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Metric conditions, graphic contractions and weakly Picard operators

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Abstract. In the paper of S. Park (*Almost all about Rus-Hicks-Rhoades maps in quasi-metric spaces*, Adv. Theory Nonlinear Anal. Appl., **7**(2023), No. 2, 455-472), the author solves the following problem: *Which metric conditions imposed on f imply that f is a graphic contraction?* In this paper we study the following problem: *Which metric conditions imposed on f imply that f satisfies the conditions of Rus saturated principle of graphic contractions?*

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1. Introduction and preliminaries

Let (X,d) be an L-space and $f : X \to X$ be a mapping. By following [17], [16] and [15], we present the following notions and notations, which will be used in the sequel of this paper.

By definition, f is a pre-weakly Picard mapping ($PWPM$) if the sequence $\{f^n(x)\}_{n\in\mathbb{N}}$ converges for all $x \in X$. If f is $PWPM$, then we consider the mapping, $f^\infty : X \to X$ defined by, $f^\infty(x) := \lim\limits_{n\to\infty} f^n(x)$, for all $x \in X$.

If f is a $PWPM$ and $f^\infty(x) \in F_f$, for any $x \in X$, then by definition, f is a weakly Picard mapping (WPM). Each WPM generates a partition of X. Let $x^* \in F_f$

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and $X_{x^*} := \{x \in X \mid f^n(x) \to x^* \text{ as } n \to \infty\}$. Then, $X = \bigcup_{x^* \in F_f} X_{x^*}$ is a partition of X. In this case, we have that: $f(X_{x^*}) \subset X_{x^*}$ and $X_{x^*} \cap F_f = \{x^*\}$, for all $x^* \in F_f$.

If f is WPM and $F_f = \{x^*\}$, then by definition, f is Picard mapping (PM).

The following result was given by I.A. Rus in [15].

Theorem 1.1 (Saturated principle of graphic contractions). *Let (X, d) be a complete metric space and $f : X \to X$ be a graphic l-contraction, i.e., $0 < l < 1$ and $d(f^2(x), f(x)) \leq l d(x, f(x))$, for all $x \in X$. Then we have that:*

(i) *$\{f^n(x)\}_{n\in\mathbb{N}}$ converges for all $x \in X$ and $\sum_{n\in\mathbb{N}} d(f^n(x), f^{n+1}(x)) < +\infty$, for all $x \in X$.*

If in addition, $\lim_{n\to\infty} f(f^n(x)) = f(\lim_{n\to\infty} f^n(x))$, for all $x \in X$, then:

(ii) *$F_f = F_{f^n} \neq \emptyset$, for all $n \in \mathbb{N}^*$.*

(iii) *f is a WPM.*

(iv) *$d(x, f^\infty(x)) \leq \frac{1}{1-l} d(x, f(x))$, for all $x \in X$.*

(v) *The fixed point equation corresponding to f is Ulam-Hyers stable.*

(vi) *$x^* \in F_f$, $y_n \in X_{x^*}$, $d(y_n, f(y_n)) \to 0$ as $n \to \infty \Rightarrow y_n \to x^*$ as $n \to \infty$, i.e., the fixed point problem for f is well-posed.*

(vii) *If in addition, $l < \frac{1}{3}$, then, $x^* \in F_f$, $y_n \in X_{x^*}$, $d(y_{n+1}, f(y_n)) \to 0$ as $n \to \infty \Rightarrow y_n \to x^*$ as $n \to \infty$, i.e., f has the Ostrowski property.*

Notice that if $card F_f \leq 1$, then Theorem 1.1 takes the following form:

Theorem 1.2. *Let (X, d) be a complete metric space and $f : X \to X$ be a graphic l-contraction. We assume that $card F_f \leq 1$. Then we have that:*

(i) *$\{f^n(x)\}_{n\in\mathbb{N}}$ converges for all $x \in X$ and $\sum_{n\in\mathbb{N}} d(f^n(x), f^{n+1}(x)) < +\infty$, for all $x \in X$.*

If in addition, $\lim_{n\to\infty} f(f^n(x)) = f(\lim_{n\to\infty} f^n(x))$, for all $x \in X$, then:

(ii) *$F_f = F_{f^n} = \{x^*\}$, for all $n \in \mathbb{N}^*$.*

(iii) *f is a PM.*

(iv) *$d(x, x^*) \leq \frac{1}{1-l} d(x, f(x))$, for all $x \in X$.*

(v) *The fixed point equation corresponding to f is Ulam-Hyers stable.*

(vi) *$y_n \in X$, $d(y_n, f(y_n)) \to 0$ as $n \to \infty \Rightarrow y_n \to x^*$ as $n \to \infty$, i.e., the fixed point problem for f is well-posed.*

(vii) *If in addition, $l < \frac{1}{3}$, then $y_n \in X$, $d(y_{n+1}, f(y_n)) \to 0$ as $n \to \infty \Rightarrow y_n \to x^*$ as $n \to \infty$, i.e., f has the Ostrowski property.*

On the other hand, in the metric fixed point theory, there is a large number of metric conditions (see [12], [9], [2], [18], [14], [13], [1], [4], [20], ...).

In the paper [10], S. Park solves the following problem: *Which metric conditions imposed on f imply that f is a graphic contraction?*

In this paper we study the following problem: *Which metric conditions imposed on f imply that f satisfies the conditions of Rus saturated principle of graphic contractions?*

Throughout this paper we follow the notation and terminology used in [15], [17], [3] and [19].

2. Conditions with respect to a standard metric

Let (X, d) be a metric space and $f : X \to X$ be a mapping. In many fixed point results, there are several standard metric conditions imposed on f with respect to d, which imply that f is a graphic contraction. For example, we have the Banach, Kannan, Ćirić-Reich-Rus, Ćirić, Berinde, Zamfirescu's metric conditions. More of them can be found in the paper of S. Park [10].

In this section we will focus on some other interesting metric conditions, implying the graphic contraction property of the mapping f.

