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Strongly nonlinear periodic parabolic equation in Orlicz spaces

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Abstract. In this paper, we prove the existence of a weak solution to the following nonlinear periodic parabolic equations in Orlicz-spaces:

$$\frac{\partial u}{\partial t} - div(a(x,t,\nabla u)) = f(x,t)$$

where $-div(a(x,t,\nabla u))$ is a Leray-Lions operator defined on a subset of $W_0^{1,x}L_M(Q)$. The Δ_2-condition is not assumed and the data f belongs to $W^{-1,x}E_{\overline{M}}(Q)$.
The Galerkin method and the fixed point argument are employed in the proof.

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1. Introduction

Let Ω be a bounded subset of $\mathbb{R}^N$, and let Q be the cylinder $\Omega \times (0,T)$ with some given $T > 0$. In this paper we deal with the following periodic parabolic boundary value problem:

$$\begin{cases} \frac{\partial u}{\partial t} - div(a(x,t,\nabla u)) = f(x,t) \ \text{ in } Q \ , \\ u(x,t) = 0 \ \text{ on } \partial\Omega \times (0,T) \ , \\ u(x,0) = u(x,T) \ \text{ in } \Omega \ , \end{cases} \tag{1.1}$$

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where A is a second-order operator in divergence form

$$A(u) = -div(a(x,t,\nabla u)),$$

with the coefficient a satisfying Leray-Lions conditions related to some N-function.
The study of nonlinear partial differential equations in Orlicz-spaces is motivated by numerous phenomena of physics, namely the problems related to non-Newtonian fluids of strongly inhomogeneous behavior with a high ability of increasing their viscosity under a different stimulus, like the shear rate, magnetic or electric field (see for examples [1], [10], [14], [15], [16] and [21]).
Consider first the case where a have polynomial growth with respect to u and ∇u. Therefore A is a bounded operator from $L^p(0,T,W^{1,p}(\Omega))$, $1 < p < \infty$, into its dual. In this setting, Brézis and Browder in cite16 proved the existence of problem (1) when $p > 2$ and the periodic condition is replaced by the initial one, and by Landes and Mustonen when $1 < p < 2$ [19].
Specifically, when we have the periodicity condition Boldrini and Crema in [4] studied the following problem:

$$\begin{cases} \frac{\partial u}{\partial t} - \Delta_p u = m(t)g(u) + h(x,t) \text{ in } Q_T; \\ u(x,t) = 0 \text{ on } \partial\Omega \times (0,T); \\ u(x,T) = u(x,0) \text{ in } \Omega; \end{cases} \tag{1.2}$$

g is a continuous function such that $|g(v)| \leq a(|v|s+1)$, where s and a are positive constants. The existence of a solution to this problem is established under the condition $0 \leq s < p-1$, and for $s = p-1$ by using Schauder's fixed point theorem. Related topics can be found in [7], [8], [9]. However, when attempting to relax the restriction on a, we replace the space $L^p(0,T,W_0^{1,p})$ with an inhomogeneous Orlicz-Sobolev space $W_0^{1,x}L_M(Q)$, constructed from an Orlicz space L_M instead of L^p, where the N-function M is related to the actual growth of a. Several studies have explored this setting, considering $u(x,0) = u_0$ and a depending on u and ∇u, see for instance, the works of Donaldson in [6] and Robert in [20], who proved the existence of a solution for a nonlinear parabolic problem under the Δ_2 condition, $u^2 \leq cM(ku)$, with c and k are positive constants, and A is monotone. Additionally, in cases where the Δ_2 condition is not assumed and under various assumptions, other authors have demonstrated the existence of solutions to diverse parabolic problems (see [2], [14], [17], [19]).
The objective of this paper is to establish the existence of a solution to problem (1.1) when f belongs to $W^{-1,x}E_{\overline{M}}(Q)$, without assuming the Δ_2 condition. Moreover, we consider the periodicity condition instead of the initial one, which necessitates demonstrating the existence of the approximate problem once more. To achieve this, we assume that $u^2 \leq cM(ku)$ with c and k are positive constants.
We employ the Galerkin method due to Landes and Mustonen, along with the fixed point argument due to Schauder.
The paper is structured as follows: In Section 2, we provide a review of some preliminary concepts concerning Orlicz-Sobolev spaces, along with various inequalities and compactness results. Section 3 is dedicated to stating the assumptions and presenting the main result. In the fourth section, we prove the existence theorem. In the appendix we prove the existence of a solution to the approximate problem.

