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Existence and asymptotic stability for a semilinear damped wave equation with dynamic boundary conditions involving variable nonlinearity

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Abstract. We investigate the solvability of a class of quasilinear elliptic equations characterized by a $(p(x), k(x))$ growth structure and nonlinear boundary conditions, specifically in the context of Kelvin-Voigt damping with arbitrary data. Our approach involves analyzing the problem within appropriate functional spaces, utilizing Lebesgue and Sobolev spaces with variable exponents. In the first step, we establish the existence and uniqueness of results for solutions to the model, provided the data meet certain regularity conditions. Our methodology primarily relies on fixed-point theory and Faedo-Galerkin techniques, incorporating some novel strategies. In the second part, we consider scenarios with sufficiently large data sets and show that the system's energy grows exponentially.

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1. Introduction

Let $\Omega \subset \mathbb{R}^n (n \geq 1)$ be a bounded domain with a smooth boundary $\Gamma = \partial\Omega$. In this work, we deal with the existence and asymptotic behavior of weak solutions of a weakly damped wave equation with dynamic boundary conditions and source terms

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involving nonlinearities with variable exponents. More specifically, let's look at the problem

$$\begin{array}{ll} u_{tt}(x,t) - \Delta u(x,t) - \gamma \Delta u_t(x,t) = |u|^{p(x)-2} u(x,t), & x \in \Omega,\ t>0, \\ u(x,t) = 0, & x \in \Gamma_0,\ t>0, \\ u_{tt}(x,t) = -a\left(\frac{\partial u}{\partial \nu}(x,t) + \gamma \frac{\partial u_t}{\partial \nu}(x,t) + r|u_t|^{k(x)-2} u_t(x,t)\right), & x \in \Gamma_1,\ t>0, \\ u(x,0) = u_0(x),\ u_t(x,0) = u_1(x), & x \in \Omega, \end{array} \tag{1.1}$$

where $u = u(x,t)$, $t \geq 0$, γ, a and r are positive real numbers and $-\Delta$ represent the Laplace operator with respect to the spatial variable. The boundary Γ of Ω is assumed to be regular and the union of two closed and disjoint parts Γ_0, Γ_1, where $\Gamma_0 \neq \emptyset$. $\frac{\partial u}{\partial \nu}$ denotes the unit of the exterior normal derivative, u_0, u_1 are given functions and the exponents $k(.)$ and $p(.)$ are given measurable functions on Ω to satisfy

$$\begin{cases} 2 \leq p_1 \leq p(x) \leq p_2 < \infty, \\ 2 \leq k_1 \leq k(x) \leq k_2 < \infty, \end{cases} \tag{1.2}$$

where we fix q on Ω for any given measurable function:

$$q_2 = ess \sup_{x\in\Omega} q(x), \quad q_1 = ess \inf_{x\in\Omega} q(x). \tag{1.3}$$

We also assume that the following uniform Zhikov-Fan local continuity condition holds

$$\begin{aligned} |p(x) - p(y)| + |k(x) - k(y)| &\leq \frac{M}{|\log|x-y||}, \text{ for all } x,\ y \text{ in } \Omega, \\ \text{with } 0 &< |x-y| < \frac{1}{2},\ M>0. \end{aligned} \tag{1.4}$$

In recent years, many authors have engaged in the study of nonlinear hyperbolic, parabolic and elliptic equations with a non-standard growth condition, since they are applicable to real problems and many physical phenomena such as flows of electro-rheological fluids or fluids with temperature-dependent viscosity, nonlinear viscoelasticity, filtration processes through a porous media [3, 18], and the processing of digital images [2, 7], and can all be associated with problem (1.1), more details on the subject can be found in [19] and the other references contained therein. In the classical case of constant exponent ($k(x)$ =constant $= p$, $p(x)$ =constant $= p$), this equation has its origin in the nonlinear dynamic evolution of a viscoelastic rod that is fixed at one end and has a tip mass attached to its free end [5, 14, 13], where the source term $|u|^{p-2} u$ forces the negative-energy solutions to explode in finite time, and the dissipation term $|u_t|^{k-2} u_t$ assures the existence (in time) of global solutions. The dynamic boundary conditions represent Newton's law for the attached mass [5, 4]. In two-dimensional space, as shown in [15], boundary conditions of this kind appear when we consider the transverse motion of a flexible membrane, the boundary of which is only allowed to be affected by vibrations in one region. For other applications and related results, we refer the reader to [9, 16, 1, 17]. The aim of this article is to consider a class of nonlinear damped wave equations with dynamic boundary conditions and source terms with variable exponents and to prove a local existence theorem and sufficient conditions and initial data for the exponential energy increase to appear, indicate that this study is through the presence of the strong damping term $-\Delta u_t$ and the variable exponents differs from those previously considered. For this reason, extensive changes in the approaches are required.

2. Preliminaries

2.1. Function spaces

Throughout this paper, we assume that Ω is a bounded domain of $\mathbb{R}^n$, $n \geq 1$ with a smooth boundary $\Gamma = \partial\Omega$. Let $p(x) \geq 2$ be a measurable bounded function defined in Ω. We introduce the set of functions

$$L^{p(.)}(\Omega) = \left\{ u(x) : u \text{ is measurable in } \Omega,\ \varrho_{p(.)}(u) \equiv \int\limits_{\Omega} |u(x)|^{p(x)} \, dx < \infty \right\}.$$

The set $L^{p(.)}(\Omega)$ equipped with the norm (Luxemburg norm)

$$\|u\|_{p(.)} = \inf \left\{ \lambda > 0,\ \int\limits_{\Omega} \left| \frac{u}{\lambda} \right|^{p(x)} dx \leq 1 \right\},$$

becomes a Banach space. The set $C^\infty(\Omega)$ is dense in $L^{p(.)}(\Omega)$, provided that the exponent $p(x) \in C^0(\Omega)$. Hölder's inequality holds for the elements of these spaces in the following form:

$$\int_{\Omega} |u(x)\, v(x)| \, dx \leq \left(\frac{1}{p_1} + \frac{1}{q_1} \right) \|u\|_{p(x)} \|v\|_{q(x)},$$

for all $u \in L^{p(.)}(\Omega)$, $v \in L^{q(.)}(\Omega)$ with $p(x) \in [p_1, p_2] \subset (1, \infty)$, $q(x) = \frac{p(x)}{p(x)-1} \in [q_1, q_2] \subset (1, \infty)$. With $W_0^{1,p(.)}(\Omega)$ we denote the Banach space

$$W_0^{1,p(.)}(\Omega) = \left\{ u \in L^{p(.)}(\Omega) \mid\ |\nabla u|^{p(x)} \in L^1(\Omega),\ u = 0 \text{ on } \Gamma = \partial\Omega \right\}.$$

An equivalent norm of $W_0^{1,p(.)}(\Omega)$ is given by

$$\|u\|_{W_0^{1,p(.)}(\Omega)} = \|\nabla u\|_{p(.)} = \sum_i \|D_i u\|_{p(.)} + \|u\|_{p(.)},$$

and $W^{-1,p'(.)}(\Omega)$ is defined in the same way as the usual Sobolev spaces (see [8]). Here we note that the space $W_0^{1,p(.)}(\Omega)$ is usually defined differently for the variable exponent case. The $\left(W_0^{1,p(.)}(\Omega)\right)'$ is the dual space of $W_0^{1,p(.)}(\Omega)$ with respect to the inner product in $L^2(\Omega)$ and is defined as $W^{-1,q(.)}(\Omega)$, where $\frac{1}{p(.)} + \frac{1}{q(.)} = 1$. If $p \in C\left(\overline{\Omega}\right)$, $q : \Omega \to [1, +\infty)$ is a measurable function and $\underset{x\in\Omega}{ess\inf}\,(p^*(x) - q(x)) > 0$ with $p^*(x) = \frac{np(x)}{(n-p(x))_2}$, then $W_0^{1,p(.)}(\Omega) \hookrightarrow L^{q(.)}(\Omega)$ is continuous and compact.

Lemma 2.1. *([8]) If Ω is a bounded domain of $\mathbb{R}^n$, $p(.) \subset (1,\infty)$ is a measurable function on $\overline{\Omega}$, then*

$$\min\left(\varrho_{p(.)}(u)^{\frac{1}{p_1}}, \varrho_{p(.)}(u)^{\frac{1}{p_2}}\right) \leq \|u\|_{p(.)} \leq \max\left(\varrho_{p(.)}(u)^{\frac{1}{p_1}}, \varrho_{p(.)}(u)^{\frac{1}{p_2}}\right), \tag{2.1}$$

for any $u \in L^{p(\cdot)}(\Omega)$.

Proposition 2.2. *(See [12]) If Ω is a bounded domain in $\mathbb{R}^n$, $p \in C^{0,1}\left(\overline{\Omega}\right)$, $1 < p_1 \leq p(x) \leq p_2 < n$. Then, for every $q \in C(\Gamma)$ with $1 \leq q(x) \leq \frac{(n-1)p(x)}{n-p(x)}$, there is a continuous trace $W^{1,p(x)}(\Omega) \to L^{q(x)}(\Gamma)$, when $1 \leq q(x) << \frac{(n-1)p(x)}{n-p(x)}$, the trace is compact, and in particular, the continuous trace $W^{1,p(x)}(\Omega) \to L^{p(x)}(\Gamma)$ is compact.*

2.2. Mathematical Hypotheses

We start this section by introducing some hypotheses and our main result. In this paper we use standard function spaces and denote that $\|.\|_{q,\Gamma_1}$, $\|\cdot\|_{p(.),\Gamma_1}$ are the $L^q(\Gamma_1)$ norm and the $L^{p(\cdot)}(\Gamma_1)$ norm such that

$$\|u\|_{p(.),\Gamma_1} = \int_{\Gamma_1} |u(x)|^{p(x)} \, \mathrm{d}\Gamma.$$

And we define $(u,v) = \int_\Omega u(x)v(x)\,\mathrm{d}x$ and $(u,v)_{\Gamma_1} = \int_{\Gamma_1} u(x)v(x)\,\mathrm{d}\Gamma$. Furthermore, we use standard functional spaces and denote that $(.,.)$, $\|.\|$ the inner products and norms are represented in $L^2(\Omega)$ and $H_0^1(\Omega)$ and they are given by

$$(u,v) = \int_\Omega u(x)v(x)\,\mathrm{d}x \text{ and } \|u\|_{L^2(\Omega)}^2 = \int_\Omega |u|^2\,\mathrm{d}x.$$

We adopt the fixed definition of the $H_0^1(\Omega)$ norm as

$$\|u\|_{H_0^1(\Omega)}^2 = \|u\|_{L^2(\Omega)}^2 + \|\nabla u\|_{L^2(\Omega)}^2, \quad \forall u \in H_0^1(\Omega).$$

Next we give the assumptions for the problem (1.1).

(H) Hypotheses on $p(.)$, $k(.)$**.** Let $k(.)$ and $p(.)$ be measurable functions on $\overline{\Omega}$ that satisfy the following condition:

$$2 < p_1 \leq p(x) \leq p_2 < \infty, \text{ and } 2 \leq k_1 \leq k(x) \leq k_2 < \infty. \tag{2.2}$$

We will use the embedding $H_{\Gamma_0}^1(\Omega) \hookrightarrow L^q(\Gamma_1)$, $2 \leq q \leq \overline{q}$, where $\overline{q} = \frac{2n-2}{n-2}$, $n > 2$ and $1 \leq \overline{q} < \infty$ if $n = 2$ where

$$H_{\Gamma_0}^1(\Omega) = \left\{u \in H^1(\Omega) : u\,|_{\Gamma_0} = 0\right\},$$

equipped with the Hilbert structure induced by $H^1(\Omega)$ is a Hilbert space.