- *Hardy-Rogers' metric condition* (see [6] and also [13]).
  f is called *HR mapping* if there exist three constants $a, b, c \in \mathbb{R}_+$, with $a + 2b + 2c \in (0, 1)$, such that

$$d(f(x), f(y)) \leq ad(x, y) + b[d(x, f(x)) + d(y, f(y))]+ \\ + c[d(x, f(y)) + d(y, f(x))], \text{ for all } x, y \in X.$$

- *Khojasteh, Abbas and Costache's metric condition* (see [8]).
  f is called *KAC mapping* if

$$d(f(x), f(y)) \leq \frac{d(y, f(x)) + d(x, f(y))}{d(x, f(x)) + d(y, f(y)) + 1} d(x, y), \text{ for all } x, y \in X.$$

- *Interpolative Kannan's metric condition* (see [7]).
  f is called *IK mapping* if there exist two constants $\lambda \in [0, 1)$ and $\alpha \in (0, 1)$ such that

$$d(f(x), f(y)) \leq \lambda[d(x, f(x))]^{\alpha} \cdot [d(y, f(y))]^{1-\alpha}, \text{ for all } x, y \in X \setminus F_f.$$

2.1. The case of HR mappings

Lemma 2.1. *Let (X, d) be a metric space. Let $f : X \to X$ be a HR mapping. Then f is a graphic l_{HR}-contraction, i.e.,*

$$d(f(x), f^2(x)) \leq l_{HR} \cdot d(x, f(x)), \text{ for all } x \in X,$$

where the constant $l_{HR} = \frac{a+b+c}{1-b-c}$, with $a, b, c \in \mathbb{R}_+$ and $a + 2b + 2c \in (0, 1)$.

Proof. The conclusion follows by replacing y with $f(x)$ in the Hardy-Rogers' metric condition. □

Lemma 2.2. *Let (X, d) be a complete metric space. Let $f : X \to X$ be a HR mapping. Then $f(f^n(x)) \to f(f^{\infty}(x))$ as $n \to \infty$, for all $x \in X$.*

Proof. By replacing x with $f^n(x)$ and y with $f^\infty(x)$ in the Hardy-Rogers' metric condition, we get that

$$\begin{aligned} d(f(f^n(x)), f(f^\infty(x))) \leq\ & ad(f^n(x), f^\infty(x))+ \\ & + b[d(f^n(x), f(f^n(x))) + d(f^\infty(x), f(f^\infty(x)))]+ \\ & + c[d(f^n(x), f(f^\infty(x))) + d(f^\infty(x), f(f^n(x)))]. \end{aligned}$$

Next, by using the triangle inequality satisfied by the metric d we get

$$\begin{aligned} d(f(f^n(x)), f(f^\infty(x))) \leq\ & ad(f^n(x), f^\infty(x))+ \\ & + b[d(f^n(x), f(f^n(x))) + d(f^\infty(x), f(f^n(x))) + d(f(f^n(x)), f(f^\infty(x)))]+ \\ & + c[d(f^n(x), f(f^n(x))) + d(f(f^n(x)), f(f^\infty(x))) + d(f^\infty(x), f(f^n(x)))]. \end{aligned}$$

By letting $n \to \infty$ in the above inequality and taking into account the continuity of the metric d and the fact that the operator f is a graphic l_{HR}-contraction via Lemma 2.1, we get that $d(f^n(x), f^\infty(x)) \to 0$ as $n \to \infty$ and $d(f^n(x), f(f^n(x))) \to 0$ as $n \to \infty$. It follows that $f(f^n(x)) \to f(f^\infty(x))$ as $n \to \infty$, for all $x \in X$. □

In the paper [6], G.E. Hardy and T.D. Rogers showed that any HR mapping is a PM. In the following theorem, we give a simple proof of this result and several other conclusions concerning HR mappings.

Theorem 2.3 (Saturated principle of HR mappings). *Let (X, d) be a complete metric space and $f : X \to X$ be a HR mapping. Then we have that:*

(i) $F_f = F_{f^n} = \{x^*\}$, *for all* $n \in \mathbb{N}^*$.
(ii) f *is a* PM.
(iii) $d(x, x^*) \leq \frac{1}{1-l_{HR}} d(x, f(x))$, *for all* $x \in X$.
(iv) The fixed point equation corresponding to f *is Ulam-Hyers stable.*
(v) $y_n \in X$, $d(y_n, f(y_n)) \to 0$ *as* $n \to \infty \Rightarrow y_n \to x^*$ *as* $n \to \infty$, *i.e., the fixed point problem for* f *is well-posed.*
(vi) If in addition, $l_{HR} < \frac{1}{3}$, *then* $y_n \in X$, $d(y_{n+1}, f(y_n)) \to 0$ *as* $n \to \infty \Rightarrow y_n \to x^*$ *as* $n \to \infty$, *i.e.,* f *has the Ostrowski property.*

Proof. From Lemma 2.1, f is a graphic l_{HR}-contraction.

From Lemma 2.2, it follows that $f^{n+1}(x) \to f(f^\infty(x))$ as $n \to \infty$. But $f^{n+1}(x) \to f^\infty(x)$ as $n \to \infty$. So, $f^\infty(x) \in F_f$. Hence, $F_f \neq \varnothing$.

Let $x^*, y^* \in F_f$ with $x^* \neq y^*$. By replacing x with x^* and y with y^* in the Hardy-Rogers' metric condition, we get $(1 - a - 2c)d(x^*, y^*) \leq 0$, which implies that $x^* = y^*$. So, $cardF_f = 1$. We apply next Theorem 1.2. □

2.2. The case of KAC mappings

Lemma 2.4. *Let (X, d) be a bounded metric space. Let $f : X \to X$ be a KAC mapping. Then f is a graphic l_{KAC}-contraction, i.e.,*

$$d(f(x), f^2(x)) \leq l_{KAC} \cdot d(x, f(x)), \text{ for all } x \in X,$$

where the constant $l_{KAC} = \frac{2\delta(X)}{2\delta(X)+1}$ *and* $\delta(X)$ *is the diameter functional of the space* X.

