the next result in the case of equally spaced nodes $x_k = hk$, $h=\frac{1}{N}$

$$\|\varphi\|^2_{L_2(a,b)} = \sum_{k=1}^{N}\int_{x_{k-1}}^{x_k}\varphi_k^2(x)dx = \frac{1}{\sigma^2}-\frac{2}{\sigma^3 h}\cdot\frac{e^{\sigma h}-1}{e^{\sigma h}+1}.$$

6. Conclusion

In this work, we constructed an optimal quadrature formula in the space $W_{2,\sigma}^{(1,0)}(a,b)$, where $W_{2,\sigma}^{(1,0)}(a,b)$ is the Hilbert space of absolutely continuous functions whose first-order derivatives are square-integrable on the interval $[a,b]$. Here the quadrature sum consists of a linear combination of the values $f(x_k)$ of the function $f(x)$ at the nodes $x_k\in[a,b]$, where $a=x_0<x_1<\ldots<x_n=b$. The error of the quadrature formula under consideration is estimated from above using the product of the norm of the integrand and the L_2 norm of the particular φ function from the space $W_{2,\sigma}^{(1,0)}(a,b)$. Moreover, this φ function is determined by an unknown factor on each subinterval. The optimal quadrature formula is obtained by choosing these factors, which provide the smallest value of the L_2-norm of the φ function. In this work, we found such a φ - function. Explicit coefficients for optimal quadrature are found using φ-function. The resulting quadrature formula is exact for the functions $e^{\sigma x}$ and $e^{-\sigma x}$. In particular, well-known results are obtained from the results of this work.

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Abdullo Hayotov
[1]V.I.Romanovskiy Institute of Mathematics,
9, University Street, 100174 Tashkent, Uzbekistan
[2]Central Asian University, 264, Milliy Bog Street, 111221 Tashkent, Uzbekistan
[3]Bukhara State University, 11, M. Ikbol Street, 200117 Bukhara, Uzbekistan
e-mail: hayotov@mail.ru

Samandar Babaev
[1]V.I.Romanovskiy Institute of Mathematics,
9, University Street, 100174 Tashkent, Uzbekistan
[2]Tashkent International University,
7, Kichik Khalka Yoli Street, 100084 Tashkent, Uzbekistan
[3]Bukhara State University, 11, M. Ikbol Street, 200117 Bukhara, Uzbekistan
e-mail: bssamandar@gmail.com

Alibek Abduakhadov
Bukhara State University, 11, M. Ikbol Street, 200117 Bukhara, Uzbekistan
e-mail: alibekabduaxadov@gmail.com

Javlon Davronov
[1]V.I. Romanovskiy Institute of Mathematics,
9, University Street, 100174 Tashkent, Uzbekistan
[2]Tashkent International University,
7, Kichik Khalka Yoli Street, 100084 Tashkent, Uzbekistan
e-mail: javlondavronov77@gmail.com