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On the class of analytic functions defined by Robertson associated with nephroid domain

Kavitha Sivasubramanian

Abstract. The primary focus of this article is to explore a novel subclass, denoted as $\mathscr{G}_{\mathcal{N}}$, of analytic functions. These functions exhibit starlike properties concerning a boundary point within a nephroid domain. The author obtains representation theorems, establishes growth and distortion theorems, and investigates various implications related to differential subordination. In addition to the investigation of coefficient estimates, the study also explores specific consequences of differential subordination.

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1. Introduction

Let $\mathscr{H}$ be the class of all holomorphic functions in the open unit disc $\mathscr{D} := \{z \in \mathbb{C} : |z| < 1\}$. Further, let $\mathscr{A}$ represent the subclass of $\mathscr{H}$ entailing of functions h with the normalization $h(0) = h'(0) - 1 = 0$. Hence, the class of all functions $h \in \mathscr{A}$ will be of the form

$$h(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad z \in \mathscr{D}.$$

By $\mathscr{S}$, we mean the subclass of $\mathscr{A}$ comprising of univalent functions. A function $f \in \mathscr{H}$ is subordinate to another function $g \in \mathscr{H}$ written as $f \prec g$ if there exists a function $\omega \in \mathscr{H}$ satisfying $\omega(0) = 0$, $\omega(\mathscr{D}) \subset \mathscr{D}$ and such that $f(z) = g(\omega(z))$ for every $z \in \mathscr{D}$. In precise, if g is univalent, then $f \prec g$ if and only if $f(0) = g(0)$ and $f(\mathscr{D}) \subset g(\mathscr{D})$.

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Two well-established subclasses of $\mathscr{A}$ are the starlike functions and convex functions of order γ $(0 \le \gamma < 1)$, which were introduced by Robertson [20]. These classes are defined analytically as follows:

The starlike functions of order γ, denoted as $\mathscr{S}^*(\gamma)$, consist of functions in $\mathscr{A}$ for which $\Re\left(\frac{zh'(z)}{h(z)}\right) > \gamma$ for all $z \in \mathscr{D}$. The convex functions of order γ, denoted as $\mathscr{C}(\gamma)$, comprise functions in $\mathscr{A}$ satisfying $\Re\left(1 + \frac{zh''(z)}{h'(z)}\right) > \gamma$ for all $z \in \mathscr{D}$.

It is well-known that $\mathscr{S}^*(\gamma) \subset \mathscr{S}$ and $\mathscr{C}(\gamma) \subset \mathscr{S}$. Additionally, based on Alexander's relation, if $h \in \mathscr{C}(\gamma)$, then $zh'(z)$ belongs to $\mathscr{S}^*(\gamma)$ for each $0 \le \gamma < 1$. For $\gamma = 0$, $\mathscr{S}^*$ corresponds to the normalized starlike univalent functions, and $\mathscr{C}$ represents the normalized convex univalent functions.

A function $h \in \mathscr{H}$ is said to be close-to-convex if and only if there exists a function $\psi \in \mathscr{C}$ and $\beta \in (-\pi/2, \pi/2)$ such that

$$\Re\left(\frac{\mathrm{e}^{\mathrm{i}\beta} h'(z)}{\psi'(z)}\right) > 0, \quad z \in \mathscr{D}.$$

The class of close-to-convex functions was defined in [11]. Further, it is also known that the class of close-to-convex functions generally are normalized.Let $\mathscr{P}$ denote the class of functions p holomorphic in $\mathscr{D}$, satisfying $p(0) = 1$ and $\Re\,(p(z)) > 0$ for $z \in \mathscr{D}$. This class is referred to as the class of functions with a positive real part, commonly known as Class of Caratheodory functions.