Proof. By taking $y = f(x)$ in the Khojasteh, Abbas and Costache's metric condition, we obtain the following estimation:

$$d(f(x), f^2(x)) \leq \frac{d(x, f^2(x))}{d(x, f(x)) + d(f(x), f^2(x)) + 1} d(x, f(x)), \text{ for all } x \in X.$$

Let us notice that the number $\frac{d(x,f^2(x))}{d(x,f(x))+d(f(x),f^2(x))+1}$ is not a constant, since it depends on $x \in X$. However, we can find an upper bound for it, by considering the diameter functional of the space X,

$$\delta(X) := \sup\{d(x, y) \mid x, y \in X\}.$$

Let us consider the function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$, defined by $\psi(x) := \frac{x}{x+1}$, for all $x \in \mathbb{R}_+$. By calculating its first derivative, we conclude that the function ψ is increasing on $\mathbb{R}_+$. We have the following estimations:

$$\frac{d(x, f^2(x))}{d(x, f(x)) + d(f(x), f^2(x)) + 1} \leq \frac{d(x, f(x)) + d(f(x), f^2(x))}{d(x, f(x)) + d(f(x), f^2(x)) + 1} =$$

$$= \psi(d(x, f(x)) + d(f(x), f^2(x))) \leq \psi(2\delta(X)) = \frac{2\delta(X)}{2\delta(X) + 1}, \text{ for all } x \in X.$$

Hence, for $l_{KAC} = \frac{2\delta(X)}{2\delta(X)+1}$ we have that $d(f(x), f^2(x)) \leq l_{KAC} \cdot d(x, f(x))$, for all $x \in X$. □

Lemma 2.5. *Let (X, d) be a bounded complete metric space and $f : X \to X$ be a KAC mapping. Then, $f(f^n(x)) \to f(f^\infty(x))$ as $n \to \infty$, for all $x \in X$ and $f^\infty(x) \in F_f$, for all $x \in X$.*

Proof. We have that $d(f(f^n(x)), f(f^\infty(x))) = d(f^{n+1}(x), f(f^\infty(x))) \leq$

$$\leq \frac{d(f^\infty(x), f^{n+1}(x)) + d(f^n(x), f(f^\infty(x)))}{d(f^n(x), f^{n+1}(x)) + d(f^\infty(x), f(f^\infty(x))) + 1} d(f^n(x), f^\infty(x)).$$

By letting $n \to \infty$, it follows that,

$$d(f^\infty(x), f(f^\infty(x))) \leq \frac{d(f^\infty(x), f(f^\infty(x)))}{d(f^\infty(x), f(f^\infty(x))) + 1} \cdot 0 = 0.$$

So, $f(f^n(x)) \to f(f^\infty(x))$ as $n \to \infty$ and $f^\infty(x) \in F_f$, for all $x \in X$. □

Theorem 2.6 (Saturated principle of KAC mappings). *Let (X, d) be a complete bounded metric space and $f : X \to X$ be a KAC mapping. Then we have that:*

(i) $\{f^n(x)\}_{n\in\mathbb{N}}$ *converges for all* $x \in X$ *and* $\sum_{n\in\mathbb{N}} d(f^n(x), f^{n+1}(x)) < +\infty$, *for all* $x \in X$.
(ii) $F_f = F_{f^n} \neq \varnothing$, *for all* $n \in \mathbb{N}^*$.
(iii) f *is a* WPM.
(iv) $d(x, f^\infty(x)) \leq \frac{1}{1-l_{KAC}} d(x, f(x))$, *for all* $x \in X$.
(v) The fixed point equation corresponding to f *is Ulam-Hyers stable.*
(vi) $x^* \in F_f$, $y_n \in X_{x^*}$, $d(y_n, f(y_n)) \to 0$ *as* $n \to \infty \Rightarrow y_n \to x^*$ *as* $n \to \infty$, *i.e., the fixed point problem for* f *is well-posed.*

(vii) *If in addition,* $l_{KAC} < \frac{1}{3}$, *then,* $x^* \in F_f$, $y_n \in X_{x^*}$, $d(y_{n+1}, f(y_n)) \to 0$ *as* $n \to \infty \Rightarrow y_n \to x^*$ *as* $n \to \infty$, *i.e.,* f *has the Ostrowski property.*

Proof. The conclusions follow from the saturated principle of graphic contractions. □

2.3. The case of IK mappings

Lemma 2.7. *Let* (X, d) *be a metric space. Let* $f : X \to X$ *be an* IK *mapping. Then* f *is a graphic* l_{IK}*-contraction, i.e.,*

$$d(f(x), f^2(x)) \leq l_{IK} \cdot d(x, f(x)), \text{ for all } x \in X,$$

where $l_{IK} = \lambda^{\frac{1}{\alpha}}$, *with* $\lambda \in [0, 1)$ *and* $\alpha \in (0, 1)$.

Proof. By replacing y with $f(x)$ in the interpolative Kannan's metric condition, we get the conclusion. □

Lemma 2.8. *Let* (X, d) *be a metric space. Let* $f : X \to X$ *be an* IK *mapping. Then* $f(f^n(x)) \to f(f^\infty(x))$ *as* $n \to \infty$, *for all* $x \in X$.

Proof. If $f^\infty(x) \notin F_f$ then, by replacing x with $f^n(x)$ and y with $f^\infty(x)$ in the interpolative Kannan's metric condition, we have

$$d(f(f^n(x)), f(f^\infty(x))) \leq \lambda[d(f^n(x), f^{n+1}(x))]^\alpha [d(f^\infty(x), f(f^\infty(x)))]^{1-\alpha},$$

for all $x \in X$. By letting $n \to \infty$ and taking into account the Lemma 2.7, $d(f^n(x), f^{n+1}(x)) \to 0$ as $n \to \infty$, for all $x \in X$. The conclusion follows. □

Theorem 2.9 (Saturated principle of IK mappings). *Let* (X, d) *be a complete metric space and* $f : X \to X$ *be an* IK *mapping. Then we have that:*