2. Preliminaries

2.1. Orlicz-Sobolev Spaces-Notations and Properties

- let $M : \mathbb{R}^+ \to \mathbb{R}^+$ be an N-function, i.e continuous, convex, with $M(t) > 0$ for $t > 0, M(t)/t \to 0$ as $t \to 0$ and $M(t)/t \to \infty$ as $t \to \infty$.
  Equivalently, M admits the representation: $M(t) = \int_0^t m(\tau)d\tau$ where $m : \mathbb{R}^+ \to \mathbb{R}^+$ is non-decreasing, right continuous, with $m(0) = 0,\ m(t) > 0$ for $t > 0$ and $m(t) \to \infty$ as $t \to \infty$.
  The N-function $\overline{M}$ conjugate to M is defined by $\overline{M}(t) = \int_0^t \overline{m}(\tau)d\tau$ where $\overline{m} : \mathbb{R}^+ \to \mathbb{R}^+$ is given by $m(t) = \sup\{s : m(s) \le t\}$.
  The N-function M is said to satisfy a Δ_2 condition if, for some $k > 0$:
  $$M(2t) \le kM(t) \quad \forall t \ge 0$$
  When this inequality holds only for $t \ge t_0 > 0$, M is said to satisfy the Δ_2-condition near infinity.
- Let Ω be an open subset of $\mathbb{R}^N$. The Orlicz class $\mathcal{L}_M(\Omega)$ (resp. the Orlicz space $L_M(\Omega)$) is defined as the set of (equivalence classes of) real-valued measurable functions u on Ω such that $\int_\Omega M(u(x))dx < +\infty$ (resp. $\int_\Omega M(u(x)/\lambda)dx < +\infty$ for some $\lambda > 0$).
  $L_M(\Omega)$ is a Banach space under the norm:
  $$\|u\|_{M,\Omega} = \inf\left\{\lambda > 0 : \int_\Omega M\left(\frac{u(x)}{\lambda}\right)dx \le 1\right\}$$
  and $\mathcal{L}_M(\Omega)$ is a convex subset of $L_M(\Omega)$. The closure in $L_M(\Omega)$ of the set of bounded measurable functions with compact support in $\overline{\Omega}$ is denoted by $E_M(\Omega)$.
  The equality $E_M(\Omega) = L_M(\Omega)$ holds if and only if M satisfies the Δ_2 condition, for all t or for t large according to whether Ω has infinite measure or not.
  The dual of $E_M(\Omega)$ can be identified with $L_M(\Omega)$ by means of the pairing $\int_\Omega u(x)v(x)dx$, and the dual norm on $L_{\overline{M}}(\Omega)$ is equivalent to $\|\cdot\|_{\overline{M},\Omega}$.
  The space $L_M(\Omega)$ is reflexive if and only if M and $\overline{M}$ satisfy the Δ_2 condition (near infinity only if Ω has finite measure).

- We now turn to the Orlicz-Sobolev spaces. $W^1L_M(\Omega)$ (resp. $W^1E_M(\Omega)$) is the space of all functions u such that u and its distributional derivatives up to order 1 lie in $L_M(\Omega)$ (resp. $E_M(\Omega)$). It is a Banach space under the norm:
  $$\|u\|_{1,M,\Omega} = \sum_{|\alpha|\le 1} \|D^\alpha u\|_{M,\Omega} \,.$$
  Thus $W^1L_M(\Omega)$ and $W^1E_M(\Omega)$ can be identified with subspace of the product of $(N+1)$ copies of $L_M(\Omega)$. Denoting this product by ΠL_M, we will use the weak topologies $\sigma\left(\Pi L_M, \Pi E_{\overline{M}}\right)$ and $\sigma\left(\Pi L_M, \Pi L_{\overline{M}}\right)$.
  The space $W_0^1E_M(\Omega)$ is defined as the (norm) closure of the Schwartz space $\mathcal{D}(\Omega)$ in $W^1E_M(\Omega)$ and the space $W_0^1L_M(\Omega)$ as the $\sigma\left(\Pi L_M, \Pi E_{\overline{M}}\right)$ closure of $\mathcal{D}(\Omega)$ in $W^1L_M(\Omega)$.