A significant development emerged in [21], where a novel class $\mathscr{G}$ of functions $G(z)$ was introduced. These functions are analytic within $|z| < 1$, normalized such that $G(0) = 1$, $G(1) = \lim_{r\to 1^-} G(r) = 0$, and they satisfy the condition that $\Re(e^{i\delta}G(z)) > 0$ for $z \in \mathscr{D}$. Additionally, $G(z)$ maps $\mathscr{D}$ univalently to a domain that is starlike with respect to $G(1)$. Notably, the constant function 1 is included in the class $\mathscr{G}$.

A significant conjecture proposed by Robertson [21] is that the class $\mathscr{G}$ of functions f of the form:

$$f(z) = 1 + \sum_{n=1}^{\infty} d_n z^n, \tag{1.1}$$

holomorphic and nonvanishing in $\mathscr{D}$ and such that

$$\Re\left\{\frac{2zf'(z)}{f(z)} + \frac{1+z}{1-z}\right\} > 0, z \in \mathscr{D} \tag{1.2}$$

coincides with $\mathscr{G}^*$. The above hypothesis was confirmed by Lyzzaik [17] in 1984. Robertson [21] also proved that if the function $f \in \mathscr{G}$ and $g \neq 1$ then f is close-to-convex and univalent in $\mathscr{D}$. It is worth to be mentioned here that the analytic condition (1.2) was known to Styer [24] earlier. In [10], a class closely related to $\mathscr{G}$ denoted by $\mathscr{G}(M)$, $M > 1$, of functions g of the form (1.1) holomorphic and nonvanishing in $\mathscr{D}$ was introduced and such that

$$\Re\left\{\frac{2zf'(z)}{g(z)} + z\frac{P_M'(z)}{P_M(z)}\right\} > 0, z \in \mathscr{D},$$

where $P_M(z)$ denotes the Pick function. The class

$$\mathscr{G}(1)=\left\{f \text{ of the form (1.1)} : f(z)\neq 0 \text{ and } \Re\left\{2z\frac{f'(z)}{f(z)}+1\right\}>0, z\in\mathscr{D}\right\}$$

was also considered in [10]. In an another investigation, Obradovic and Owa [19] investigated the class $\mathscr{G}(\gamma)$, $0\leq\gamma<1$, of functions g of the form (1.1) holomorphic in the disc $\mathscr{D}$, $g(z)\neq 0$ for $z\in\mathscr{D}$ and satisfying the condition

$$\Re\left\{\frac{zf'(z)}{f(z)}+(1-\gamma)\frac{1+z}{1-z}\right\}>0,\ z\in\mathscr{D}.$$

Todorov [25] established a connection between the class $\mathscr{G}$ and a functional expression $\frac{f(z)}{1-z}$, resulting in a well-structured formula and precise coefficient estimates. Silverman and Silvia [22] offered a comprehensive exploration of the class of univalent functions within the domain $\mathscr{D}$ whose images take on a star-like configuration concerning a boundary point. Subsequently, this category of functions exhibiting star-like behavior with respect to boundary points has garnered significant interest among geometric function theorists and researchers from diverse backgrounds. Among the works in this direction, Abdullah et al. [1] obtained certain properties of functions belonging to $\mathscr{G}$ and derived a set of inequalities pertaining to functional coefficients. Distortion results associated with star-like functions concerning boundary points were further examined, and were presented in both [3] and [6]. Moreover, dynamic characterizations of functions demonstrating star-like characterizations concerning boundary points can be found in [8].

Lecko [13] introduced an alternative representation of functions manifesting star-like qualities concerning a boundary point. Additionally, Lecko and Lyzzaik [14] contributed diverse characterizations of the class $\mathscr{G}$. Furthermore, Aharonov et al. [2] provided a definition for spiral-shaped domains concerning a boundary point, outlined as follows:

Let $\mathscr{G}_\mu$ denote the class of functions $f\in H(\mathscr{D})$, and non-vanishing in $\mathscr{D}$ with $f(0)=1$, and for $\mu\in\mathbb{C}$, $\left|\frac{\mu}{\pi}-1\right|\leq 1$ satisfying

$$\Re\left\{\frac{2\pi zf'(z)}{\mu f(z)}+\frac{1+z}{1-z}\right\}>0, z\in\mathscr{D}.$$

In the work by Elin [8], a set of fundamental properties and several equivalent descriptions of the class $\mathscr{G}\mu$ are formulated (also see [7]). When μ is chosen to be π, the class $\mathscr{G}\mu$ aligns with the class initially introduced by Robertson [21], who sparked interest in this class and related categories. It's worth noting that functions within $\mathscr{G}_\mu$ are either close-to-convex or simply the constant function 1.