3. Existence of weak solutions

In this section we prove the existence of weak solutions to our problem (1.1). Our proof method is based on the Faedo-Galerkin approximation, the fixed point theory in Banach spaces, and the concept of compactness, which we discussed in this section. For the sake of simplicity, $a = 1$.

Theorem 3.1. *Let* $2 \leq p_1 \leq p(x) \leq p_2 \leq \overline{q}$ *and* $\max\left(2, \frac{\overline{q}}{\overline{q}+1-p_2}\right) \leq k_1 \leq k(x) \leq k_2 \leq \overline{q}$. *Then given* $(u_0, u_1) \in H^1_{\Gamma_0}(\Omega) \times L^2(\Omega)$, *there exists* $T > 0$ *and a unique solution* u *of the problem* (1.1) *on* $(0, T)$ *such that*

$$\begin{gathered} u \in C\left(0, T; H^1_{\Gamma_0}(\Omega)\right) \cap C^1\left(0, T; L^2(\Omega)\right), \\ u_t \in L^2\left(0, T; H^1_{\Gamma_0}(\Omega)\right) \cap L^{k(.)}\left((0, T) \times \Gamma_1\right). \end{gathered}$$

In order to prove the main theorem, we need the local existence and uniqueness of the solution to the following related problem:

$$\begin{gathered} v_{tt}(x,t) - \Delta v(x,t) - \gamma \Delta v_t(x,t) = |u|^{p(x)-2} u(x,t) \text{ in } \Omega \times \mathbb{R}^+, \\ v = 0 \text{ on } \Gamma_0 \times (0, +\infty), \\ v_{tt}(x,t) = -\left[\frac{\partial v}{\partial \nu}(x,t) + \gamma \frac{\partial v_t}{\partial \nu}(x,t) + r |v_t|^{k(x)-2} v_t(x,t)\right] \\ \text{on } \Gamma_1 \times (0, +\infty), \\ v(x,0) = u_0(x), \;\; v_t(x,0) = u_1(x), \;\; x \in \Omega. \end{gathered} \tag{P4}$$

We now have to give the following existence result of the local solution of problem (P4) for an arbitrary initial value $(u_0, u_1) \in H^1_{\Gamma_0}(\Omega) \times L^2(\Omega)$.

Lemma 3.2. *Let* $2 \leq p_1 \leq p(x) \leq p_2 \leq \overline{q}$ *and* $\max\left(2, \frac{\overline{q}}{\overline{q}+1-p_2}\right) \leq k_1 \leq k(x) \leq k_2 \leq \overline{q}$. *Then given* $(u_0, u_1) \in H^1_{\Gamma_0}(\Omega) \times L^2(\Omega)$ *there exists* $T > 0$ *and a unique solution* v *of the problem* (P4) *on* $(0, T)$ *such that*

$$\begin{gathered} v \in C\left(0, T; H^1_{\Gamma_0}(\Omega)\right) \cap C^1\left(0, T; L^2(\Omega)\right), \\ v_t \in L^2\left(0, T; H^1_{\Gamma_0}(\Omega)\right) \cap L^{k(.)}\left((0, T) \times \Gamma_1\right). \end{gathered}$$

To justify Lemma (3.2), we first investigate the following problem for every $T > 0$ and $f \in H^1(0, T; L^2(\Omega))$

$$\begin{gathered} v_{tt}(x,t) - \Delta v(x,t) - \gamma \Delta v_t(x,t) = f(x,t) \text{ in } \Omega \times \mathbb{R}^+, \\ v(x,t) = 0 \text{ on } \Gamma_0 \times (0, +\infty), \\ v_{tt}(x,t) = -\left[\frac{\partial v}{\partial \nu}(x,t) + \gamma \frac{\partial v_t}{\partial \nu}(x,t) + r |v_t|^{k(x)-2} v_t(x,t)\right] \\ \text{on } \Gamma_1 \times (0, +\infty), \\ v(x,0) = u_0(x), \;\; v_t(x,0) = u_1(x), \;\; x \in \Omega. \end{gathered} \tag{P5}$$

At this point, as reported by Doronin et al. [11], we need to know exactly what kind of solutions to problem (P5) we need

Definition 3.3. We say that a function v is a local generalized solution to problem (P5) if

- **(i).** $v \in L^\infty\left(0, T; H^1_{\Gamma_0}(\Omega)\right)$,
- **(ii).** $v_t \in L^2\left(0, T; H^1_{\Gamma_0}(\Omega)\right) \cap L^{k(.)}\left((0, T) \times \Gamma_1\right) \cap L^\infty\left(0, T; H^1_{\Gamma_0}(\Omega)\right) \cap L^\infty\left(0, T; L^2(\Gamma_1)\right)$,
- **(iii).** $v_{tt} \in L^\infty\left(0, T; L^2(\Omega)\right) \cap L^\infty\left(0, T; L^2(\Gamma_1)\right)$,
- **(iv).** $v(x,0) = u_0(x), \;\; v_t(x,0) = u_1(x)$,

(v). for all $\varphi \in H^1_{\Gamma_0}(\Omega) \cap L^{k(\cdot)}(\Gamma_1)$ and a.e. $t \in [0,T]$ with $\phi \in C(0,T)$ and $\phi(T) = 0$, the following identity hold:

$$\begin{aligned}\int_0^T (f,\varphi)(t)\,\phi(t)\,\mathrm{d}t = & \int_0^T [(v_{tt},\varphi)(t) + (\nabla v,\nabla\varphi)(t) + \gamma(\nabla v_t,\nabla\varphi)(t)]\,\phi(t)\,\mathrm{d}t \\ & + \int_0^T \phi(t)\int_{\Gamma_1}\left[v_{tt}(t) + r|v_t|^{k(x)-2}v_t(t)\right]\varphi \mathrm{d}\Gamma\mathrm{d}t.\end{aligned}$$

Using the Galerkin arguments, we prove the following lemma on the existence and uniqueness of a local solution of (P5) in time.

Lemma 3.4. *Let* $2 \le p_1 \le p(x) \le p_2 \le \overline{q}$ *and* $2 \le k_1 \le k(x) \le k_2 \le \overline{q}$. *Then, for all* $(u_0,u_1) \in H^2(\Omega) \cap H^1_{\Gamma_0}(\Omega) \cap L^{k(\cdot)}(\Gamma_1) \times H^2(\Omega)$ *and* $f \in H^1(0,T;L^2(\Omega))$, *there is a unique solution* v *of problem* (P5) *in the sense of definition* (3.3).

The proof of the above lemma depends on the Faedo-Galerkin method, which consists of constructing approximations of the solution. Then we get the necessary a priori estimates to ensure the convergence of these approximations. It seems difficult to get second-order estimates for $v_{tt}(0)$. To obtain them we relied on the ideas of Doronin and Larkin in [10] and Cavalcanti et al. [6] be inspired.

Proof of Lemma (3.4). We propose the following modification of variables:

$$\widetilde{v}(t,x) = v(t,x) - \omega(t,x) \text{ with } \omega(t,x) = u_0(x) + tu_1(x).$$

Hence we have the following problem with the unknown $\widetilde{v}(t,x)$ and null initial conditions

$$\begin{cases}\widetilde{v}_{tt} - \Delta\widetilde{v} - \gamma\Delta\widetilde{v}_t = f(x,t) + \Delta\omega + \gamma\Delta\omega_t \text{ in } \Omega\times\mathbb{R}^+, \\ \widetilde{v} = 0 \text{ on } \Gamma_0\times(0,+\infty), \\ \widetilde{v}(x,t) = -\left[\begin{array}{c}\frac{\partial(\widetilde{v}+\omega)}{\partial\nu}(x,t) + \gamma\frac{\partial(\widetilde{v}_t+\omega_t)}{\partial\nu}(x,t) \\ +r|\widetilde{v}_t+\omega_t|^{k(x)-2}(\widetilde{v}_t+\omega_t)(x,t)\end{array}\right] \text{ on } \Gamma_1\times(0,+\infty), \\ \widetilde{v}(x,0) = 0,\;\; v_t(x,0) = 0,\;\; x\in\Omega.\end{cases} \tag{P6}$$

Therefore we first prove the existence and uniqueness of the local solution for (P5). Let (w_j), $j = 1,2,...,$ be a complete orthonormal system in $L^2(\Omega) \cap L^2(\Gamma_1)$ with the following properties:
* $\forall j; w_j \in H^1_{\Gamma_0}(\Omega) \cap L^{k(\cdot)}(\Gamma_1)$;
* The family $\{w_1,w_2,...,w_m\}$ is linearly independent;
* V_m the space generated by $\{w_1,w_2,...,w_m\}$, $\underset{m}{\cup} V_m$ is dense in $H^1_{\Gamma_0}(\Omega) \cap L^{k(\cdot)}(\Gamma_1)$.

We construct approximate solutions, $\widetilde{v}_m$ $(m = 1,2,3,...)$ in V_m in the form

$$\widetilde{v}_m(t) = \sum_{i=1}^{m} K_{jm}(t)w_i,\;\; m = 1,2,..., \tag{3.1}$$

where $K_{jm}(t)$ are determined by the following ordinary differential equation:

$$\begin{aligned}&\left(\tfrac{\mathrm{d}^2}{\mathrm{d}t^2}\widetilde{v}_m(t),w_j\right) + (\nabla(\widetilde{v}_m+\omega),\nabla w_j) + \gamma(\nabla(\widetilde{v}_m+\omega)_t,\nabla w_j) \\ &+ \left(\tfrac{\mathrm{d}^2}{\mathrm{d}t^2}\widetilde{v}_m(t) + r|(\widetilde{v}_m+\omega)_t|^{k(x)-2}(\widetilde{v}_m+\omega)_t,w_j\right)_{\Gamma_1} = (f(t),w_j),\;\; j = 1,2,...,\end{aligned}$$

and is completed by the following initial conditions $v_m(0)$, $v_{tm}(0)$ that satisfy

$$\widetilde{v}_m(0) = \widetilde{v}_{tm}(0) = 0. \tag{3.2}$$

Then

$$\begin{aligned}&\left(\tfrac{\mathrm{d}^2}{\mathrm{d}t^2}\widetilde{v}_m(t),v\right)+\left(\nabla\left(\widetilde{v}_m+\omega\right),\nabla v\right)+\gamma\left(\nabla\left(\widetilde{v}_m+\omega\right)_t,\nabla v\right)\\&+\left(\tfrac{\mathrm{d}^2}{\mathrm{d}t^2}\widetilde{v}_m(t)+r\left|\left(\widetilde{v}_m+\omega\right)_t\right|^{k(x)-2}\left(\widetilde{v}_m+\omega\right)_t,v\right)_{\Gamma_1}=\left(f\left(t\right),v\right),\end{aligned}\tag{3.3}$$

it holds for any given $v\in Span\left\{w_1,w_2,...,w_m\right\}$, due to the theory of ordinary differential equations, the system (3.1)-(3.3) has a unique local solution, which is extended to maximal intervals $[0,t_m]$.

A solution $\widetilde{v}$ of problem (1.1) in an interval $[0,t_m]$ is obtained as the limit of $\widetilde{v}_m$ as $m\to\infty$. Then, as a consequence of the a priori estimates to be proved in the next step, this solution can be extended to the entire interval $[0,T]$ for all $T>0$. In this section, $C>0$ and $c_*>0$ denote various positive constants that vary from line to line, are independent of the natural number m, and only (possibly) depend on the initial value.