(i) $\{f^n(x)\}_{n\in\mathbb{N}}$ *converges for all* $x \in X$ *and* $\sum_{n\in\mathbb{N}} d(f^n(x), f^{n+1}(x)) < +\infty$, *for all* $x \in X$.
(ii) $F_f = F_{f^n} \neq \varnothing$, *for all* $n \in \mathbb{N}^*$.
(iii) f *is a* WPM.
(iv) $d(x, f^\infty(x)) \leq \frac{1}{1-l_{IK}} d(x, f(x))$, *for all* $x \in X$.
(v) *The fixed point equation corresponding to* f *is Ulam-Hyers stable.*
(vi) $x^* \in F_f$, $y_n \in X_{x^*}$, $d(y_n, f(y_n)) \to 0$ *as* $n \to \infty \Rightarrow y_n \to x^*$ *as* $n \to \infty$, *i.e., the fixed point problem for* f *is well-posed.*
(vii) *If in addition,* $l_{IK} < \frac{1}{3}$, *then,* $x^* \in F_f$, $y_n \in X_{x^*}$, $d(y_{n+1}, f(y_n)) \to 0$ *as* $n \to \infty \Rightarrow y_n \to x^*$ *as* $n \to \infty$, *i.e.,* f *has the Ostrowski property.*

Proof. The conclusions follow from Lemmas 2.7, 2.8 and Theorem 1.1. □

In the case when $cardF_f \leq 1$, Theorem 2.9 takes the following form:

Theorem 2.10. *Let* (X, d) *be a complete metric space and* $f : X \to X$ *be an* IK *mapping. We assume that* $cardF_f \leq 1$. *Then we have that:*

(i) $\{f^n(x)\}_{n\in\mathbb{N}}$ *converges for all* $x \in X$ *and* $\sum_{n\in\mathbb{N}} d(f^n(x), f^{n+1}(x)) < +\infty$*, for all* $x \in X$.
(ii) $F_f = F_{f^n} = \{x^*\}$*, for all* $n \in \mathbb{N}^*$.
(iii) f *is a* PM.
(iv) $d(x, x^*) \leq \frac{1}{1-l_{IK}} d(x, f(x))$*, for all* $x \in X$.
(v) *The fixed point equation corresponding to* f *is Ulam-Hyers stable.*
(vi) $y_n \in X$, $d(y_n, f(y_n)) \to 0$ *as* $n \to \infty \Rightarrow y_n \to x^*$ *as* $n \to \infty$*, i.e., the fixed point problem for* f *is well-posed.*
(vii) *If in addition,* $l_{IK} < \frac{1}{3}$*, then,* $y_n \in X$, $d(y_{n+1}, f(y_n)) \to 0$ *as* $n \to \infty \Rightarrow y_n \to x^*$ *as* $n \to \infty$*, i.e.,* f *has the Ostrowski property.*

Proof. The conclusions follow from Lemmas 2.7, 2.8 and Theorem 1.2. □

Remark 2.11. Not any metric condition yields a graphic contraction. For instance, if we consider a metric space (X, d) and the mapping $f : X \to X$ with the property that there exist two constants $\theta \in [0, 1)$ and $L \geq 0$ such that

$$d(f(x), f(y)) \leq \theta d(x, y) + L[d(x, f(x)) + d(y, f(y))],$$

for all $x, y \in X$, then f is not a graphic contraction.

Indeed, by choosing $y := f(x)$ we get $d(f(x), f^2(x)) \leq \frac{\theta+L}{1-L} d(x, f(x))$, for all $x \in X$. By taking $L = \frac{1}{2}$, the condition $\frac{\theta+L}{1-L} < 1$ implies $\theta < 0$, which is a contradiction with $\theta \in [0, 1)$.

3. Conditions with respect to a dislocated metric

Let us recall first the notion of dislocated metric.

Definition 3.1. Let X be a nonempty set. A functional $d : X \times X \to \mathbb{R}_+$ is called dislocated metric on X if the following conditions hold:

(i) $d(x, y) = d(y, x) = 0 \Rightarrow x = y$.
(ii) $d(x, y) = d(y, x)$, for all $x, y \in X$.
(iii) $d(x, y) \leq d(x, z) + d(z, y)$, for all $x, y, z \in X$.

If X is a nonempty set and $d : X \times X \to \mathbb{R}_+$ is a dislocated metric on X, then the couple (X, d) is called dislocated metric space.

In the above setting, we have the following results.

Theorem 3.2 (Saturated principle of graphic contraction). *Let* (X, d) *be a complete dislocated metric space and* $f : X \to X$ *be a graphic* l*-contraction, i.e.,* $0 < l < 1$ *and* $d(f^2(x), f(x)) \leq l d(x, f(x))$*, for all* $x \in X$*. Then we have that:*

(i) $\{f^n(x)\}_{n\in\mathbb{N}}$ *converges for all* $x \in X$ *and* $\sum_{n\in\mathbb{N}} d(f^n(x), f^{n+1}(x)) < +\infty$*, for all* $x \in X$.

If in addition, $\lim_{n\to\infty} f(f^n(x)) = f(\lim_{n\to\infty} f^n(x))$*, for all* $x \in X$*, then:*

(ii) $F_f = F_{f^n} \neq \varnothing$*, for all* $n \in \mathbb{N}^*$.

(*iii*) *f is a WPM.*
(*iv*) $d(x, f^\infty(x)) \leq \frac{1}{1-l} d(x, f(x))$, *for all* $x \in X$.
(*v*) *The fixed point equation corresponding to f is Ulam-Hyers stable.*
(*vi*) $x^* \in F_f$, $y_n \in X_{x^*}$, $d(y_n, f(y_n)) \to 0$ *as* $n \to \infty \Rightarrow y_n \to x^*$ *as* $n \to \infty$, *i.e., the fixed point problem for f is well-posed.*
(*vii*) *If in addition,* $l < \frac{1}{3}$, *then,* $x^* \in F_f$, $y_n \in X_{x^*}$, $d(y_{n+1}, f(y_n)) \to 0$ *as* $n \to \infty \Rightarrow y_n \to x^*$ *as* $n \to \infty$, *i.e., f has the Ostrowski property.*

Proof. (*i*). Let $x \in X$. We construct the sequence of successive approximations, $\{f^n(x)\}_{n\in\mathbb{N}}$, for f starting from x.