In recent times, the investigation of star-like functions concerning boundary points has attracted attention from researchers such as Cho et al. [4], Dhurai et al. [5], Lecko et al. [15, 16, 9], and Sivasubramanian [23] (also see [12]). The purpose of this paper is to introduce and investigate a new class of the aforesaid type involving nephroid domains with respect to a boundary point.

Definition 1.1. Let $\mathscr{G}_{\mathscr{N}}$, denote the class of functions f of the form (1.1) holomorphic and nonvanishing in disc $\mathscr{D}$ and such that

$$\Re\left\{\frac{2zf'(z)}{f(z)}+1+z-\frac{z^3}{3}\right\}>0, z\in\mathscr{D}, \tag{1.3}$$

which can be rewritten as

$$\Re\left\{\frac{2zf'(z)}{f(z)}+Q_{\mathscr{N}}(z)\right\}>0, z\in\mathscr{D}, \tag{1.4}$$

where

$$Q_{\mathscr{N}}(z)=1+z-\frac{z^3}{3}, z\in\mathscr{D}. \tag{1.5}$$

It is to be observed that the function $Q_{\mathscr{N}}$ of the form (1.5) maps $\mathscr{D}$ onto the region bounded by the nephroid $\left((u-1)^2+v^2-\frac{4}{9}\right)^3-\frac{4v^2}{3}=0$ which is symmetric about the real axis and lies completely inside the right-half plane $u>0$. The nephroid domain was introduced and studied by Wani and Swaminathan [26]. Let us first construct few examples for the new class of functions to show that the class is non empty.

Examples. The functions

$$f_0(z)=\exp\left(\frac{1}{2}\left(-z+\frac{z^3}{9}\right)\right),\ z\in\mathscr{D}, \tag{1.6}$$

and

$$f_1(z)=\frac{\exp\left(\frac{1}{2}\left(-z+\frac{z^3}{9}\right)\right)}{1-z},\ z\in\mathscr{D} \tag{1.7}$$

belong to the class $\mathscr{G}_{\mathscr{N}}$.

To see this, one may compute

$$\Re\left\{\frac{2zf_0'(z)}{f_0(z)}+1+z-\frac{z^3}{3}\right\}=1>0, z\in\mathscr{D},$$

and

$$\Re\left\{\frac{2zf_1'(z)}{f_1(z)}+1+z-\frac{z^3}{3}\right\}=1>0, z\in\mathscr{D},$$

A straight forward computations will show that both the functions $f_0(z)$ and $f_1(z)$ belong to the class $\mathscr{G}_{\mathscr{N}}$. However, it is of interest to observe that although the functions f_0 and f_1 belong to $\mathcal{G}_N$, the functions f_0 and f_1 need not be necessarily univalent and hence $\mathcal{G}_N \not\subseteq \mathcal{G}$.