Estimates for $\widetilde{v}_{tm}(t)$

By taking $v=\widetilde{v}_{tm}(t)$ in (3.3), we have for $t\in(0,t_m)$

$$\begin{aligned}&\tfrac{1}{2}\left(\int_\Omega\left|\widetilde{v}_{tm}\right|^2\mathrm{d}x+\int_\Omega\left|\nabla\widetilde{v}_m\right|^2\mathrm{d}x+\left\|\widetilde{v}_{tm}\right\|_{\Gamma_1}^2\right)+\gamma\int_0^t\int_\Omega\left|\nabla\widetilde{v}_{tm}\right|^2\mathrm{d}x\mathrm{d}s\\&\quad+\int_0^t\left(\nabla\omega,\nabla\widetilde{v}_{tm}\right)\mathrm{d}s+\gamma\int_0^t\left(\nabla\omega_t,\nabla\widetilde{v}_{tm}\right)\mathrm{d}s\\&+r\int_0^t\left(\left|\left(\widetilde{v}_m+\omega\right)_t\right|^{k(x)-2}\left(\widetilde{v}_m+\omega\right)_t,\widetilde{v}_{tm}\right)_{\Gamma_1}\mathrm{d}s=\int_0^t\left(f,\widetilde{v}_{tm}\right)\mathrm{d}s.\end{aligned}\tag{3.4}$$

Using Young's inequality, there are $\delta_1>0$ (actually small enough) so they hold

$$\gamma\int_0^t\left(\nabla\omega_t,\nabla\widetilde{v}_{tm}\right)\mathrm{d}s\le\delta_1\int_0^t\int_\Omega\left|\nabla\widetilde{v}_{tm}\right|^2\mathrm{d}x\mathrm{d}s+\frac{1}{4\delta_1}\int_0^t\int_\Omega\left|\nabla\omega_t\right|^2\mathrm{d}x\mathrm{d}s,\tag{3.5}$$

and

$$\int_0^t\left(\nabla\omega,\nabla\widetilde{v}_{tm}\right)\mathrm{d}s\le\delta_1\int_0^t\int_\Omega\left|\nabla\widetilde{v}_{tm}\right|^2\mathrm{d}x\mathrm{d}s+\frac{1}{4\delta_1}\int_0^t\int_\Omega\left|\nabla\omega\right|^2\mathrm{d}x\mathrm{d}s.\tag{3.6}$$

By the inequalities of Hölder and Young there is $C>0$ such that

$$\int_0^t\left(f,\widetilde{v}_{tm}\right)\mathrm{d}s\le C\int_0^t\int_\Omega\left(\left|f\right|^2+\left|\widetilde{v}_{tm}\left(s\right)\right|^2\right)\mathrm{d}x\mathrm{d}s.\tag{3.7}$$

The last term on the left of Equation (3.4) can be written as follows

$$\begin{aligned}&\int_0^t\left(\left|\left(\widetilde{v}_m+\omega\right)_t\right|^{k(x)-2}\left(\widetilde{v}_m+\omega\right)_t,\widetilde{v}_{tm}\right)_{\Gamma_1}\mathrm{d}s\\&=\int_0^t\left(\left|\left(\widetilde{v}_m+\omega\right)_t\right|^{k(x)-2}\left(\widetilde{v}_m+\omega\right)_t,\left(\widetilde{v}_m+\omega\right)_t\right)_{\Gamma_1}\mathrm{d}s\\&-\int_0^t\left(\left|\left(\widetilde{v}_m+\omega\right)_t\right|^{k(x)-2}\left(\widetilde{v}_m+\omega\right)_t,\omega_t\right)_{\Gamma_1}\mathrm{d}s\\&=\int_0^t\int_{\Gamma_1}\left|\left(\widetilde{v}_m+\omega\right)_t\right|^{k(x)}\mathrm{d}\Gamma\mathrm{d}s-\int_0^t\left(\left|\left(\widetilde{v}_m+\omega\right)_t\right|^{k(x)-2}\left(\widetilde{v}_m+\omega\right)_t,\omega_t\right)_{\Gamma_1}\mathrm{d}s.\end{aligned}$$

Therefore, Young's inequality grants us for $\delta_2>0$

$$\begin{aligned}\left|\int_0^t\left(\left|\left(\widetilde{v}_m+\omega\right)_t\right|^{k(x)-2}\left(\widetilde{v}_m+\omega\right)_t,\omega_t\right)_{\Gamma_1}\mathrm{d}s\right|&\le\tfrac{1}{k_1}\int_0^t\int_{\Gamma_1}\delta_2^{k(x)}\left|\left(\widetilde{v}_m+\omega\right)_t\right|^{k(x)}\mathrm{d}\Gamma\mathrm{d}s\\&+\tfrac{k_2-1}{k_1}\int_0^t\int_{\Gamma_1}\delta_2^{-\frac{k(x)}{k(x)-1}}\left|\omega_t\right|^{k(x)}\mathrm{d}\Gamma\mathrm{d}s.\end{aligned}\tag{3.8}$$

So if we apply inequalities (3.5), (3.6), (3.7) and (3.8) to Equation (3.4) and make δ_1 and δ_2 small enough, we can conclude

$$\int_\Omega |\widetilde{v}_{tm}|^2 \, \mathrm{d}x + \int_\Omega |\nabla \widetilde{v}_m|^2 \, \mathrm{d}x + \|\widetilde{v}_{tm}\|^2_{\Gamma_1} + \int_0^t |\nabla \widetilde{v}_{tm}|^2 \, \mathrm{d}s + \int_0^t \int_{\Gamma_1} |(\widetilde{v}_m + \omega)_t|^{k(x)} \, \mathrm{d}\Gamma \mathrm{d}s \leq C_T, \tag{3.9}$$

where C_T is a positive constant independent of m. Thus the solution can be extended to $[0, T)$ and also hold

$$(\widetilde{v}_m) \text{ is a bounded sequence in } L^\infty\left(0, T; H^1_{\Gamma_0}(\Omega)\right), \tag{3.10}$$

$$(\widetilde{v}_{tm}) \text{ is a bounded sequence in } L^\infty\left(0, T; L^2(\Omega)\right) \cap L^2\left(0, T; H^1_{\Gamma_0}(\Omega)\right) \cap L^\infty\left(0, T; L^2(\Gamma_1)\right), \tag{3.11}$$

Now applying the following algebraic inequality:

$$A^\lambda = (A + B - B)^\lambda \leq 2^{\lambda - 1}\left((A + B)^\lambda + B^\lambda\right), \;\; A, \;\; B > 0 \text{ and } \lambda \geq 1,$$

there are $C_1 >$ and $C_2 > 0$ such that

$$\int_0^t \int_{\Gamma_1} |\widetilde{v}_{tm}|^{k(x)} \, \mathrm{d}\Gamma \mathrm{d}s = \int_0^t \int_{\Gamma_1} |(\widetilde{v}_m + \omega)_t - \omega_t|^{k(x)} \, \mathrm{d}\Gamma_1 \mathrm{d}s$$
$$\leq C_1 \int_0^t \int_{\Gamma_1} |(\widetilde{v}_m + \omega)_t|^{k(x)} \, \mathrm{d}\Gamma \mathrm{d}s + C_2 \int_0^t \int_{\Gamma_1} |\omega_t|^{k(x)} \, \mathrm{d}\Gamma \mathrm{d}s.$$

Hence, from inequalities (3.9) and (3.6), there are $C'_T > 0$ such that

$$\int_0^t \int_{\Gamma_1} |\widetilde{v}_{tm}|^{k(x)} \, \mathrm{d}\Gamma \mathrm{d}s \leq C'_T.$$

Thus

$$(\widetilde{v}_{tm}) \text{ is a bounded sequence in } L^{k(.)}\left((0, T) \times \Gamma_1\right). \tag{3.12}$$

Estimates for $\widetilde{v}_{ttm}(t)$

First we estimate $\widetilde{v}_{ttm}(0)$. Putting $t = 0$ and $v = \widetilde{v}_{ttm}(0)$ in (3.3) and considering (3.11), we get

$$\int_\Omega |\widetilde{v}_{ttm}(0)|^2 \, \mathrm{d}x + \|\widetilde{v}_{ttm}(0)\|^2_{2,\Gamma_1} + (\nabla \omega(0), \nabla \widetilde{v}_{ttm}(0))$$
$$+ \gamma (\nabla \omega_t(0), \nabla \widetilde{v}_{ttm}(0)) + \left(r |\omega_t(0)|^{k(x)-2} \omega_t(0), \widetilde{v}_{ttm}(0)\right)_{\Gamma_1} = (f(0), \widetilde{v}_{ttm}(0)).$$

Knowing that the following inequalities hold:

$$(\nabla \omega_t(0), \nabla \widetilde{v}_{ttm}(0)) = -(\Delta \omega_t(0), \widetilde{v}_{ttm}(0)) + \left(\omega_t(0), \frac{\partial \widetilde{v}_{ttm}(0)}{\partial \nu}\right)_{\Gamma_1},$$

$$(\nabla \omega(0), \nabla \widetilde{v}_{ttm}(0)) = -(\Delta \omega(0), \widetilde{v}_{ttm}(0)) + \left(\omega(0), \frac{\partial \widetilde{v}_{ttm}(0)}{\partial \nu}\right)_{\Gamma_1},$$

and from $2(k_1-1)\le 2(k_2-1)\le \frac{2n}{n-2}$,

$$\begin{aligned}
&\left|\left(r\left|\omega_t(0)\right|^{k(x)-2}\omega_t(0),\widetilde{v}_{ttm}(0)\right)_{\Gamma_1}\right| \\
&\le r\max\begin{pmatrix} \int_{\Gamma_1}\left|\omega_t(0)\right|^{k_2-2}\left|\omega_t(0)\right|\left|\widetilde{v}_{ttm}(0)\right|\mathrm{d}\Gamma, \\ \int_{\Gamma_1}\left|\omega_t(0)\right|^{k_1-2}\left|\omega_t(0)\right|\left|\widetilde{v}_{ttm}(0)\right|\mathrm{d}\Gamma \end{pmatrix} \\
&\le r\max\begin{pmatrix} \|\omega_t(0)\|_{(k_2-2)n,\Gamma_1}^{k_2-2}\|\omega_t(0)\|_{\frac{2n}{n-2},\Gamma_1}\|\widetilde{v}_{ttm}(0)\|_{2,\Gamma_1}, \\ \|\omega_t(0)\|_{(k_1-2)n,\Gamma_1}^{k_1-2}\|\omega_t(0)\|_{\frac{2n}{n-2},\Gamma_1}\|\widetilde{v}_{ttm}(0)\|_{2,\Gamma_1} \end{pmatrix} \\
&\le Cr\max\begin{pmatrix} \|\nabla\omega_t(0)\|_{2,\Gamma_1}^{k_2-2}\|\nabla\omega_t(0)\|_{2,\Gamma_1}, \\ \|\nabla\omega_t(0)\|_{2,\Gamma_1}^{k_1-2}\|\nabla\omega_t(0)\|_{2,\Gamma_1} \end{pmatrix}\|\widetilde{v}_{ttm}(0)\|_{2,\Gamma_1} \\
&\le Cr\max\left(\left|\omega_t(0)\right|^{k_2-1},\left|\omega_t(0)\right|^{k_1-1}\right)\|\widetilde{v}_{ttm}(0)\|_{2,\Gamma_1}.
\end{aligned} \tag{3.13}$$

Then from $(u_0,u_1)\in H^2(\Omega)\cap H^1_{\Gamma_0}(\Omega)\cap L^{k(\cdot)}(\Gamma_1)\times H^2(\Omega)$ and $f\in H^1(0,T;L^2(\Omega))$, by applying Young's inequality and embedding $H^1(\Omega)\hookrightarrow L^{k_2}(\Gamma_1)$ and $H^1(\Omega)\hookrightarrow L^{k_1}(\Gamma_1)$ we conclude that there is $C>0$ independent of m such that

$$\int_\Omega\left|\widetilde{v}_{ttm}(0)\right|^2\mathrm{d}x+\|\widetilde{v}_{ttm}(0)\|_{2,\Gamma_1}^2\le C. \tag{3.14}$$