- We say that u_n converges to u for the modular convergence in $W^1 L_M(\Omega)$ if for some $\lambda > 0$
$$\int_\Omega M\left(\left(D^\alpha u_n - D^\alpha u\right)/\lambda\right) dx \to 0 \text{ for all } |\alpha| \leq 1$$
This implies convergence for $\sigma\left(\Pi L_M, \Pi L_{\overline{M}}\right)$. Note that, if $u_n \to u$ in $L_M(\Omega)$ for the modular convergence and $v_n \to v$ in $L_M(\Omega)$ for the modular convergence, we have
$$\int_\Omega u_n v_n dx \to \int_\Omega uv dx \quad \text{ as } n \to \infty$$
If M satisfies the Δ_2-condition on $\mathbb{R}^+$, then modular convergence coincides with norm convergence.

- Let $W^{-1}L_{\overline{M}}(\Omega)$ [resp. $W^{-1}E_{\overline{M}}(\Omega)$ denote the space of distributions on Ω which can be written as sums of derivatives of order at most 1 of functions in $L_{\overline{M}}(\Omega)$ [resp. $W^{-1}E_{\overline{M}}(\Omega)$]. It is a Banach space under the usual quotient norm.

- If the open set Ω has the segment property then the space $\mathcal{D}(\Omega)$ is dense in $W_0^1 L_M(\Omega)$ for the modular convergence and thus for the topology $\sigma(\Pi L_M, \Pi L_{\overline{M}})$ (cf. [13], [19]). Consequently, the action of a distribution S in $W^{-1}L_{\overline{M}}(\Omega)$ on an element of $W_0^1 L_M(\Omega)$ is well defined, it will be noted by $< S, u >$.

2.2. The Inhomogeneous Orlicz-Sobolev

Let Ω be a bounded open subset of $\mathbb{R}^N$, $T > 0$ and set $Q = \Omega \times]0, T[$. Let M be an N-function. For each $\alpha \in \mathbb{N}^N$, denote by D_x^α the distributional derivative on Q of order α with respect to the variable $x \in \mathbb{R}^N$. The inhomogeneous Orlicz-Sobolev spaces of order 1 are defined as follows
$$W^{1,x}L_M(Q) = \{u \in L_M(Q) : D_x^\alpha u \in L_M(Q), \forall |\alpha| \leq 1\}$$
and
$$W^{1,x}E_M(Q) = \{u \in E_M(Q) : D_x^\alpha u \in E_M(Q), \forall |\alpha| \leq 1\}$$
The last space is a subspace of the former. Both are Banach spaces under the norm
$$\|u\| = \sum_{|\alpha| \leq 1} \|D_x^\alpha u\|_{M,Q}\,.$$
The space $W_0^{1,x}L_M(Q)$ is defined as the (norm) closure in $W^{1,x}L_M(Q)$ of $\mathcal{D}(Q)$ and we have.
$$W_0^{1,x}L_M(Q) = \overline{\mathcal{D}(Q)}^{\sigma\left(\Pi L_M, \Pi L_{\overline{M}}\right)}.$$
We can easily show that they form a complementary system when Ω satisfies the segment property. These spaces are considered as subspaces of the product space $\Pi L_M(Q)$ which has (N+1) copies. We shall also consider the weak topologies $\sigma(\Pi L_M \Pi E_M)$ and $\sigma(\Pi L_M, \Pi L_M)$. If $u \in W^{1,x}L_M(Q)$, then the function: $t \mapsto u(t) = u(., t)$ is defined on $(0, T)$ with values in $W^1 L_M(\Omega)$. If, further, $u \in W^{1,x}E_M(Q)$, then $u(., t)$ is $W^1 E_M(\Omega)$-valued and is strongly measurable.
Furthermore, the following continuous imbedding holds: $W^{1,x}E_M(Q) \subset L^1(0, T)$,