Since f is a graphic l-contraction, we have the following estimations:

$$\begin{aligned} d(f(x), f^2(x)) &\leq l d(x, f(x)), \\ d(f^2(x), f^3(x)) &\leq l d(f(x), f^2(x)) \leq l^2 d(x, f(x)), \\ &\vdots \\ d(f^n(x), f^{n+1}(x)) &\leq l d(f^{n-1}(x), f^n(x)) \leq \ldots \leq l^n d(x, f(x)), \end{aligned}$$

for all $x \in X$ and $n \in \mathbb{N}$. By summing up the left hand side of the above inequalities, we have

$$\sum_{n\in\mathbb{N}} d(f^n(x), f^{n+1}(x)) \leq \sum_{n\in\mathbb{N}} l^n d(x, f(x)) = \frac{1}{1-l} d(x, f(x)) < +\infty.$$

It follows that $\{f^n(x)\}_{n\in\mathbb{N}}$ is a Cauchy sequence in (X, d) and, since (X, d) is complete, we get that $\{f^n(x)\}_{n\in\mathbb{N}}$ is convergent in (X, d).

(*ii*) + (*iii*). Since $\{f^n(x)\}_{n\in\mathbb{N}}$ is convergent in (X, d), there exists $f^\infty(x) = \lim\limits_{n\to\infty} f^n(x) \in X$. By using the assumption $\lim\limits_{n\to\infty} f(f^n(x)) = f(\lim\limits_{n\to\infty} f^n(x))$, for all $x \in X$, it follows that $f^\infty(x) \in F_f$ and also that $f^\infty(x) \in F_{f^n}$. So (*ii*) holds. By definition, f is a WPM. So, (*iii*) also holds.

(*iv*). Let $x \in X$. Since f is a graphic l-contraction, we have

$$\begin{aligned} d(x, f^\infty(x)) &\leq \sum_{k=0}^{n} d(f^k(x), f^{k+1}(x)) + d(f^{n+1}(x), f^\infty(x)) \\ &\leq \sum_{k=0}^{n} l^k d(x, f(x)) + d(f^{n+1}(x), f^\infty(x)). \end{aligned}$$

By letting $n \to \infty$, the conclusion follows.

(*v*). We recall that the fixed point equation $x = f(x)$, $x \in X$ is Ulam-Hyers stable if there exists a constant $c > 0$ such that for any $\varepsilon > 0$ and any ε-solution z of the fixed point equation, i.e., $d(z, f(z)) \leq \varepsilon$, there exists $x^* \in F_f$ such that $d(z, x^*) \leq c\varepsilon$.

Let $\varepsilon > 0$ and let z be the ε-solution of the fixed point equation $x = f(x)$, for all $x \in X$. Since $f^\infty(x) \in F_f$, by using the inequality (*iv*), we have

$$d(z, f^\infty(x)) \leq \frac{1}{1-l} d(z, f(z)) \leq \frac{1}{1-l}\varepsilon$$

So, there exists $c := \frac{1}{1-l} > 0$ such that $d(z, f^\infty(x)) \leq c\varepsilon$.

(vi). Let $x^* \in F_f$ and $y_n \in X_x^* := \{x \in X \mid f^n(x) \to x^*,\ \text{as } n \to \infty\}$, such that $d(y_n, f(y_n)) \to 0$ as $n \to \infty$. We have the following estimations

$$\begin{aligned} d(y_n, x^*) &\leq \sum_{k=0}^{n-1} d(f^k(y_n), f^{k+1}(y_n)) + d(f^n(y_n), x^*) \\ &\leq \sum_{k=0}^{n-1} l^k d(y_n, f(y_n)) + d(f^n(y_n), x^*). \end{aligned}$$

By letting $n \to \infty$, it follows that $d(y_n, x^*) \to 0$. So, (vi) holds.

(vii). First, we show that

$$d(f(x), f^\infty(x)) \leq \frac{l}{1-2l} d(x, f^\infty(x)), \text{ for all } x \in X. \tag{3.1}$$

Indeed, for any $x \in X$, we have the following estimations

$$\begin{aligned} d(f(x), f^\infty(x)) &\leq \sum_{k=0}^{\infty} d(f^k(x), f^{k+1}(x)) - d(x, f(x)) \\ &\leq \sum_{k=0}^{\infty} l^k d(x, f(x)) - d(x, f(x)) = \left(\frac{1}{1-l} - 1\right) d(x, f(x)) \\ &\leq \frac{l}{1-l} d(x, f^\infty(x)) + \frac{l}{1-l} d(f^\infty(x), f(x)). \end{aligned}$$

It follows that $\frac{1-2l}{1-l} d(f(x), f^\infty(x)) \leq \frac{l}{1-l} d(x, f^\infty(x))$. Hence (3.1) holds. Notice also that the constant $\frac{l}{1-2l} < 1$ if and only if $l < \frac{1}{3}$.

Now, let $x^* \in F_f$ and $y_n \in X_x^* := \{x \in X \mid f^n(x) \to x^*,\ \text{as } n \to \infty\}$, such that $d(y_{n+1}, f(y_n)) \to 0$ as $n \to \infty$. By using (3.1), we have

$$\begin{aligned} d(y_{n+1}, x^*) &\leq d(y_{n+1}, f(y_n)) + d(f(y_n), x^*) \\ &\leq d(y_{n+1}, f(y_n)) + \frac{l}{1-2l} d(y_n, x^*) \\ &\leq d(y_{n+1}, f(y_n)) + \frac{l}{1-2l} d(y_n, f(y_{n-1})) + \frac{l}{1-2l} d(f(y_{n-1}), x^*) \\ &\leq d(y_{n+1}, f(y_n)) + \frac{l}{1-2l} d(y_n, f(y_{n-1})) + \left(\frac{l}{1-2l}\right)^2 d(y_{n-1}, x^*) \\ &\vdots \\ &\leq d(y_{n+1}, f(y_n)) + \frac{l}{1-2l} d(y_n, f(y_{n-1})) + \\ &\quad + \ldots + \left(\frac{l}{1-2l}\right)^n d(y_1, f(y_0)) + \left(\frac{l}{1-2l}\right)^n d(f(y_0), x^*). \end{aligned}$$

By letting $n \to \infty$ and applying a Cauchy (or Toeplitz) lemma, we obtain $d(y_{n+1}, x^*) \to 0$. The conclusion follows. □