2. Main results

We start this section with the following representation theorem

Theorem 2.1. *Let f be a holomorphic function in $\mathscr{D}$ such that $f(0)=1$. Then $f \in \mathscr{G}_{\mathscr{N}}$ if and only if there exists a function $h \in \mathscr{S}^*$ such that*

$$f(z) = \sqrt{\frac{h(z)}{z}} \exp\left(\frac{1}{2}\left(-z + \frac{z^3}{9}\right)\right), z \in \mathscr{D}, \tag{2.1}$$

Proof. Let F be a function satisfying the relation

$$\frac{zF'(z)}{F(z)} = 1 + z - \frac{z^3}{3}, z \in \mathscr{D}. \tag{2.2}$$

Then $F \in \mathscr{S}^*_{\mathscr{N}}$, where

$$\mathscr{S}^*_{\mathscr{N}} = \left\{ F : F(z) = z + \sum_{n=2}^{\infty} a_n z^n, z \in \mathscr{D} \text{ and } \frac{zF'(z)}{F(z)} \prec Q_{\mathscr{N}}(z) \right\}.$$

From (2.2), one may easily see after a simple calculation that

$$F(z) = z \cdot \exp\left(\int_0^z \frac{Q_{\mathscr{N}}(\zeta) - 1}{\zeta} d\zeta\right), z \in \mathscr{D},. \tag{2.3}$$

and therefore

$$F(z) = z \cdot \exp\left(z - \frac{z^3}{9}\right), z \in \mathscr{D}. \tag{2.4}$$

From (1.4) and (1.5), it can be easily seen that for some function $f \in \mathscr{G}_{\mathscr{N}}$ there exists a starlike function h of the class $\mathscr{S}^*$ such that $(f(z))^2 F(z) = h(z), z \in \mathscr{D}$ and conversely. □

Remark 2.2. Let us consider the function f_3, $f_3(0) = 1$, satisfying the equation

$$\frac{2zf_3'(z)}{f_3(z)} + 1 + z - \frac{z^3}{3} = \frac{1+z^2}{1-z^2}, z \in \mathscr{D}.$$

In view of (1.3) and (1.5) it is obvious that $f_3 \in \mathscr{G}_{\mathscr{N}}$.

Theorem 2.3. *Let f be a holomorphic function in $\mathscr{D}$ such that $f(0)=1$. Then $f \in \mathscr{G}_{\mathscr{N}}$ if and only if there exists a function $h \in \mathscr{S}^*(1/2)$ such that*

$$f(z) = \frac{h(z)}{z} \exp\left(\frac{1}{2}\left(-z + \frac{z^3}{9}\right)\right), z \in \mathscr{D}. \tag{2.5}$$

Proof. It is a known that,

$$h \in \mathscr{S}^*(1/2) \Leftrightarrow h = \frac{f^2}{I}, I(z) \equiv z.$$

An application of (2.1) essentially completes the proof of Theorem 2.3. □

Remark 2.4. It follows immediately from the Herglotz representation that for $\mathscr{S}^*\left(\frac{1}{2}\right)$ that $g \in \mathscr{G}_{\mathscr{N}}$ if and only if

$$f(z) = \exp\left(-z + \frac{z^3}{6}\right)\left(\frac{\mu}{\pi}\int_{-\pi}^{\pi} \log\left(\frac{1}{1 - ze^{-it}}\right) d\mu(t)\right), \ z \in \mathscr{D}. \tag{2.6}$$

where $\mu(t)$ is a probability measure on $[-\pi, \pi]$.

From Theorem 2.1 and from the known estimates of the respective functionals in the class $\mathscr{S}^*$ we have the following theorem.

Theorem 2.5. *If $f \in \mathscr{G}_{\mathscr{N}}$, then the following sharp estimate*

$$\frac{1}{1+|z|}\exp\left(-\frac{1}{2}\Re\left(z - \frac{z^3}{9}\right)\right) \le |f(z)| \le \frac{1}{1-|z|}\exp\left(-\frac{1}{2}\Re\left(z - \frac{z^3}{9}\right)\right) \tag{2.7}$$

hold. The extremal function for the upper estimate (2.7) is the function f_ε^* of the form

$$f_\varepsilon^*(z) = \exp\left(-\frac{1}{2}\Re\left(z - \frac{z^3}{9}\right)\right)\sqrt{\frac{k_\varepsilon(z)}{z}},$$

where $\varepsilon = e^{-i\varphi}$, while for the lower estimate is the function g_ε^* for $\varepsilon = -e^{-i\varphi}$ with