By differentiating equation (3.3) with respect to t and replacing v with $\widetilde{v}_{ttm}(t)$, we get

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_\Omega\left|\widetilde{v}_{ttm}\right|^2\mathrm{d}x+\int_\Omega\left|\nabla\widetilde{v}_{tm}\right|^2\mathrm{d}x+\|\widetilde{v}_{ttm}\|_{2,\Gamma_1}^2\right)+\gamma\int_\Omega\left|\nabla\widetilde{v}_{ttm}\right|^2\mathrm{d}x+(\nabla\omega_t,\nabla\widetilde{v}_{ttm}) \\
&+r\int_{\Gamma_1}(k(x)-1)\left|(\widetilde{v}_m+\omega)_t\right|^{k(x)-2}(\widetilde{v}_m+\omega)_{tt}\,\widetilde{v}_{ttm}\mathrm{d}\Gamma=(f_t,\widetilde{v}_{ttm}).
\end{aligned} \tag{3.15}$$

Since $\omega_{tt}=0$, the last term on the left-hand side of Equation (3.15) can be expressed as follows

$$\begin{aligned}
&\int_{\Gamma_1}\left|(\widetilde{v}_m+\omega)_t\right|^{k(x)-2}(\widetilde{v}_m+\omega)_{tt}\,\widetilde{v}_{ttm}\mathrm{d}\Gamma \\
&=\int_{\Gamma_1}\frac{4}{k^2(x)}\left(\frac{\partial}{\partial t}\left(\left|\widetilde{v}_{tm}(t)+\omega_t\right|^{\frac{k(x)-2}{2}}(\widetilde{v}_{tm}(t)+\omega_t)\right)\right)^2\mathrm{d}\Gamma.
\end{aligned}$$

Now Equation (3.15) is integrated over $(0,t)$ using estimate (3.14) and the Young and Poincare's inequalities (as in (3.8)) there is $\widetilde{C}_T>0$ such that

$$\begin{aligned}
&\left(\int_\Omega\left|\widetilde{v}_{ttm}\right|^2\mathrm{d}x+\int_\Omega\left|\nabla\widetilde{v}_{tm}\right|^2\mathrm{d}x+\|\widetilde{v}_{ttm}\|_{2,\Gamma_1}^2\right)+\gamma\int_0^t\left|\nabla\widetilde{v}_{ttm}\right|^2\mathrm{d}s \\
&+r\frac{4(k_1-1)}{(k_2)^2}\int_0^t\int_{\Gamma_1}\left(\frac{\partial}{\partial t}\left(\left|\widetilde{v}_{tm}(t)+\omega_t\right|^{\frac{k(x)-2}{2}}(\widetilde{v}_{tm}(t)+\omega_t)\right)\right)^2\mathrm{d}\Gamma\mathrm{d}s\quad\le\widetilde{C}_T.
\end{aligned}$$

Consequently we come to the following results:

$$\begin{aligned}
&(\widetilde{v}_{ttm})\text{ is a bounded sequence in }L^\infty\left(0,T;L^2(\Omega)\right), \\
&(\widetilde{v}_{ttm})\text{ is a bounded sequence in }L^\infty\left(0,T;L^2(\Gamma_1)\right), \\
&(\widetilde{v}_{tm})\text{ is a bounded sequence in }L^\infty\left(0,T;H^1_{\Gamma_0}(\Omega)\right).
\end{aligned} \tag{3.16}$$

From (3.10), (3.11), (3.12) and (3.16), we have that $(\widetilde{v}_m)$ is bounded in $L^\infty\left(0,T;H^1_{\Gamma_0}(\Omega)\right)$. Then, $(\widetilde{v}_m)$ is bounded in $L^2\left(0,T;H^1_{\Gamma_0}(\Omega)\right)$. Since $(\widetilde{v}_{tm})$ is bounded in $L^\infty\left(0,T;L^2(\Omega)\right)$, $(\widetilde{v}_{tm})$ is bounded in $L^2\left(0,T;L^2(\Omega)\right)$. Thus, $(\widetilde{v}_m)$ is bounded in $H^1(Q)$. Since the embedding $H^1(Q) \hookrightarrow L^2(Q)$ is compact, by the Aubin-Lions theorem we have that there is a subsequence of $(\widetilde{v}_m)$, still denoted by $(\widetilde{v}_m)$, so that

$$\widetilde{v}_m \to v \text{ strongly in } L^2(Q).$$

Therefore

$$\widetilde{v}_m \to v \text{ strongly and a.e. on } (0,T)\times\Omega.$$

Using Lion's Lemma, we get

$$|\widetilde{v}_m|^{p(.)-2}\,\widetilde{v}_m \to |\widetilde{v}|^{p(.)-2}\,\widetilde{v} \text{ strongly and a.e. on } (0,T)\times\Omega.$$

On the other hand, we have from (3.11)

$$(\widetilde{v}_{tm}) \text{ is a bounded sequence in } L^\infty\left(0,T;L^2(\Gamma_1)\right).$$

From (3.10) and (3.16), since

$$\|\widetilde{v}_m\|_{H^{\frac{1}{2}}(\Gamma_1)} \le C\,\|\nabla\widetilde{v}_m\|_2 \text{ and } \|\widetilde{v}_{tm}\|_{H^{\frac{1}{2}}(\Gamma_1)} \le C\,\|\nabla\widetilde{v}_{tm}\|_2,$$

we derive that

$$\begin{aligned}(\widetilde{v}_m) &\text{ is a bounded sequence in } L^2\left(0,T;H^{\frac{1}{2}}(\Gamma_1)\right),\\ (\widetilde{v}_{tm}) &\text{ is a bounded sequence in } L^2\left(0,T;H^{\frac{1}{2}}(\Gamma_1)\right),\\ (\widetilde{v}_{ttm}) &\text{ is a bounded sequence in } L^2\left(0,T;L^2(\Gamma_1)\right).\end{aligned}$$

Since the embedding $H^{\frac{1}{2}}(\Gamma_1) \hookrightarrow L^2(\Gamma_1)$ is compact, again using the Aubin-Lions theorem, we conclude that we can extract a subsequence of $(\widetilde{v}_m)$ still denoted by $(\widetilde{v}_m)$ so that

$$\widetilde{v}_{tm} \to v_t \text{ strongly in } L^2\left(0,T;L^2(\Gamma_1)\right). \tag{3.17}$$

So we get that from (3.12)

$$|\widetilde{v}_{tm}|^{k(.)-2}\,\widetilde{v}_{tm} \to \varkappa \text{ weakly in } L^{\frac{k(.)}{k(.)-1}}\left((0,T)\times\Gamma_1\right).$$

It is enough to prove that $\varkappa = |\widetilde{v}_t|^{k(.)-2}\,\widetilde{v}_t$.
Clearly, from (3.17) we get

$$|\widetilde{v}_{tm}|^{k(.)-2}\,\widetilde{v}_{tm} \to |\widetilde{v}_t|^{k(.)-2}\,\widetilde{v}_t \text{ strongly and a.e. on } (0,T)\times\Gamma_1.$$

Again, using the Lions lemma, we get $\varkappa = |\widetilde{v}_t|^{k(.)-2}\,\widetilde{v}_t$. The proof can now be completed as follows

Proof of uniqueness:

Let u_1 and u_2 be two solutions of the problem (P5) with the same initial data. Let us denote $w = u_1 - u_2$. It is easy to see that w satisfies

$$\begin{aligned}&\left(\int_\Omega |w_t|^2\,\mathrm{d}x + \int_\Omega |\nabla w|^2\,\mathrm{d}x + \|w_t\|_{\Gamma_1}^2\right) + 2\gamma\int_0^t |\nabla w_t|^2\,\mathrm{d}s\\ &+2r\int_0^t\int_{\Gamma_1}\left(|u_{1t}|^{k(x)-2}\,u_{1t} - |u_{2t}|^{k(x)-2}\,u_{2t}\right) w_t(s)\,\mathrm{d}\Gamma\mathrm{d}s \quad = 0.\end{aligned}$$

By using the inequality

$$\left(|a|^{k(x)-2}\,a - |b|^{k(x)-2}\,b\right).(a-b) \ge 0, \tag{3.18}$$

for all a, $b \in \mathbb{R}^n$ and a.e. $x \in \Omega$, we have

$$\int_\Omega |w_t|^2 \, dx + \int_\Omega |\nabla w|^2 \, dx + \|w_t\|_{2,\Gamma_1}^2 = 0,$$

which implies that $w = C = 0$. Hence, the uniqueness follows.

This completes the proof of the lemma (3.4). □

Proof of lemma (3.2). First we approximate $u \in \left(C[0,T], H^1_{\Gamma_0}(\Omega)\right) \cap C^1([0,T], L^2(\Omega))$ equipped with the norm $\|u\| = \max_{t\in[0,T]} \left(\|u_t\|_2 + \|u\|_{H^1(\Omega)}\right)$, by a sequence $(u^\mu) \in C^\infty\left(([0,T]) \times \overline{\Omega}\right)$ by standard convolution arguments. It is clear that $|u^\mu|^{p_1-2} u^\mu$ and $|u^\mu|^{p_2-2} u^\mu \in H^1([0,T], L^2(\Omega))$, since $2(p_1-1) \le 2(p_2-1) \le \frac{2n}{n-2}$. Next, we approximate the initial data $u_1 \in L^2(\Omega)$ by a sequence (u_1^μ) in $C_0^\infty(\Omega)$ since the space $H^2(\Omega) \cap H^1_{\Gamma_0}(\Omega) \cap L^{k(.)}(\Gamma_1)$ is dense in $H^1_{\Gamma_0}(\Omega)$ for the H^1 endowed norm we approximate $u_0 \in H^1_{\Gamma_0}(\Omega)$ by a sequence (u_0^μ) in $H^2(\Omega) \cap H^1_{\Gamma_0}(\Omega) \cap L^{k(.)}(\Gamma_1)$. We examine the set of the following approximation problems:

$$\begin{gathered} v_{tt}^\mu - \Delta v^\mu - \gamma \Delta v_t^\mu = |u^\mu|^{p(x)-2} u^\mu \text{ in } \Omega \times \mathbb{R}^+, \\ v^\mu = 0 \text{ on } \Gamma_0 \times (0,+\infty), \\ v_{tt}^\mu(x,t) = -\left[\frac{\partial v^\mu}{\partial \nu}(x,t) + \gamma \frac{\partial v_t^\mu}{\partial \nu}(x,t) + r\left|v_t^\mu\right|^{k(x)-2} v_t^\mu(x,t)\right] \\ \text{on } \Gamma_1 \times (0,+\infty), \\ v^\mu(x,0) = u_0^\mu(x),\ v_t^\mu(x,0) = u_1^\mu(x),\ x \in \Omega. \end{gathered} \tag{3.19}$$

Since Lemma (3.4) is hypothesized, we can find a sequence of unique solutions (v^μ) to problem (3.19). We will show that the sequence $\{(v^\mu, v_t^\mu)\}$ is a Cauchy sequence in space

$$\mathcal{W}_T = \left\{ \begin{array}{c} (v, v_t) \mid v \in C\left([0,T], H^1_{\Gamma_0}(\Omega)\right) \cap C^1([0,T], L^2(\Omega)), \\ v_t \in L^\infty\left(0,T; H^1_{\Gamma_0}(\Omega)\right) \cap L^{k(.)}\left((0,T) \times \Gamma_1\right), \end{array} \right\}$$

endowed with the norm

$$\begin{aligned} \|(v,v_t)\|_{\mathcal{W}_T}^2 &= \max_{t\in[0,T]} \left(\|v_t\|_2^2 + \|\nabla v\|_2^2\right) + \|v_t\|_{L^{k(.)}((0,T)\times\Gamma_1)}^2 \\ &+ \int_0^t \|\nabla v_t(s)\|_2^2 \, ds. \end{aligned}$$