$W^1E_M(\Omega)$. The space $W^{1,x}L_M(Q)$ is not in general separable; if $u \in W^{1,x}L_M(Q)$, we cannot conclude that the function $u(t)$ is measurable from $(0,T)$ into $W^{1,x}L_M(\Omega)$. However the scalar function $t \mapsto \|D_x^\alpha u(t)\|_{M,\Omega}$ is in $L^1(0,T)$ for all $|\alpha| \leq 1$. Furthermore, $W_0^{1,x}E_M(Q) = W_0^{1,x}L_M(Q) \cap \Pi E_M$. Poincare's inequality also holds in $W_0^{1,x}L_M(Q)$ and then there is a constant $C > 0$ such that for all $u \in W_0^{1,x}L_M(Q)$ one has

$$\sum_{|\alpha|\leq 1} \|D_x^\alpha u\|_{M,Q} \leq C \sum_{|\alpha|=1} \|D_x^\alpha u\|_{M,Q}$$

thus both sides of the last inequality are equivalent norms on $W_0^{1,x}L_M(Q)$. We have then the following complementary system

$$\begin{pmatrix} W_0^{1,x}L_M(Q) & F \\ W_0^{1,x}E_M(Q) & F_0 \end{pmatrix}$$

F being the dual space of $W_0^{1,x}E_M(Q)$. It is also, up to an isomorphism, the quotient of $\Pi L_{\overline{M}}$ by the polar set $W_0^{1,x}E_M(Q)^\perp$, and will be denoted by $F = W^{-1,x}L_{\overline{M}}(Q)$ and it is shown that

$$W^{-1,x}L_M(Q) = \left\{ f = \sum_{|\alpha|\leq 1} D_x^\alpha f_\alpha \ : \ f_\alpha \in L_{\overline{M}}(Q) \right\}$$

This space will be equipped with the usual quotient norm:

$$\|f\| = \inf \sum_{|\alpha|\leq 1} \|f_\alpha\|_{\overline{M},Q}$$

where the infinum is taken on all possible decompositions

$$f = \sum_{|\alpha|\leq 1} D_x^\alpha f_\alpha, f_\alpha \in L_{\overline{M}}(Q)$$

The space F_0 is then given by

$$F_0 = \left\{ f = \sum_{|\alpha|\leq 1} D_x^\alpha f_\alpha : f_\alpha \in E_{\overline{M}}(Q) \right\}$$

and is denoted by $F_0 = W^{-1,x}E_{\overline{M}}(Q)$.

2.3. Some inequalities

Lemma 2.1. [17] *Let M be an N-function, we have the following inequality:*

$$st \leq M(s) + \overline{M}(t)$$

called Young inequality.

Lemma 2.2. [17] *The generalized Holder inequality*

$$\left| \int_\Omega u(x)v(x)|dx \right| \leq 2\|u\|_M \|v\|_{\overline{M}}$$

hold for any pair function $u \in L_M(\Omega)$ and $v \in L_{\overline{M}}(\Omega)$.

Proof. The proof of this inequalities is detailed in [17] (see pages 18 for the first one 111 for the second). □

2.4. Approximation theorem and trace result

Let Ω is an open subset of $\mathbb{R}^N$ with the segment property and I is a sub-interval of $\mathbb{R}$ (both possibly unbounded) and $Q = \Omega \times I$. It is easy to see that Q also satisfies the segment property.

Definition 2.3. [12] We say that $u_n \to u$ in $W^{-1,x}L_{\overline{M}}(Q) + L^2(Q)$ for the modular convergence if we can write

$$u_n = \sum_{|\alpha|\leq 1} D_x^\alpha u_n^\alpha + u_n^0$$

$$u = \sum_{|\alpha|\leq 1} D_x^\alpha u^\alpha + u^0$$

with $u_n^\alpha \to u^\alpha$ in $L_{\overline{M}}(Q)$ for the modular convergence for all $|\alpha| \leq 1$ and $u_n^0 \to u^0$ strongly in $L^2(Q)$.