$$k(z) = \frac{z}{(1-z)^2}.$$

Theorem 2.6. *If $f(z) = 1 + \sum_{n=1}^{\infty} d_n z^n \in \mathscr{G}_{\mathscr{N}}$, then the coefficients d_n satisfy the following sharp coefficient inequalities*

$$|d_1| \le \frac{3}{2} \tag{2.8}$$

$$|2d_1 + 1| \le 2 \tag{2.9}$$

$$\left|2d_2 - d_1^2\right| \le 1 \tag{2.10}$$

$$\left|6(3d_3 - 3d_1d_2 + d_1^3) - 1\right| \le 6 \tag{2.11}$$

$$\left|4d_2 - 2d_1^2(1+2\gamma) - 4\gamma d_1 - \gamma\right| \le \begin{cases} 2 - \gamma\,|2d_1+1|^2 & \left(\gamma \le \frac{1}{2}\right) \\ 2 - (1-\gamma)\,|2d_1+1|^2 & \left(\gamma \ge \frac{1}{2}\right). \end{cases} \tag{2.12}$$

and finally

$$\left|6\left(3d_3 - 7d_1d_2 - 2d_2 + 3d_1^3 + d_1^2\right) - 1\right| \le 6. \tag{2.13}$$

Proof. Let $d_0 = 1$ and $p(z) = \dfrac{2zg'(z)}{g(z)} + 1 + z - \dfrac{z^3}{3}$ and

$$p(z) = 1 + p_1 z + p_2 z^2 + \cdots.$$

On expanding the right hand side of the above function and comparing with $p(z)$ we get,

$$1+p_1z+p_2z^2+\cdots = 1+(2d_1+1)\,z+\left(2(2d_2-d_1^2)\right)z^2+\left(2(3d_3-3d_1d_2+d_1^3)-\frac{1}{3}\right)z^3+\cdots.$$

Hence,

$$2d_1 + 1 = p_1$$
$$2(2d_2 - d_1^2) = p_2$$

$$2(3d_3 - 3d_1 d_2 + d_1^3) - \frac{1}{3} = p_3$$

It is a known fact that if $p(z) = 1 + p_1 z + p_2 z^2 + \cdots \in \mathscr{P}$, then $|p_i| \le 2, i = 1, 2 \cdots$. By virtue of this inequality one may easily get (2.8),(2.9), (2.10) and (2.11). Inequality (2.12) follows from the fact that

$$|p_2 - \gamma p_1^2| \le \begin{cases} 2 - \gamma |p_1|^2 & \left(\gamma \le \frac{1}{2}\right) \\ 2 - (1-\gamma)|p_1|^2 & \left(\gamma \ge \frac{1}{2}\right). \end{cases} \tag{2.14}$$

By applying a less unknown familiar inequality $|p_3 - p_1 p_2| \le 2$ and performing a computation yields the inequality (2.13). □

It is known that for each function $h \in \mathscr{S}^*$ the functions

$$z \to \frac{1}{\rho} h(\rho z), z \to e^{i\varphi} h(e^{-i\varphi} z), 0 < \rho < 1, \varphi \in \mathbb{R}, z \in \mathscr{D}, \tag{2.15}$$

also belong to $\mathscr{S}^*$. From Theorem 2.1 and estimation (2.9) we obtain:

Theorem 2.7. *The region of values of the coefficient d_1, i.e. $\{d_1 : g \in \mathscr{G}_{\mathscr{N}}, g(z) = 1 + d_1 z + \cdots\}$ has the form*

$$\left\{ w \in \mathbb{C} : \left| w + \frac{1}{2} \right| \le 1 \right\}.$$

In this section, we establish specific differential subordination findings related to the class $\mathscr{G}_N$.

To prove differential subordination results, we recall the following lemma (see [18, Theorem 3.4h, p. 132]).