For this purpose we set $U = u^\mu - u^\tau$, $V = v^\mu - v^\tau$. It is easy to see that V satisfies

$$\begin{gathered} V_{tt} - \Delta V - \gamma \Delta V_t = |u^\mu|^{p(x)-2} u^\mu - |u^\tau|^{p(x)-2} u^\tau \text{ in } \Omega \times \mathbb{R}^+, \\ V = 0 \text{ on } \Gamma_0 \times (0,+\infty), \\ V_{tt}(x,t) = -\left[\begin{array}{c} \frac{\partial V}{\partial \nu}(x,t) + \gamma \frac{\partial V_t}{\partial \nu}(x,t) \\ +r\left(\left|v_t^\mu\right|^{k(x)-2} v_t^\mu(x,t) - \left|v_t^\tau\right|^{k(x)-2} v_t^\tau(x,t)\right) \end{array} \right] \\ \text{on } \Gamma_1 \times (0,+\infty), \\ V(x,0) = u_0^\mu(x) - u_0^\tau(x),\ V_t(x,0) = u_1^\mu(x) - u_1^\tau(x),\ x \in \Omega. \end{gathered}$$

Multiply the above differential equations by V_t for all $t \in (0,T)$ and integrate over $(0,t)\times\ \Omega$ we get

$$\begin{aligned}&\left(|V_t|^2 + |\nabla V|^2 + \|V_t\|_{2,\Gamma_1}^2\right) + 2\gamma \int_0^t \int_\Omega |\nabla V_t|^2 \,\mathrm{d}x\mathrm{d}s \\ &+2r \int_0^t \int_{\Gamma_1} \left(|v_t^\mu|^{k(x)-2} v_t^\mu (x,t) - |v_t^\tau|^{k(x)-2} v_t^\tau (x,t)\right) (v_t^\mu - v_t^\tau)(s)\,\mathrm{d}\Gamma\mathrm{d}s \\ &= \left(|V_t(0)|^2 + |\nabla V(0)|^2 + \|V_t(0)\|_{2,\Gamma_1}^2\right) \\ &+2 \int_0^t \int_\Omega \left(|u^\mu|^{p(x)-2} u^\mu - |u^\tau|^{p(x)-2} u^\tau\right) (v_t^\mu - v_t^\tau)(s)\,\mathrm{d}x\mathrm{d}s.\end{aligned}$$

Using the inequality (3.18), we get

$$\begin{aligned}&\left(|V_t|^2 + |\nabla V|^2 + \|V_t\|_{\Gamma_1}^2\right) + 2\gamma \int_0^t \int_\Omega |\nabla V_t|^2 \,\mathrm{d}x\mathrm{d}s \\ &\leq \left(|V_t(0)|^2 + |\nabla V(0)|^2 + \|V_t(0)\|_{\Gamma_1}^2\right) \\ &+2 \int_0^t \int_\Omega \left(|u^\mu|^{p(x)-2} u^\mu - |u^\tau|^{p(x)-2} u^\tau\right) (v_t^\mu - v_t^\tau)(s)\,\mathrm{d}x\mathrm{d}s.\end{aligned} \tag{3.20}$$

Let's estimate the last term of the second member of the above inequality

$$\begin{aligned}&\int_\Omega \left(|u^\mu|^{p(x)-2} u^\mu - |u^\tau|^{p(x)-2} u^\tau\right) (v_t^\mu - v_t^\tau)\,\mathrm{d}x \\ &\leq (p_2-1) \int_\Omega \sup\left(|u^\mu|^{p(x)-2}, |u^\tau|^{p(x)-2}\right) |u^\mu - u^\tau|\,|v_t^\mu - v_t^\tau|\,\mathrm{d}x \\ &\leq c \max \begin{pmatrix} \left(||u^\mu(t)||_{(p_2-2)n}^{p_2-2} + ||u^\mu(t)||_{(p_1-2)n}^{p_1-2}\right) ||U||_{\frac{2n}{n-2}} \|V_t\|_2, \\ \left(||u^\tau(t)||_{(p_2-2)n}^{p_2-2} + ||u^\tau(t)||_{(p_1-2)n}^{p_1-2}\right) ||U||_{\frac{2n}{n-2}} \|V_t\|_2 \end{pmatrix} \\ &\leq cc_* \max \begin{pmatrix} \int_\Omega \left(|\nabla u^\mu(t)|^{p_2-2} + |\nabla u^\mu(t)|^{p_1-2}\right) \mathrm{d}x, \\ \int_\Omega \left(|\nabla u^\tau(t)|^{p_2-2} + |\nabla u^\tau(t)|^{p_1-2}\right) \mathrm{d}x \end{pmatrix} ||\nabla U||_2 \|V_t\|_2.\end{aligned} \tag{3.21}$$

Then the estimate (3.20) takes the form

$$\begin{aligned}&\left(\int_\Omega |V_t|^2 \,\mathrm{d}x + \int_\Omega |\nabla V|^2 \,\mathrm{d}x + \|V_t\|_{2,\Gamma_1}^2\right) + 2\gamma \int_0^t \int_\Omega |\nabla V_t|^2 \,\mathrm{d}x\mathrm{d}s \\ &\leq \left(\int_\Omega |V_t(0)|^2 \,\mathrm{d}x + \int_\Omega |\nabla V(0)|^2 \,\mathrm{d}x + \|V_t(0)\|_{2,\Gamma_1}^2\right) \\ &+2cc_* \int_0^t \max \begin{pmatrix} \int_\Omega \left(|\nabla u^\mu(t)|^{p_2-2} + |\nabla u^\mu(t)|^{p_1-2}\right) \mathrm{d}x, \\ \int_\Omega \left(|\nabla u^\tau(t)|^{p_2-2} + |\nabla u^\tau(t)|^{p_1-2}\right) \mathrm{d}x \end{pmatrix} ||\nabla U||_2 \|V_t\|_2 \,\mathrm{d}s.\end{aligned} \tag{3.22}$$

From (3.10) , (3.22) becomes

$$\begin{aligned}&\left(\int_\Omega |V_t|^2 \,\mathrm{d}x + \int_\Omega |\nabla V|^2 \,\mathrm{d}x + \|V_t\|_{\Gamma_1}^2\right) + 2\gamma \int_0^t \|\nabla V_t\|_2^2 \,\mathrm{d}s \\ &\leq \left(\int_\Omega |V_t(0)|^2 \,\mathrm{d}x + \int_\Omega |\nabla V(0)|^2 \,\mathrm{d}x + \|V_t(0)\|_{2,\Gamma_1}^2\right) \qquad +C \int_0^t ||\nabla U||_2 \|V_t\|_2 \,\mathrm{d}s.\end{aligned}$$

Thus, applying Young's and Gronwall inequalities, there is C that depending only on Ω, p_1 and p_2 such that

$$\|V\|_{\mathcal{W}_T} \leq C \left(\int_\Omega |V_t(0)|^2 \,\mathrm{d}x + \int_\Omega |\nabla V(0)|^2 \,\mathrm{d}x + \|V_t(0)\|_{2,\Gamma_1}^2\right) + CT \|U\|_{\mathcal{W}_T}.$$

Since $\{(u_0^\mu)\}$, $\{(u_1^\mu)\}$ and $\{(u^\mu)\}$ Cauchy in $H^1_{\Gamma_0}(\Omega)$, $L^2(\Omega)$ and $\left(C[0,T], H^1_{\Gamma_0}(\Omega)\right) \cap C^1([0,T], L^2(\Omega))$ we vonclude that $\{v_t^\mu\}$ and $\{v^\mu\}$ are Cauchy in $\mathcal{W}_T$. Thus, (v^μ, v_t^μ) converges to a limit $(v, v_t) \in \mathcal{W}_T$.

We now prove that the limit $(v(x,t), v_t(x,t))$ is a weak solution of (P4).

To this end, we multiply equation (3.19) by ψ in $D(\Omega)$ and integrate over Ω; then, we get

$$\begin{aligned}&\tfrac{\mathrm{d}^2}{\mathrm{d}t^2}\int_\Omega v^\mu \psi \mathrm{d}x + \tfrac{\mathrm{d}}{\mathrm{d}t}\int_{\Gamma_1} v_t^\mu \psi \mathrm{d}\Gamma + \int_\Omega \nabla v^\mu \nabla \psi \mathrm{d}x + \gamma \int_\Omega \nabla v_t^\mu \nabla \psi \mathrm{d}x \\ &+ r\int_{\Gamma_1} |v_t^\mu|^{k(x)-2} v_t^\mu(t)\psi \mathrm{d}\Gamma = \int_\Omega |u^\mu|^{p(x)-2} u^\mu \psi \mathrm{d}x.\end{aligned}$$

As $\mu \to \infty$, the followings hold in $C([0,T])$:

$$\begin{aligned}&\tfrac{\mathrm{d}^2}{\mathrm{d}t^2}\int_\Omega v^\mu \psi \mathrm{d}x \to \int_\Omega v\psi \mathrm{d}x; && \int_\Omega \nabla v^\mu \nabla\psi \mathrm{d}x \to \int_\Omega \nabla v \nabla \psi \mathrm{d}x;\\ &\int_\Omega \nabla v_t^\mu \nabla \psi \mathrm{d}x \to \int_\Omega \nabla v_t \nabla\psi \mathrm{d}x; && \int_{\Gamma_1} v_t^\mu \psi \mathrm{d}\Gamma \to \int_{\Gamma_1} v_t \psi \mathrm{d}\Gamma;\\ &\int_\Omega |u^\mu|^{p(x)-2} u^\mu \psi \mathrm{d}x \to \int_\Omega |u|^{p(x)-2} u\psi \mathrm{d}x; && \int_{\Gamma_1} |v_t^\mu|^{k(x)-2} v_t^\mu(t)\psi \mathrm{d}\Gamma \to \\ & && \int_{\Gamma_1} |v_t|^{k(x)-2} v_t(t)\psi \mathrm{d}\Gamma.\end{aligned}$$

It follows that $\int_\Omega v_{tt}\psi \mathrm{d}x = \lim_{\mu\to\infty} \int_\Omega v_{tt}^\mu \psi \mathrm{d}x$ is an absolutely continuous function on $[0,T]$, hence $(v(x,t), v_t(x,t))$ is a weak solution to the problem (P4) for almost all $t \in [0,T]$.