This implies, in particular, that $u_n \to u$ in $W^{-1,x}L_{\overline{M}}(Q) + L^2(Q)$ for the weak topology $\sigma(\Pi L_M + L^2, \Pi L_M \cap L^2)$ in the sense that $< u_n, v > \to < u, v >$ for all $v \in W_0^{1,x}L_M(Q) \cap L^2(Q)$, where here and throughout the paper, $< ., . >$ means either the pairing between $W_0^{1,x}L_M(Q)$ and $W^{-1,x}L_{\overline{M}}(Q)$, or the pairing between $W_0^{1,x}L_M(Q) \cap L^2(Q)$ and $W^{-1,x}L_{\overline{M}}(Q) + L^2(Q)$. Indeed,

$$\langle u_n, v\rangle = \sum_{|\alpha|\leqslant 1} (-1)^{|\alpha|} \int_Q u_n^\alpha D_x^\alpha v \ \mathrm{d}x \ \mathrm{d}t + \int_Q u_n^0 v \ \mathrm{d}x \ \mathrm{d}t$$

and since for all $|\alpha| \leqslant 1, u_n^\alpha \to u^\alpha$ in $L_{\overline{M}}(Q)$ for the modular convergence, and so for $\sigma\left(L_{\overline{M}}, L_M\right)$, we have

$$\sum_{|\alpha|\leqslant 1} (-1)^{|\alpha|} \int_Q u_n^\alpha D_x^\alpha v \ \mathrm{d}x \ \mathrm{d}t + \int_Q u_n^0 v \ \mathrm{d}x \ \mathrm{d}t$$
$$\to \sum_{|\alpha|\leqslant 1} (-1)^{|\alpha|} \int_Q u^\alpha D_x^\alpha v \ \mathrm{d}x \ \mathrm{d}t + \int_Q u^0 v \ \mathrm{d}x \ \mathrm{d}t = \langle u, v\rangle.$$

Moreover, if $v_n \to v$ in $W_0^{1,x}L_M(Q) \cap L^2(Q)$ for the modular convergence (i.e. $v_n \to v$ in $W_0^{1,x}L_M(Q)$ for the modular convergence and in $L^2(Q)$ strong), we have $\langle u_n, v_n\rangle \to \langle u, v\rangle$ as $n \to \infty$.

Theorem 2.4. [12] *If $u \in W^{1,x}L_M(\Omega) \cap L^2(\Omega)$ (respectively $W_0^{1,x}L_M(\Omega) \cap L^2(\Omega)$) and $\frac{\partial u}{\partial t} \in W^{-1,x}L_{\overline{M}}(Q) + L^2(Q)$, then there exists a sequence (v_j) in $\mathcal{D}(\overline{Q})$ (respectively $\mathcal{D}(I, \mathcal{D}(\Omega))$) such that*

$$v_j \to u \ \text{in} \ W^{1,x}L_M(\Omega) \cap L^2(\Omega)$$
$$\frac{\partial v_j}{\partial t} \to \frac{\partial u}{\partial t} \ \text{in} \ W^{-1,x}L_{\overline{M}}(Q) + L^2(Q)$$

for the modular convergence.

Remark 2.5. If in the statement of theorem (2.4), one considers $\Omega \times \mathbb{R}$ instead of Q we have $\mathcal{D}(\Omega \times \mathbb{R})$ is dense in

$$\{u \in W_0^{1,x}(\Omega \times \mathbb{R}) \cap L^2(\Omega \times \mathbb{R}) : \frac{\partial u}{\partial t} \in W^{-1,x}L_{\overline{M}}(\Omega \times \mathbb{R}) + L^2(\Omega \times \mathbb{R})\}$$

for the modular convergence.

A first application of Theorem (2.4) is the following trace result generalizing a classical result which states that if u belongs to $L^2(a, b; H_0^1(\Omega))$ and $\frac{\partial u}{\partial t}$ belongs to $L^2(a, b; H^{-1}(\Omega))$, then u is in $C(a, b; L^2(\Omega))$.

Lemma 2.6. [12] *Let $a < b \in \mathbb{R}$ and let Ω be a bounded subset of $\mathbb{R}^N$ with the segment property, then*

$$\{u \in W_0^{1,x}L_M(\Omega \times (a,b)) \cap L^2(\Omega \times (a,b)); \frac{\partial u}{\partial t} \in W^{-1,x}L_{\overline{M}}(\Omega \times (a,b)) + L^2(\Omega \times (a,b))\}$$

is a subset of $C([a,b], L^2(\Omega))$.