Lemma 2.8. *Let q be univalent in $\mathscr{D}$, θ and φ be holomorphic in a domain D containing $q(\mathscr{D})$ with $\varphi(w) \ne 0$ when $w \in q(\mathscr{D})$. Let $Q(z) := zq'(z)\varphi(q(z))$ and $h(z) := \theta(q(z)) + Q(z)$ for $z \in \mathscr{D}$. Suppose that either*

(i) *Q is starlike univalent in $\mathscr{D}$, or*
(ii) *h is convex univalent in $\mathscr{D}$.*

Assume also that

(iii)

$$\Re \frac{zh'(z)}{Q(z)} > 0, \quad z \in \mathscr{D}.$$

If $p \in \mathscr{H}$ with $p(0) = q(0)$, $p(\mathscr{D}) \subset D$, and

$$\theta(p(z)) + zp'(z)\varphi(p(z)) \prec \theta(q(z)) + zq'(z)\varphi(q(z)), \quad z \in \mathscr{D},$$

then $p \prec q$ and q is the best dominant.

Theorem 2.9. *Let $f \in \mathscr{H}$ and $f(0) = 1$. If f satisfies the subordination condition*

$$\frac{2zf'(z)}{f(z)} + 1 + z - \frac{z^3}{3} \prec \frac{1+z}{1-z}, \quad z \in \mathscr{D}, \tag{2.16}$$

then

$$\frac{(f(z))^2}{\exp\left(-z+\frac{z^3}{9}\right)} \prec \frac{1}{(1-z)^2}, \quad z \in \mathscr{D}. \tag{2.17}$$

That is, if $f \in \mathscr{G}_{\mathscr{N}}$, *then* $\dfrac{(f(z))^2}{\exp\left(-z+\frac{z^3}{9}\right)} \prec \dfrac{1}{(1-z)^2}, \quad z \in \mathscr{D}.$

Proof. Let us define a function $p(z) = \dfrac{(f(z))^2}{\exp\left(-z+\frac{z^3}{9}\right)}$. Let $q(z) = \dfrac{1}{(1-z)^2}, z \in \mathscr{D}$. Subsequently, one can readily notice that $p(0) = q(0) = 1$, $p(z) \neq 0$ for $z \in \mathscr{D}$, and p is holomorphic. Further,

$$1 + \frac{zp'(z)}{p(z)} = \frac{2zf'(z)}{f(z)} + 1 + z - \frac{z^3}{3}, \quad z \in \mathscr{D}.$$

Let $f \in \mathscr{H}$ with $f(0) = 1$ and $f(z)$ be nonzero for $z \in \mathscr{D}$ satisfying (2.16). Let a function p be defined as in (2.17). Let $D := \mathbb{C} \setminus \{0\}$. Let $\theta(w) := 1,\ w \in \mathbb{C}$, and $\varphi(w) := 1/w,\ w \in D$. Note that $q(\mathscr{D}) \subset D$ and θ and φ are holomorphic in D. Thus

$$Q(z) := zq'(z)\varphi(q(z)) = \frac{zq'(z)}{q(z)} = \frac{2z}{1-z}, \quad z \in \mathscr{D}, \tag{2.18}$$

is well defined and holomorphic. Clearly, Q is a univalent. Additionally, a straightforward calculation will demonstrate that Q is a starlike function as well. Hence for a function $h(z) := \theta(q(z)) + Q(z) = \dfrac{1+z}{1-z}$, $z \in \mathscr{D}$, we obtain

$$\Re\frac{zh'(z)}{Q(z)} = \Re\frac{zQ'(z)}{Q(z)} = \frac{1}{1-z} > 0, \quad z \in \mathscr{D}.$$

Therefore, for any given function p belonging to $\mathscr{H}$ with $p(0) = q(0) = 1$ such that $p(\mathscr{D}) \subset D$, i.e., for p non-vanishing in $\mathscr{D}$, by applying Lemma 2.8 we infer that from the subordination

$$1 + \frac{zp'(z)}{p(z)} \prec 1 + \frac{zq'(z)}{q(z)} = \frac{1+z}{1-z}, \quad z \in \mathscr{D}, \tag{2.19}$$

implies the subordination $p \prec \dfrac{1}{(1-z)^2}$ is true.