Remaining to prove uniqueness, we denote that v^μ, v^ν are the corresponding solutions of problem (P4) to u^μ, u^ν, respectively. Then obviously $V = v^\mu - v^\nu$ satisfies

$$\left(\int_\Omega |V_t|^2 \mathrm{d}x + \int_\Omega |\nabla V|^2 \mathrm{d}x + \|V_t\|^2_{2,\Gamma_1}\right) + 2\gamma \int_0^t \|\nabla V_t\|_2^2 \mathrm{d}s \le C\int_0^t ||\nabla U||_2 \|V_t\|_2 \mathrm{d}s.$$

This shows that $V = 0$ for $u^\mu = u^\nu$ which implies the uniqueness. □

Proof of theorem (3.1). Let us define for $T > 0$ the convex closed subset of $\mathcal{W}_T$

$$Y_T = \{(v, v_t) \in \mathcal{W}_T \text{ such that } v(0) = u_0 \text{ and } v_t(0) = u_1\}.$$

Let's denote

$$B_R(Y_T) = \left\{(v, v_t) \in \mathcal{W}_T \text{ such that } \|(v, v_t)\|_{\mathcal{W}_T} \le R\right\}.$$

Then Lemma (3.2) implies that for every $u \in Y_T$ we define $v = \Phi(u)$ as the unique solution of problem (P4) corresponding to u. We want to show that this is a satisfying contractive map

$$\Phi(B_R(Y_T)) \subset B_R(Y_T).$$

Let $u \in B_R(Y_T)$ and $v = \Phi(u)$. Then for all $t \in [0,T]$

$$\begin{aligned}&\left(\int_\Omega |v_t|^2 \mathrm{d}x + \int_\Omega |\nabla v|^2 \mathrm{d}x + \|v_t\|^2_{2,\Gamma_1}\right) + 2\gamma\int_0^t |\nabla v_t|^2 \mathrm{d}s + 2r\int_0^t\int_{\Gamma_1} |v_t|^{k(x)} \mathrm{d}\Gamma \mathrm{d}s \\ &= \left(\int_\Omega |v_t(0)|^2 \mathrm{d}x + \int_\Omega |\nabla v(0)|^2 \mathrm{d}x + \|v_t(0)\|^2_{2,\Gamma_1}\right) + 2\int_0^t\int_\Omega |u|^{p(x)-2} u v_t(s)\mathrm{d}x\mathrm{d}s.\end{aligned} \tag{3.23}$$

Using Hölder's inequality, we can examine the last term on the right-hand side of inequality (3.23) as follows

$$\begin{aligned}
\int_0^t \int_\Omega |u|^{p(x)-2} uv_t(s)\,\mathrm{d}x\mathrm{d}s &\leq (p_2-1) \int_\Omega \max\left(|u|^{p_2-2}, |u|^{p_1-2}\right) |u|\,|v_t|\,\mathrm{d}x \\
&\leq c\max\left(\left(||u(t)||^{p_2-2}_{(p_2-2)n}, ||u(t)||^{p_1-2}_{(p_1-2)n}\right) ||u||_{\frac{2n}{n-2}} \|v_t\|_2\right) \\
&\leq cc_* \max\left(\int_\Omega |\nabla u(t)|^{p_2-2}\,\mathrm{d}x, \int_\Omega |\nabla u(t)|^{p_1-2}\,\mathrm{d}x\right) ||\nabla u||_2 \|v_t\|_2 \\
&= cc_* \max\left(||\nabla u||_2^{p_2-1}, ||\nabla u||_2^{p_1-1}\right) \|v_t\|_2 .
\end{aligned}$$

Since $(p_1-2)\,n \leq (p_2-2)\,n \leq \frac{2n}{n-2}$. Thus, by Young's and Sobolev's inequalities, we get $\forall \delta > 0$, $\exists C(\delta) > 0$, such that $\forall t \in (0,T)$,

$$\int_0^t \int_\Omega |u|^{p(x)-2} uv_t(s)\,\mathrm{d}x\mathrm{d}s \leq C(\delta)t\max\left(R^{2(p_2-1)}, R^{2(p_1-1)}\right) + \delta \int_0^t |\nabla v_t|^2\,\mathrm{d}s.$$

Plugging the last estimate into inequality (3.23) and choosing δ small enough we get

$$\begin{aligned}
\|v\|^2_{Y_T} &\leq \left(\int_\Omega |v_t(0)|^2\,\mathrm{d}x + \int_\Omega |\nabla v(0)|^2\,\mathrm{d}x + \|v_t(0)\|^2_{2,\Gamma_1}\right) \\
&\quad + CT\max\left(R^{2(p_2-1)}, R^{2(p_1-1)}\right).
\end{aligned} \tag{3.24}$$

By choosing R large enough so that

$$\int_\Omega |v_t(0)|^2\,\mathrm{d}x + \int_\Omega |\nabla v(0)|^2\,\mathrm{d}x + \|v_t(0)\|^2_{2,\Gamma_1} \leq \frac{1}{2}R^2,$$

then T sufficiently small so that $CT\max\left(R^{2(p_2-1)}, R^{2(p_1-1)}\right) \leq \frac{1}{2}R^2$, it follows that $\|v\|_{Y_T} \leq R$ from (3.24), hence $v \in B_R(Y_T)$. Next, we have to check that it is a contraction. To this point, we set $U = u - \overline{u}$, $V = v - \overline{v}$ where $v = \Phi(u)$ and $\overline{v} = \Phi(\overline{u})$

$$\begin{cases}
V_{tt} - \Delta V - \gamma\Delta V_t = |u|^{p(x)-2} u - |\overline{u}|^{p(x)-2}\,\overline{u} \text{ in } \Omega\times\mathbb{R}^+, \\
V = 0 \text{ on } \Gamma_0 \times (0,+\infty), \\
V_{tt}(x,t) = -\left[\begin{array}{c} \frac{\partial V}{\partial\nu}(x,t) + \gamma\frac{\partial V_t}{\partial\nu}(x,t) \\ +r\left(|v_t|^{k(x)-2} v_t(x,t) - |\overline{v}_t|^{k(x)-2}\,\overline{v}_t(x,t)\right) \end{array}\right] \\
\text{on } \Gamma_1 \times (0,+\infty), \\
V(x,0) = 0,\ V_t(x,0) = 0,\ x\in\Omega.
\end{cases} \tag{3.25}$$

Multiplying the first equation in (3.25) by V_t, integrating over $(0,t)\times\Omega$, and using the algebraic inequality in (3.18) and the estimate (3.21) yields

$$\begin{aligned}
&\left(\int_\Omega |V_t|^2\,\mathrm{d}x + \int_\Omega |\nabla V|^2\,\mathrm{d}x + \|V_t\|^2_{2,\Gamma_1}\right) + 2\gamma\int_0^t |\nabla V_t|^2\,\mathrm{d}s \\
&\leq 2cc_* \int_0^t \max\left(\begin{array}{c} \int_\Omega \left(|\nabla u(t)|^{p_2-2} + |\nabla u(t)|^{p_1-2}\right)\mathrm{d}x, \\ \int_\Omega \left(|\nabla \overline{u}(t)|^{p_2-2} + |\nabla \overline{u}(t)|^{p_2-2}\right)\mathrm{d}x \end{array}\right) ||\nabla U||_2\,||V_t||_2\,\mathrm{d}s.
\end{aligned}$$

So

$$\|V\|^2_{Y_T} \leq 4cc_*T\left(R^{p_2-2} + R^{p_1-2}\right)\|U\|^2_{Y_T} \leq CTR^{p_2-2}\|U\|^2_{Y_T}. \tag{3.26}$$

If one chooses T small enough to have $CTR^{p_2-2} < 1$, estimate (3.26) shows that Φ is a contraction. The contraction mapping theorem guarantees the existence of a unique solution v that satisfies $v = \Phi(v)$. This completes the proof of Theorem (3.1). □

4. Exponential growth

In this section we consider the problem (1.1) from an energetic point of view: The energy grows exponentially and with it the L^{p_1} and L^{p_2} norms. To state and prove the result, we declare the following notations. From Corollary 3.3.4 in [8] we know $L^{p_2}(\Omega) \hookrightarrow L^{p(\cdot)}(\Omega)$. Hence it is a consequence of embedding $H_0^1(\Omega) \hookrightarrow L^{p_2}(\Omega)$ and Poincar's inequality

$$\|u\|_{p(.)} \leq B \|\nabla u\|_2 , \tag{4.1}$$

where B is the best constant of the embedding $H_0^1(\Omega) \hookrightarrow L^{p(\cdot)}(\Omega)$ determined by

$$B^{-1} = \inf\left\{\|\nabla u\|_2 : u \in H_0^1(\Omega),\ \ \|u\|_{p(.)} = 1\right\}.$$

We also define the following constant which will play an important role in the proof of our result.

Let B_1, α_1, α_0, E_1, and $E(0)$ be satisfying constants

$$\begin{aligned} B_1 = \max(1, B), \quad \alpha_1 = B_1^{\frac{-2p_1}{p_1-2}}, \\ \alpha_0 = \|\nabla u_0\|_2^2, \quad E_1 = \left(\tfrac{1}{2} - \tfrac{1}{p_1}\right)\alpha_1. \\ E(0) = \tfrac{1}{2}\|u_1\|_2^2 + \tfrac{1}{2}\|\nabla u_0\|_2^2 + \tfrac{1}{2}\|u_1\|_{2,\Gamma_1}^2 - \int_\Omega \tfrac{1}{p(x)} |u_0|^{p(x)}\, \mathrm{d}x. \end{aligned} \tag{4.2}$$

For the sake of simplicity, we also write $\varrho(u)$ instead of $\varrho_{p(.)}(u)$.

For this purpose we start with the following lemma, which defines the energy of the solution.

Lemma 4.1. *We define the energy of a solution u of (1.1) as:*

$$E(t) = \frac{1}{2}\|u_t\|_2^2 + \frac{1}{2}\|\nabla u\|_2^2 + \frac{1}{2}\|u_t\|_{2,\Gamma_1}^2 - \int_\Omega \frac{1}{p(x)} |u|^{p(x)}\, \mathrm{d}x. \tag{4.3}$$

If we multiply the first equation in (1.1) by u_t and integrate over Ω and with respect to t, we get

$$E(t) - E(s) = -\int_s^t \left(\gamma \|\nabla u_t(\tau)\|_2^2 + r\|u_t(\tau)\|_{k(.),\Gamma_1}\right) \mathrm{d}\tau \leq 0,\ \forall\ 0 < s \leq t < T. \tag{4.4}$$

Thus the function E is decrease along the trajectories.

Theorem 4.2. *Let $k_2 < p_1 \leq p(x) \leq p_2$ with $2 < p_1 \leq p(x) \leq p_2 \leq \overline{q}$. Assume that the initial value u_0 is chosen suvh that $E(0) < E_1$ and $B_1^{-2} \geq \|\nabla u_0\|_2^2 > \alpha_1$ hold. Then, under the above conditions, the solution to problem (1.1) will grow exponentially in the norms L^{p_1} and L^{p_2}.*

We conclude from (4.3) and (4.1)

$$\begin{aligned} E(t) &\geq \tfrac{1}{2}\|\nabla u\|_2^2 - \tfrac{1}{p_1}\max\left(\|u\|_{p(.)}^{p_2}, \|u\|_{p(.)}^{p_1}\right) \\ &\geq \tfrac{1}{2}\|\nabla u\|_2^2 - \tfrac{1}{p_1}\max\left((B_1\|\nabla u\|_2)^{p_2}, (B_1\|\nabla u\|_2)^{p_1}\right) \\ &= \tfrac{1}{2}\alpha - \tfrac{1}{p_1}\max\left(\left(B_1^2\alpha\right)^{\frac{p_2}{2}}, \left(B_1^2\alpha\right)^{\frac{p_1}{2}}\right) := g(\alpha) \ \forall \alpha \in [0, +\infty), \end{aligned} \tag{4.5}$$

where $\alpha = \|\nabla u\|_2^2$.

Lemma 4.3. *Let $h : [0, +\infty) \to \mathbb{R}$ be defined by*

$$h(\alpha) = \frac{1}{2}\alpha - \frac{1}{p_1}\left(B_1^2\alpha\right)^{\frac{p_1}{2}}. \tag{4.6}$$

Then the following claims hold under the hypotheses of Theorem (4.2):

(i). *h is increasing for $0 < \alpha \leq \alpha_1$ and decreasing for $\alpha \geq \alpha_1$;*

(ii). $\lim_{\alpha\to+\infty} h(\alpha) = -\infty$ *and* $h(\alpha_1) = E_1$.