In the case when $cardF_f \leq 1$, Theorem 3.2 takes the following form:

Theorem 3.3. *Let (X, d) be a complete dislocated metric space and $f : X \to X$ be a graphic l-contraction. We assume that $cardF_f \leq 1$. Then we have that:*

(i) $\{f^n(x)\}_{n\in\mathbb{N}}$ *converges for all $x \in X$ and* $\sum_{n\in\mathbb{N}} d(f^n(x), f^{n+1}(x)) < +\infty$, *for all $x \in X$.*

If in addition, $\lim_{n\to\infty} f(f^n(x)) = f(\lim_{n\to\infty} f^n(x))$, *for all $x \in X$, then:*

(ii) $F_f = F_{f^n} = \{x^*\}$, *for all $n \in \mathbb{N}^*$.*

(iii) *f is a PM.*

(iv) $d(x, x^*) \leq \frac{1}{1-l} d(x, f(x))$, *for all $x \in X$.*

(v) *The fixed point equation corresponding to f is Ulam-Hyers stable.*

(vi) $y_n \in X$, $d(y_n, f(y_n)) \to 0$ *as* $n \to \infty \Rightarrow y_n \to x^*$ *as* $n \to \infty$, *i.e., the fixed point problem for f is well-posed.*

(vii) *If in addition, $l < \frac{1}{3}$, then,* $y_n \in X$, $d(y_{n+1}, f(y_n)) \to 0$ *as* $n \to \infty \Rightarrow y_n \to x^*$ *as* $n \to \infty$, *i.e., f has the Ostrowski property.*

Remark 3.4. Notice that the Theorem 2.3 for HR mappings and the Theorems 2.9 and 2.10 for IK mappings also hold in the context of a complete dislocated metric space. Theorem 2.6 for KAC mappings also holds in the context of a complete bounded dislocated metric space.

4. Conditions with respect to an $\mathbb{R}^m_+$-metric

In this section we follow the terminology and notations given in [5], concerning vector-valued metric ($\mathbb{R}^m_+$-metric) and matrices convergent to zero. Regarding the properties of these matrices, we recall the following result (see [5]).

Theorem 4.1. *Let $S \in \mathcal{M}_m(\mathbb{R}_+)$. The following assertions are equivalent:*

(1) *S is convergent to zero;*

(2) $S^n \to O_m$ *as* $n \to \infty$;

(3) *the spectral radius $\rho(S)$ is strictly less than 1;*

(4) *the matrix $(I_m - S)$ is nonsingular and*

$$(I_m - S)^{-1} = I_m + S + S^2 + \ldots + S^n + \ldots;$$

(5) *the matrix $(I_m - S)$ is nonsingular and $(I_m - S)^{-1}$ has nonnegative elements;*

(6) $S^n x \to 0 \in \mathbb{R}^m$ *as* $n \to \infty$, *for all* $x \in \mathbb{R}^m$.

The main result of this section is the following one.

Theorem 4.2 (Saturated principle of graphic contraction). *Let (X, d) be a complete $\mathbb{R}^m_+$-metric space and $f : X \to X$ be a graphic S-contraction, i.e., there exists a matrix convergent to zero, $S \in \mathcal{M}_m(\mathbb{R}_+)$, such that $d(f^2(x), f(x)) \leq Sd(x, f(x))$, for all $x \in X$. Then we have that:*

(i) $\{f^n(x)\}_{n\in\mathbb{N}}$ *converges for all $x \in X$ and* $\sum_{n\in\mathbb{N}} d(f^n(x), f^{n+1}(x)) < +\infty$, *for all $x \in X$.*

If in addition, $\lim_{n\to\infty} f(f^n(x)) = f(\lim_{n\to\infty} f^n(x))$, *for all* $x \in X$, *then:*

(*ii*) $F_f = F_{f^n} \neq \varnothing$, *for all* $n \in \mathbb{N}^*$.
(*iii*) f *is a* WPM.
(*iv*) $d(x, f^\infty(x)) \le (I_m - S)^{-1} d(x, f(x))$, *for all* $x \in X$.
(*v*) *The fixed point equation corresponding to* f *is Ulam-Hyers stable.*
(*vi*) $x^* \in F_f$, $y_n \in X_{x^*}$, $d(y_n, f(y_n)) \to 0$ *as* $n \to \infty \Rightarrow y_n \to x^*$ *as* $n \to \infty$, *i.e., the fixed point problem for* f *is well-posed.*
(*vii*) *If in addition, the matrix* $[2I_m - (I_m - S)^{-1}]^{-1}[(I_m - S)^{-1} - I_m]$ *converges to zero, then,* $x^* \in F_f$, $y_n \in X_{x^*}$, $d(y_{n+1}, f(y_n)) \to 0$ *as* $n \to \infty \Rightarrow y_n \to x^*$ *as* $n \to \infty$, *i.e.,* f *has the Ostrowski property.*

Proof. $(i)+(ii)+(iii)+(iv)$. We follow the proof given for Theorem 3.2, by replacing the constant l with the matrix S. We also take into account the assertions (4) and (5) of Theorem 4.1.

(v). We say that the fixed point equation $x = f(x)$, $x \in X$ is Ulam-Hyers stable if there exists a matrix $C \in \mathcal{M}_m(\mathbb{R}_+)$ such that for any $\varepsilon \in \mathbb{R}^m_+$ and any ε-solution z of the fixed point equation, i.e., $d(z, f(z)) \le \varepsilon$, there exists $x^* \in F_f$ such that $d(z, x^*) \le C\varepsilon$.

Let $\varepsilon \in \mathbb{R}^m_+$ and let z be the ε-solution of the fixed point equation $x = f(x)$, for all $x \in X$. Since $f^\infty(x) \in F_f$, by using the inequality (iv), we have

$$d(z, f^\infty(x)) \le (I_m - S)^{-1} d(z, f(z)) \le (I_m - S)^{-1}\varepsilon$$

So, there exists $C := (I_m - S)^{-1} \in \mathcal{M}_m(\mathbb{R}_+)$ such that $d(z, f^\infty(x)) \le C\varepsilon$.