□

Theorem 2.10. *Let* $f \in \mathscr{H}$ *with* $f(0) = 1$. *If* f *satisfies*

$$\frac{2zf'(z)}{f(z)} + 1 + z - \frac{z^3}{3} \prec \frac{1+z}{1-z}, \quad z \in \mathscr{D}, \tag{2.20}$$

then

$$p(z) := z\left(\frac{f(z)}{1-z}\right)^2 \left(\int_0^z \left(\frac{f(\zeta)}{1-\zeta}\right)^2 d\zeta\right)^{-1} \prec \frac{1}{(1-z)^2}, \quad z \in \mathscr{D}. \tag{2.21}$$

That is $f \in \mathscr{G}_N$ *then* $z\left(\dfrac{f(z)}{1-z}\right)^2 \left(\displaystyle\int_0^z \left(\frac{f(\zeta)}{1-\zeta}\right)^2 d\zeta\right)^{-1} \prec \dfrac{1}{(1-z)^2}.$

Proof. Let $D := \mathbb{C} \setminus \{0\}$. Let $\phi(w) := w,\ w \in \mathbb{C}$, and $\psi(w) := 1/w,\ w \in D$. Note that $q(\mathscr{D}) \subset D$ and ϕ and ψ are holomorphic in D. Thus the function Q defined by (2.18), i.e., the identity function, is univalent starlike. Hence for a function

$$h(z) := \theta(q(z)) + Q(z) = q(z) + Q(z),\ z \in \mathscr{D},$$

we obtain

$$\Re\frac{zh'(z)}{Q(z)} = \Re\frac{zq'(z)}{Q(z)} + \Re\frac{zQ'(z)}{Q(z)} = \Re q(z) + \Re\frac{zQ'(z)}{Q(z)} > 0, \quad z \in \mathscr{D}.$$

Thus for any function $p \in \mathscr{H}$ with $p(0) = q(0) = 1$ such that $p(\mathscr{D}) \subset D$, i.e., $p(z) \neq 0$ for $z \in \mathscr{D}$, by applying Lemma 2.8 we can conclude that from the subordination

$$p(z) + \frac{zp'(z)}{p(z)} \prec q(z) + \frac{zq'(z)}{q(z)} = \frac{1+z}{1-z}, \quad z \in \mathscr{D}, \tag{2.22}$$

it follows the subordination $p \prec \dfrac{1}{(1-z)^2}$.

Let now take $f \in \mathscr{H}$ with $f(0) = 1$ and $f(z) \neq 0$ for $z \in \mathscr{D}$ satisfying (2.16). Define a function p as in (2.21). We see that

$$\begin{aligned} p(0) &= \lim_{z\to 0} z \left(\frac{f(z)}{1-z}\right)^2 \left(\int_0^z \left(\frac{f(\zeta)}{1-\zeta}\right)^2 d\zeta\right)^{-1} \\ &= (f(0))^2 \lim_{z\to 0} z \left(\int_0^z \left(\frac{f(\zeta)}{1-\zeta}\right)^2 d\zeta\right)^{-1} = 1 = q(0), \end{aligned}$$

$p(z) \neq 0$ for $z \in \mathscr{D}$ and p is holomorphic. Since

$$p(z) + \frac{zp'(z)}{p(z)} = \frac{2zf'(z)}{f(z)} + \frac{1+z}{1-z}, \quad z \in \mathscr{D},$$

from (2.22), (2.20) follows which completes the proof. □

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Kavitha Sivasubramanian
Department of Mathematics,
SDNB Vaishnav College for Women,
Chennai-600064, Tamil Nadu, India
e-mail: kavi080716@gmail.com