Proof. By the assumption that $B_1 > 1$ and $p_1 > 2$, one can see that $h(\alpha) = g(\alpha)$, for $0 < \alpha \leq B_1^{-2}$. Furthermore, $h(\alpha)$ is differentiable and continuous in $[0, +\infty)$. We can see that

$$h'(\alpha) = \frac{1}{2} - \frac{1}{2}B_1^{p_1}\alpha^{\frac{p_1-2}{2}}, \quad 0 \leq \alpha < B_1^{-2}.$$

Then follows **(i)**. Since $p_1 - 2 > 0$, we have $\lim_{\alpha\to+\infty} h(\alpha) = -\infty$. A common calculation gives $h(\alpha_1) = E_1$. Then **(ii)** holds. □

Lemma 4.4. *Under the assumptions of Theorem (4.2), there exists a positive constant $\alpha_2 > \alpha_1$ such that*

$$\|\nabla u\|_2^2 \geq \alpha_2, \ t \geq 0, \tag{4.7}$$

$$\int_\Omega |u(x,t)|^{p(x)}\,\mathrm{d}x \geq \left(B_1^2\alpha_2\right)^{\frac{p_1}{2}}. \tag{4.8}$$

Proof. Since $E(0) < E_1$, Lemma (4.3) implies that there is a positive constant $\alpha_2 > \alpha_1$ such that $E(0) = h(\alpha_2)$. By (4.5) we have $h(\alpha_0) = g(\alpha_0) \leq E(0) = h(\alpha_2)$, from Lemma (4.3)**(i)** it follows that $\alpha_0 \geq \alpha_2$ so (4.7) holds for $t = 0$. Now we prove (4.7) by contradiction. Suppose $\|\nabla u(t^*)\|_2^2 < \alpha_2$ for some $t^* > 0$. Suppose that $\|\nabla u(t^*)\|_2^2 < \alpha_2$ for some $t^* > 0$. By the continuity of $\|\nabla u(.,t)\|_2$ and $\alpha_2 > \alpha_1$, we can assume t^* such that $\alpha_2 > \|\nabla u(t^*)\|_2^2 > \alpha_1$, then (4.5) yildes

$$E(0) = h(\alpha_2) < h\left(\|\nabla u(t^*)\|_2^2\right) \leq E(t^*),$$

which contradicts to Lemma (3.2), and (4.7) holds.

By (4.3) and (4.4), we obtain

$$\begin{aligned} \tfrac{1}{p_1}\textstyle\int_\Omega |u(x,t)|^{p(x)}\,\mathrm{d}x &\geq \textstyle\int_\Omega \tfrac{1}{p(x)}|u(x,t)|^{p(x)}\,\mathrm{d}x \geq \tfrac{1}{2}\|\nabla u\|_2^2 - E(0) \\ &\geq \tfrac{1}{2}\alpha_2 - E(0) = \tfrac{1}{2}\alpha_2 - h(\alpha_2) = \tfrac{1}{p_1}\left(B_1^2\alpha_2\right)^{\frac{p_1}{2}}, \end{aligned} \tag{4.9}$$

and (4.8) follows. □

Let $H(t) = E_1 - E(t)$ for $t \geq 0$, we have the following lemma:

Lemma 4.5. *Under the assumptions of Theorem* (4.2) *the function* $H(t)$ *presented above gives the following estimates:*

$$0 < H(0) \leq H(t) \leq \int_\Omega \frac{1}{p(x)} |u(x,t)|^{p(x)} \mathrm{d}x, \quad t \geq 0. \tag{4.10}$$

Proof. By Lemma (3.2), $H(t)$ is nondecreasing in t thus

$$H(t) \geq H(0) = E_1 - E(0) > 0, \quad t \geq 0. \tag{4.11}$$

If we combine (4.3), (4.2), (4.7) and $\alpha_2 > \alpha_1$, we get

$$\begin{aligned} H(t) - \int_\Omega \tfrac{1}{p(x)} |u(x,t)|^{p(x)} \mathrm{d}x &\leq E_1 - \tfrac{1}{2} \|\nabla u\|_2^2 \\ &\leq \left(\tfrac{1}{2} - \tfrac{1}{p_1}\right) \alpha_1 - \tfrac{1}{2}\alpha_1 < 0, \ t \geq 0, \end{aligned} \tag{4.12}$$

and (4.10) follows from (4.11) and (4.12). □

Based on the above three lemmas, we can provied the proof of Theorem (4.2).

Proof of Theorem (4.2). We then define the auxiliary function for the value $\varepsilon > 0$ small to be selected later

$$L(t) = H(t) + \varepsilon \int_\Omega u_t u \mathrm{d}x + \varepsilon \int_{\Gamma_1} u_t u \mathrm{d}\Gamma + \frac{1}{2}\varepsilon\gamma \int_\Omega |\nabla u|^2 \mathrm{d}x. \tag{4.13}$$

Let's consider that L is a small perturbation of the energy. Taking the time derivative of (4.13), we get

$$\begin{aligned} \tfrac{\mathrm{d}L(t)}{\mathrm{d}t} &= \gamma \int_\Omega |\nabla u_t|^2 \mathrm{d}x + \varepsilon \int_\Omega |u_t|^2 \mathrm{d}x + \varepsilon \|u_t\|_{\Gamma_1}^2 + r \int_{\Gamma_1} |u_t|^{k(x)} \mathrm{d}\Gamma \\ &+\varepsilon \int_\Omega u_{tt} u \mathrm{d}x + \varepsilon \int_{\Gamma_1} u_{tt} u \mathrm{d}\Gamma + \varepsilon\gamma \int_\Omega \nabla u \nabla u_t \mathrm{d}x. \end{aligned} \tag{4.14}$$

Using problem (1.1), we get from equation (4.14)

$$\begin{aligned} \tfrac{\mathrm{d}L(t)}{\mathrm{d}t} &= \gamma \int_\Omega |\nabla u_t|^2 \mathrm{d}x + \varepsilon \int_\Omega |u_t|^2 \mathrm{d}x + \varepsilon \|u_t\|_{2,\Gamma_1}^2 + r \int_{\Gamma_1} |u_t|^{k(x)} \mathrm{d}\Gamma \\ &-\varepsilon \int_\Omega |\nabla u|^2 \mathrm{d}x + \varepsilon \int_\Omega |u(t)|^{p(x)} \mathrm{d}x - \varepsilon r \int_{\Gamma_1} |u_t|^{k(x)} u_t u \mathrm{d}\Gamma. \end{aligned} \tag{4.15}$$

To estimate the last term on the right-hand side of the previous equation, let $\delta > 0$ shall be determined later. Young's inequality drives

$$\int_{\Gamma_1} |u_t|^{k(x)} u_t u \mathrm{d}\Gamma \leq \frac{1}{k_1} \int_{\Gamma_1} \delta^{k(x)} |u|^{k(x)} \mathrm{d}\Gamma + \frac{k_2 - 1}{k_1} \int_{\Gamma_1} \delta^{-\frac{k(x)}{k(x)-1}} |u_t|^{k(x)} \mathrm{d}\Gamma.$$

This is obtained by substituting in (4.15)

$$\begin{aligned} \tfrac{\mathrm{d}L(t)}{\mathrm{d}t} &\geq \gamma \int_\Omega |\nabla u_t|^2 \mathrm{d}x + \varepsilon \int_\Omega |u_t|^2 \mathrm{d}x + \varepsilon \|u_t\|_{2,\Gamma_1}^2 + r \int_{\Gamma_1} |u_t|^{k(x)} \mathrm{d}\Gamma \\ &-\varepsilon \int_\Omega |\nabla u|^2 \mathrm{d}x + \varepsilon \int_\Omega |u(t)|^{p(x)} \mathrm{d}x - \varepsilon r \tfrac{1}{k_1} \int_{\Gamma_1} \delta^{k(x)} |u|^{k(x)} \mathrm{d}\Gamma \\ &-\varepsilon r \tfrac{k_2-1}{k_1} \int_{\Gamma_1} \delta^{-\frac{k(x)}{k(x)-1}} |u_t|^{k(x)} \mathrm{d}\Gamma. \end{aligned} \tag{4.16}$$

Let us evoke the inequality concerning the continuity of the trace operator

$$\int_{\Gamma_1} |u|^{k(x)} \mathrm{d}\Gamma \leq \max\left(\int_{\Gamma_1} |u|^{k_2} \mathrm{d}\Gamma, \int_{\Gamma_1} |u|^{k_1} \mathrm{d}\Gamma\right) \leq C \|u\|_{H^s(\Omega)},$$

who works for

$$k_1 \geq 1 \text{ and } 0 < s < 1, \quad s \geq \tfrac{n}{2} - \tfrac{n-1}{k_2} \geq \tfrac{n}{2} - \tfrac{n-1}{k_1} > 0, \\ \text{because } k_1 \leq k_2 \leq \tfrac{2n-2}{n-2},$$

and the interpolation and Poincaré's inequalities

$$\|u\|_{H^s(\Omega)} \leq C \|u\|_2^{1-s} \|\nabla u\|_2^s \leq C \|u\|_{p(.)}^{1-s} \|\nabla u\|_2^s, \text{ according to } L^{p(\cdot)}(\Omega) \hookrightarrow L^2(\Omega).$$

So we have the following inequality:

$$\begin{aligned} \textstyle\int_{\Gamma_1} |u|^{k(x)} \mathrm{d}\Gamma &\leq C \|u\|_{p(.)}^{1-s} \|\nabla u\|_2^s \\ &\leq C \max\left(\varrho(u)^{\frac{1-s}{p_1}}, \varrho(u)^{\frac{1-s}{p_2}}\right) \|\nabla u\|_2^s, \text{ (see (2.1))}. \end{aligned}$$

If $s < \frac{2}{k_2}$ and we use the Young's inequality again, we get

$$\begin{aligned} \textstyle\int_{\Gamma_1} |u|^{k(x)} \mathrm{d}\Gamma &\leq C \|u\|_{p(.)}^{1-s} \|\nabla u\|_2^s \\ &\leq C \left[\max\left(\varrho(u)^{\frac{(1-s)k_2\mu}{p_1}}, \varrho(u)^{\frac{(1-s)k_2\mu}{p_2}}\right) + \left(\|\nabla u\|_2^2\right)^{\frac{k_2 s\theta}{2}}\right]. \end{aligned} \tag{4.17}$$

for $1/\mu + 1/\theta = 1$. Here we choose $\theta = \frac{2}{k_2 s}$ to get $\mu = 2/(2 - k_2 s)$. Therefore, the previous inequality becomes

$$\int_{\Gamma_1} |u|^{k(x)} \mathrm{d}\Gamma \leq C \left[\max\left(\varrho(u)^{\frac{2(1-s)k_2}{(2-k_2 s)p_1}}, \varrho(u)^{\frac{2(1-s)k_2}{(2-k_2 s)p_2}}\right) + \|\nabla u\|_2^2\right]. \tag{4.18}$$

Chose s such that

$$0 < s \leq \min\left(\frac{2(p_1 - k_2)}{k_2(p_1 - 2)}, \frac{2(p_2 - k_2)}{k_2(p_2 - 2)}\right),$$

we get

$$\frac{2k_2(1-s)}{(2-k_2 s)p_2} \leq \frac{2k_2(1-s)}{(2-k_2 s)p_1} \leq 1. \tag{4.19}$$