(vi). Let $x^* \in F_f$ and $y_n \in X^*_x := \{x \in X \mid f^n(x) \to x^*, \text{ as } n \to \infty\}$, such that $d(y_n, f(y_n)) \to 0$ as $n \to \infty$. We have the following estimations

$$\begin{aligned} d(y_n, x^*) &\le \sum_{k=0}^{n-1} d(f^k(y_n), f^{k+1}(y_n)) + d(f^n(y_n), x^*) \\ &\le \sum_{k=0}^{n-1} S^k d(y_n, f(y_n)) + d(f^n(y_n), x^*) \\ &\le (I_m - S)^{-1} d(y_n, f(y_n)) + d(f^n(y_n), x^*). \end{aligned}$$

By letting $n \to \infty$, it follows that $d(y_n, x^*) \to 0$. So, (vi) holds.

(vii). First, we show that

$$d(f(x), f^\infty(x)) \le \Lambda d(x, f^\infty(x)), \text{ for all } x \in X, \tag{4.1}$$

where $\Lambda := [2I_m - (I_m - S)^{-1}]^{-1}[(I_m - S)^{-1} - I_m]$.

Indeed, for any $x \in X$, we have the following estimations

$$\begin{aligned} d(f(x), f^\infty(x)) &\le \sum_{k=0}^{\infty} d(f^k(x), f^{k+1}(x)) - d(x, f(x)) \\ &\le \sum_{k=0}^{\infty} S^k d(x, f(x)) - d(x, f(x)) = [(I_m - S)^{-1} - I_m] d(x, f(x)) \\ &\le [(I_m - S)^{-1} - I_m] d(x, f^\infty(x)) + [(I_m - S)^{-1} - I_m] d(f^\infty(x), f(x)). \end{aligned}$$

It follows that

$$[2I_m - (I_m - S)^{-1}]d(f(x), f^\infty(x)) \leq [(I_m - S)^{-1} - I_m]d(x, f^\infty(x)).$$

and hence, (4.1) holds.

Now, let $x^* \in F_f$ and $y_n \in X_x^* := \{x \in X \mid f^n(x) \to x^*, \text{ as } n \to \infty\}$, such that $d(y_{n+1}, f(y_n)) \to 0$ as $n \to \infty$. By using (4.1) we have

$$\begin{aligned} d(y_{n+1}, x^*) &\leq d(y_{n+1}, f(y_n)) + d(f(y_n), x^*) \\ &\leq d(y_{n+1}, f(y_n)) + \Lambda d(y_n, x^*) \\ &\leq d(y_{n+1}, f(y_n)) + \Lambda d(y_n, f(y_{n-1})) + \Lambda d(f(y_{n-1}), x^*) \\ &\leq d(y_{n+1}, f(y_n)) + \Lambda d(y_n, f(y_{n-1})) + \Lambda^2 d(y_{n-1}, x^*) \\ &\;\;\vdots \\ &\leq d(y_{n+1}, f(y_n)) + \Lambda d(y_n, f(y_{n-1})) + \\ &\quad + \ldots + \Lambda^n d(y_1, f(y_0)) + \Lambda^n d(f(y_0), x^*). \end{aligned}$$

By letting $n \to \infty$ and applying a Cauchy (or Toeplitz) lemma, we get that $d(y_{n+1}, x^*) \to 0$. The conclusion follows. □

In the case when $cardF_f \leq 1$, Theorem 4.2 takes the following form:

Theorem 4.3. *Let (X, d) be a complete $\mathbb{R}^m_+$-metric space and $f : X \to X$ be a graphic S-contraction, i.e., there exists a matrix convergent to zero, $S \in \mathcal{M}_m(\mathbb{R}_+)$, such that $d(f^2(x), f(x)) \leq Sd(x, f(x))$, for all $x \in X$. We assume that $cardF_f \leq 1$. Then we have that:*

(i) $\{f^n(x)\}_{n\in\mathbb{N}}$ converges for all $x \in X$ and $\sum_{n\in\mathbb{N}} d(f^n(x), f^{n+1}(x)) < +\infty$, for all $x \in X$.

If in addition, $\lim_{n\to\infty} f(f^n(x)) = f(\lim_{n\to\infty} f^n(x))$, for all $x \in X$, then:

(ii) $F_f = F_{f^n} = \{x^*\}$, for all $n \in \mathbb{N}^*$.

(iii) f is a PM.

(iv) $d(x, x^*) \leq (I_m - S)^{-1}d(x, f(x))$, for all $x \in X$.

(v) The fixed point equation corresponding to f is Ulam-Hyers stable.

(vi) $y_n \in X$, $d(y_n, f(y_n)) \to 0$ as $n \to \infty \Rightarrow y_n \to x^*$ as $n \to \infty$, i.e., the fixed point problem for f is well-posed.

(vii) If in addition, the matrix $[2I_m - (I_m - S)^{-1}]^{-1}[(I_m - S)^{-1} - I_m]$ converges to zero, then, $y_n \in X$, $d(y_{n+1}, f(y_n)) \to 0$ as $n \to \infty \Rightarrow y_n \to x^*$ as $n \to \infty$, i.e., f has the Ostrowski property.

We introduce next the notion of interpolative Kannan mapping defined on a $\mathbb{R}^m_+$-metric space.

Definition 4.4. Let (X, d) be a $\mathbb{R}^m_+$-metric space. A mapping $f : X \to X$ is called interpolative Kannan mapping (IK mapping) on X, if there exists a convergent to zero matrix, $\Lambda \in \mathcal{M}_m(\mathbb{R}_+)$, and a real constant $\alpha \in (0, 1)$ such that

$$d(f(x), f(y)) \leq \Lambda[d(x, f(x))]^\alpha \cdot [d(y, f(y))]^{1-\alpha}, \text{ for all } x, y \in X \setminus F_f.$$

Lemma 4.5. *Let (X, d) be a $\mathbb{R}^m_+$-metric space. Let $f : X \to X$ be an IK mapping. Then f is a graphic L_{IK}-contraction, i.e.,*

$$d(f(x), f^2(x)) \leq L_{IK} \cdot d(x, f(x)), \text{ for all } x \in X \setminus F_f,$$

where $L_{IK} = \Lambda^{\frac{1}{\alpha}}$ is a matrix that converges to zero, having positive real values.

Proof. Let $x \in X \setminus F_f$. By replacing y with $f(x)$ in the interpolative Kannan's metric condition, the conclusion follows. □

Lemma 4.6. *Let (X, d) be a $\mathbb{R}^m_+$-metric space. Let $f : X \to X$ be an IK mapping. Then $f(f^n(x)) \to f(f^\infty(x))$ as $n \to \infty$, for all $x \in X$.*

Proof. If $f^\infty(x) \notin F_f$ then, by replacing x with $f^n(x)$ and y with $f^\infty(x)$ in the interpolative Kannan's metric condition, we have

$$d(f(f^n(x)), f(f^\infty(x))) \leq \Lambda[d(f^n(x), f^{n+1}(x))]^\alpha [d(f^\infty(x), f(f^\infty(x)))]^{1-\alpha},$$

for all $x \in X$. By letting $n \to \infty$ and taking into account the Lemma 4.5, $d(f^n(x), f^{n+1}(x)) \to 0 \in \mathbb{R}^m$ as $n \to \infty$, for all $x \in X$. The conclusion follows. □

Theorem 4.7 (Saturated principle of IK mappings). *Let (X, d) be a complete $\mathbb{R}^m_+$-metric space and $f : X \to X$ be an IK mapping. Then we have that:*

(i) $\{f^n(x)\}_{n\in\mathbb{N}}$ converges for all $x \in X$ and $\sum_{n\in\mathbb{N}} d(f^n(x), f^{n+1}(x)) < +\infty$, for all $x \in X$.
(ii) $F_f = F_{f^n} \neq \emptyset$, for all $n \in \mathbb{N}^*$.
(iii) f is a WPM.
(iv) $d(x, f^\infty(x)) \leq (I_m - L_{IK})^{-1} d(x, f(x))$, for all $x \in X$.
(v) The fixed point equation corresponding to f is Ulam-Hyers stable.
(vi) $x^* \in F_f$, $y_n \in X_{x^*}$, $d(y_n, f(y_n)) \to 0$ as $n \to \infty \Rightarrow y_n \to x^*$ as $n \to \infty$, i.e., the fixed point problem for f is well-posed.
(vii) If in addition, the matrix $[2I_m - (I_m - L_{IK})^{-1}]^{-1}[(I_m - L_{IK})^{-1} - I_m]$ converges to zero, then, $x^* \in F_f$, $y_n \in X_{x^*}$, $d(y_{n+1}, f(y_n)) \to 0$ as $n \to \infty \Rightarrow y_n \to x^*$ as $n \to \infty$, i.e., f has the Ostrowski property.

Proof. The conclusions follow from the Lemmas 4.5, 4.6 and Theorem 4.2. □

References

[1] Agarwal, R.P., Jleli, M., Samet, B., *Fixed Point Theory in Metric spaces*, Springer, 2018.

[2] Berinde, V., *Iterative Approximation of Fixed Points*, Springer-Verlag Heidelberg Berlin, 2007.

[3] Berinde, V., Petruşel, A., Rus, I.A., *Remarks on the terminology of the mappings in fixed point iterative methods in metric spaces*, Fixed Point Theory, **24**(2023), No. 2, 525-540.

[4] Filip, A.-D., *Fixed Point Theory in Kasahara Spaces*, Casa Cărţii de Ştiinţă, Cluj-Napoca, 2015.

[5] Filip, A.-D., *Fixed point theorems for nonself generalized contractions on a large Kasahara space*, Carpathian J. Math., **38**(2022), No. 3, 799-809.

[6] Hardy, G.E., Rogers, T.D., *A generalization of a fixed point theorem of Reich*, Canad. Math. Bull., **16**(1973), 201-206.

[7] Karapinar, E., Agarwal, R., Aydi, H., *Interpolative Reich-Rus-Ćirić type contractions on partial metric spaces*, Mathematics, **6**(2018) 256.

[8] Khojasteh, F., Abbas, M, Costache, S., *Two new types of fixed point theorems in complete metric spaces*, Abstr. Appl. Anal., 2014, Art. ID 325840, 5pp.

[9] Kirk, W.A., Sims, B., (eds.), *Handbook of Metric Fixed Point Theory*, Kluwer, 2001.

[10] Park, S., *Almost all about Rus-Hicks-Rhoades maps in quasi-metric spaces*, Adv. Theory Nonlinear Anal. Appl., **7**(2023), No. 2, 455-472.

[11] Petruşel, A., Rus, I.A., *Graphic contraction principle and application*, In: Th. Rassis et al. (eds.), Mathematical Analysis and Application, Springer, 2019, 395-416.

[12] Rhoades, B.E., *A comparison of various definitions of contractive mappings*, Trans. Amer. Math. Soc., **226**(1977), 257-290.

[13] Rus, I.A., *On common fixed points*, Stud. Univ. Babeş-Bolyai Math., **18**(1973), 31-33.

[14] Rus, I.A., *Generalized Contractions and Applications*, Cluj Univ. Press, Cluj-Napoca, 2001.

[15] Rus, I.A., *Relevant classes of weakly Picard operators*, An. Univ. Vest Timişoara, Mat.-Inform., **54**(2016), no. 2, 3-19.

[16] Rus, I.A., *Some variants of contraction principle, generalizations and applications*, Stud. Univ. Babeş-Bolyai Math., **61**(2016), No. 3, 343-358.

[17] Rus, I.A., *Around metric coincidence point theory*, Stud. Univ. Babeş-Bolyai Math., **68**(2023), No. 2, 449-463.

[18] Rus, I.A., Petruşel, A., Petruşel, G., *Fixed Point Theory*, Cluj Univ. Press, Cluj-Napoca, 2008.

[19] Rus, I.A., Şerban, M.-A., *Basic problems of the metric fixed point theory*, Carpathian J. Math., **29**(2013), 239-258.

[20] Tongnoi, B., *Saturated versions of some fixed point theorems for generalized contractions*, Fixed Point Theory, **21**(2020), No. 2, 755-766.

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