If inequality (4.19) is satisfied, we apply the classical algebraic inequality

$$z^d \leq (z+1) \leq \left(1 + \frac{1}{\omega}\right)(z+\omega), \ \forall z \geq 0, \ 0 < d \leq 1, \ \omega \geq 0,$$

to get the following estimate:

$$\begin{aligned} &\max\left(\varrho(u)^{\frac{2(1-s)k_2}{(2-k_2 s)p_1}}, \varrho(u)^{\frac{2(1-s)k_2}{(2-k_2 s)p_2}}\right) \\ &\quad \leq \left(1 + H(0)^{-1}\right)(\varrho(u) + H(0)) \\ &\quad \leq C(\varrho(u) + H(t)) \quad \forall t \geq 0 \end{aligned} \tag{4.20}$$

Inserting estimate (4.20) into (4.18), we get the following inequality:

$$\int_{\Gamma_1} |u|^{k(x)} \mathrm{d}\Gamma \leq C\left(\varrho(u) + 2H(t) + \|\nabla u\|_2^2\right). \tag{4.21}$$

which eventually gives

$$\begin{aligned}\int_{\Gamma_1} |u|^{k(x)}\,\mathrm{d}\Gamma &\le C\left(\varrho(u) + 2E_1 - \int_\Omega |u_t|^2\,\mathrm{d}x - \|u_t\|^2_{2,\Gamma_1} + \int_\Omega \frac{2}{p(x)}|u|^{p(x)}\,\mathrm{d}x\right)\\ &\le C\left(2E_1 - \int_\Omega |u_t|^2\,\mathrm{d}x - \|u_t\|^2_{2,\Gamma_1} + \left(1+\frac{2}{p_1}\right)\int_\Omega |u|^{p(x)}\,\mathrm{d}x\right).\end{aligned} \tag{4.22}$$

Therefore, by injecting inequality (4.22) into inequality (4.16), we get

$$\begin{aligned}\frac{\mathrm{d}L(t)}{\mathrm{d}t} &\ge \gamma\int_\Omega |\nabla u_t|^2\,\mathrm{d}x + \varepsilon\left(1+\frac{rC}{k_1}\max\left(\delta^{k_2},\delta^{k_1}\right)\right)\int_\Omega |u_t|^2\,\mathrm{d}x\\ &+\varepsilon\left(1+\frac{rC}{k_1}\max\left(\delta^{k_2},\delta^{k_1}\right)\right)\|u_t\|^2_{2,\Gamma_1}\\ &-2\frac{\varepsilon r}{k_1}\max\left(\delta^{k_2},\delta^{k_1}\right)CE_1\\ &-\varepsilon\int_\Omega |\nabla u|^2\,\mathrm{d}x + \varepsilon\left(1-\frac{rC}{k_1}\max\left(\delta^{k_2},\delta^{k_1}\right)\left(1+\frac{2}{p_1}\right)\right)\int_\Omega |u(t)|^{p(x)}\,\mathrm{d}x\\ &+r\left(1-\frac{\varepsilon(k_2-1)}{k_1}\max\left(\delta^{-\frac{k_2}{k_1-1}},\delta^{-\frac{k_1}{k_2-1}}\right)\right)\int_{\Gamma_1}|u_t|^{k(x)}\,\mathrm{d}\Gamma.\end{aligned} \tag{4.23}$$

Of inequality

$$2H(t) = -\left(\int_\Omega |u_t|^2\,\mathrm{d}x + \int_\Omega |\nabla u|^2\,\mathrm{d}x + \|u_t\|^2_{\Gamma_1} - \int_\Omega \frac{2}{p(x)}|u|^{p(x)}\,\mathrm{d}x\right),$$

we have

$$\begin{aligned}-\int_\Omega |\nabla u|^2\,\mathrm{d}x &= 2H(t) + \int_\Omega |u_t|^2\,\mathrm{d}x + \|u_t\|^2_{2,\Gamma_1} - \int_\Omega \frac{2}{p(x)}|u|^{p(x)}\,\mathrm{d}x\\ &\ge 2H(t) - 2E_1 + \int_\Omega |u_t|^2\,\mathrm{d}x + \|u_t\|^2_{2,\Gamma_1} - \frac{2}{p_1}\int_\Omega |u|^{p(x)}\,\mathrm{d}x.\end{aligned} \tag{4.24}$$

So if we inject it into (4.23) we get the following inequality:

$$\begin{aligned}\frac{\mathrm{d}L(t)}{\mathrm{d}t} &\ge \gamma\int_\Omega |\nabla u_t|^2\,\mathrm{d}x + \varepsilon\left(2+\frac{rC}{k_1}\max\left(\delta^{k_2},\delta^{k_1}\right)\right)\int_\Omega |u_t|^2\,\mathrm{d}x\\ &+\varepsilon\left(2+\frac{rC}{k_1}\max\left(\delta^{k_2},\delta^{k_1}\right)\right)\|u_t\|^2_{2,\Gamma_1}\\ &+\varepsilon\left(1-\frac{2}{p_1}-\frac{rC}{k_1}\max\left(\delta^{k_2},\delta^{k_1}\right)\left(1+\frac{2}{p_1}\right)\right)\int_\Omega |u|^{p(x)}\,\mathrm{d}x\\ &+\varepsilon\left(2H(t) - 2\left(1+\frac{r}{k_1}\max\left(\delta^{k_2},\delta^{k_1}\right)C\right)E_1\right)\\ &+r\left(1-\frac{\varepsilon(k_2-1)}{k_1}\max\left(\delta^{-\frac{k_2}{k_1-1}},\delta^{-\frac{k_1}{k_2-1}}\right)\right)\int_{\Gamma_1}|u_t|^{k(x)}\,\mathrm{d}\Gamma.\end{aligned} \tag{4.25}$$

Using the definition of α_2 and E_1 (see Equation (4.2) and Lemma (4.4)), we have

$$\begin{aligned}&-2E_1 - 4\frac{r}{k_1}\max\left(\delta^{k_2},\delta^{k_1}\right)CE_1\\ &= -2E_1\left(B_1^2\alpha_2\right)^{\frac{-p_1}{2}}\left(B_1^2\alpha_2\right)^{\frac{p_1}{2}}\\ &-4\frac{r}{k_1}\max\left(\delta^{k_2},\delta^{k_1}\right)CE_1\left(B_1^2\alpha_2\right)^{\frac{-p_1}{2}}\left(B_1^2\alpha_2\right)^{\frac{p_1}{2}}\\ &\ge \left(-2E_1\left(B_1^2\alpha_2\right)^{\frac{-p_1}{2}} - 4\frac{r}{k_1}\max\left(\delta^{k_2},\delta^{k_1}\right)CE_1\left(B_1^2\alpha_2\right)^{\frac{-p_1}{2}}\right)\int_\Omega |u|^{p(x)}\,\mathrm{d}x.\end{aligned}$$

Finally we get

$$\begin{aligned}\frac{\mathrm{d}L(t)}{\mathrm{d}t} &\geq \gamma \int_\Omega |\nabla u_t|^2 \mathrm{d}x + \varepsilon \left(2 + \frac{rC}{k_1} \max\left(\delta^{k_2}, \delta^{k_1}\right)\right) \int_\Omega |u_t|^2 \mathrm{d}x \\ &+\varepsilon \left(2 + \frac{rC}{k_1} \max\left(\delta^{k_2}, \delta^{k_1}\right)\right) \|u_t\|^2_{2,\Gamma_1} \\ &+\varepsilon \left(\begin{array}{c} 1 - \frac{2}{p_1} - 2E_1 \left(B_1^2 \alpha_2\right)^{\frac{-p_1}{2}} \\ -\frac{rC}{k_1} \max\left(\delta^{k_2}, \delta^{k_1}\right) \left[\left(1 + \frac{2}{p_1}\right) + 4E_1 \left(B_1^2 \alpha_2\right)^{\frac{-p_1}{2}}\right] \end{array}\right) \int_\Omega |u|^{p(x)} \mathrm{d}x \\ &+2\varepsilon \left(H(t) + \frac{r}{k_1} \max\left(\delta^{k_2}, \delta^{k_1}\right) C E_1\right) \\ &+r \left(1 - \frac{\varepsilon (k_2 - 1)}{k_1} \max\left(\delta^{-\frac{k_2}{k_1-1}}, \delta^{-\frac{k_1}{k_2-1}}\right)\right) \int_{\Gamma_1} |u_t|^{k(x)} \mathrm{d}\Gamma,\end{aligned} \tag{4.26}$$

because

$$1 - \frac{2}{p_1} - 2E_1 \left(B_1^2 \alpha_2\right)^{\frac{-p_1}{2}} > 0 \text{ since } \alpha_2 > B_1^{-\frac{2p_1}{p_1-2}},$$

we can now choose δ small enough such that

$$\left(\begin{array}{c} 1 - \frac{2}{p_1} - 2E_1 \left(B_1^2 \alpha_2\right)^{\frac{-p_1}{2}} \\ -\frac{rC}{k_1} \max\left(\delta^{k_2}, \delta^{k_1}\right) \left[\left(1 + \frac{2}{p_1}\right) + 4E_1 \left(B_1^2 \alpha_2\right)^{\frac{-p_1}{2}}\right] \end{array}\right) > 0.$$

Once δ is fixed, let's select ε small enough

$$\left(1 - \frac{\varepsilon (k_2 - 1)}{k_1} \max\left(\delta^{-\frac{k_2}{k_1-1}}, \delta^{-\frac{k_1}{k_2-1}}\right)\right) > 0 \text{ and } L(0) > 0.$$

Hence the inequality (4.26) becomes

$$\frac{\mathrm{d}L(t)}{\mathrm{d}t} \geq \varepsilon \eta \left[H(t) + \int_\Omega |u_t|^2 \mathrm{d}x + \|u_t\|^2_{2,\Gamma_1} + \int_\Omega |u|^{p(x)} \mathrm{d}x + E_1\right] \tag{4.27}$$

for some $\eta > 0$.

Next it is clear that by Young's inequality and Poincaré's inequality we obtain

$$L(t) \leq \lambda \left[H(t) + \int_\Omega |u_t|^2 \mathrm{d}x + \|u_t\|^2_{2,\Gamma_1} + \int_\Omega |\nabla u|^2 \mathrm{d}x\right] \text{ for some } \lambda > 0. \tag{4.28}$$

From (4.12), we have

$$\int_\Omega |\nabla u|^2 \mathrm{d}x \leq 2E_1 + \frac{2}{p_1} \int_\Omega |u(x,t)|^{p(x)} \mathrm{d}x, \quad t \geq 0.$$

So the inequality (4.28) becomes

$$L(t) \leq \zeta \left[H(t) + \int_\Omega |u_t|^2 \mathrm{d}x + \|u_t\|^2_{2,\Gamma_1} + \int_\Omega |u|^{p(x)} \mathrm{d}x + E_1\right] \text{ for some } \zeta > 0. \tag{4.29}$$

From the two inequalities (4.27) and (4.29), we finally get the differential inequality

$$\frac{\mathrm{d}L(t)}{\mathrm{d}t} \geq \mu L(t) \text{ for some } \mu > 0. \tag{4.30}$$

Integrating the previous differential inequality (4.30) on $(0, t)$ gives the following estimate for the function L:

$$L(t) \geq L(0)\, e^{\mu t}. \tag{4.31}$$

On the other hand, from the definition of the function L (and for small values of the parameter ε) follows

$$\begin{aligned} L(0)\, e^{\mu t} &\leq L(t) \leq \frac{1}{p_1} \int_\Omega |u|^{p(x)}\, \mathrm{d}x \\ &\leq \frac{1}{p_1} \max\left(\int_\Omega |u|^{p_2}\, \mathrm{d}x, \int_\Omega |u|^{p_1}\, \mathrm{d}x \right). \end{aligned} \tag{4.32}$$

From the two inequalities (4.31) and (4.32) we derive the exponential growth of the solution in the L^{p_2} and L^{p_1} norms. □

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