3. Existence result

3.1. Assumption and statement of main result

Let Ω be a bounded open subset of $\mathbb{R}^N$ $(N \geq 2)$ with the segment property, and Q be the cylinder $\Omega \times (0,T)$ with some given $T > 0$. Let M be an N-function.
Consider the second order operator $A : D(A) \subset W_0^{1,x}L_M(Q) \to W^{-1,x}L_{\overline{M}}(Q)$ of the form:

$$A(u) = -div(a(x,t,\nabla u))$$

where $a : \Omega \times (0,T) \times \mathbb{R}^N \to \mathbb{R}^N$ are a Carateodory function satisfying for almost every $(x,t) \in \Omega \times (0,T)$ and all $\xi \neq \xi^* \in \mathbb{R}^N$ we have the following assumptions:

$$|a(x,t,\xi)| \leq \beta(h_1(x,t) + \overline{M}^{-1}M(\delta|\xi|)) \; ; \tag{3.1}$$

$$[a(x,t,\xi) - a(x,t,\xi^*)][\xi - \xi^*] > 0 \; ; \tag{3.2}$$

$$a(x,t,\xi)\xi \geq \alpha M\big(\frac{|\xi|}{\lambda}\big) \; ; \tag{3.3}$$

$$f \in W^{-1,x}E_{\overline{M}}(Q) \; ; \tag{3.4}$$

where $h_1 \in L^1(Q)$, and $\beta, \delta, \alpha, \lambda > 0$.
and suppose that there exist $s' > 0$ and c, k two positive constant such that for all $s \geq s'$:

$$s^2 \leq cM(ks) \tag{3.5}$$

We shall prove the following existence theorem

Theorem 3.1. *Assume that (3.1)-(3.5) hold true then there exist a unique solution $u \in D(A) \cap W_0^{1,x}L_M(Q) \cap C(0,T,L^2(\Omega))$ of (1.1) in the following sense:*

$$< \frac{\partial u}{\partial t}, \varphi >_Q + \int_Q a(x,t,\nabla u)\nabla\varphi dxdt =< f, \varphi >_Q; \tag{3.6}$$

for every $\varphi \in W_0^{1,x}L_M(Q) \cap L^2(Q)$ *with* $\frac{\partial \varphi}{\partial t} \in W^{-1,x}L_{\overline{M}}(Q) + L^2(Q)$. *where here* $< .,. >$ *means for either the pairing between* $W_0^{1,x}L_M(Q)$ *and* $W^{-1,x}L_{\overline{M}}(Q)$, *or between* $W_0^{1,x}L_M(Q) \cap L^2(Q)$ *and* $W^{-1,x}L_{\overline{M}}(Q) + L^2(Q)$.

Integrating by part and using the periodicity condition equation (3.6) can be written as:

$$-\int_Q \frac{\partial \varphi}{\partial t} u dxdt + \int_Q a(x,t,\nabla u)\nabla \varphi dxdt =< f, \varphi >_Q \quad (3.7)$$

Remark 3.2. Note that all term in (3.7) are well defined, and by the trace result of lemma (2.6) we have that $u \in C([0,T), L^2(\Omega))$ wish make sense of the periodicity condition.

4. The proof of the main result

The proof of theorem (3.1) is divided into five steps:

Proof. **Step 1**: Firstly we have to prove that the solution u is unique. For that we suppose that there exist another solution v of problem (1.1) then v satisfy also (3.6), then by taking $\varphi = u(t) - v(t)$ we can easily see that

$$\frac{1}{2}\frac{d}{dt}\int_Q (u(t) - v(t))^2 dx + \int_Q (a(x,t,\nabla u) - a(x,t,\nabla v))(\nabla u - \nabla v)dxdt = 0 \quad (4.1)$$

Using periodicity condition and (3.2) we get $\nabla u = \nabla v$, then we have by (4.1) that $u(t) = v(t)$ for almost every $t \in (0,T)$, finally we deduce that $u = v$.

Step 2: *Approximate problem:* As in [12] we will use Galerkin method due to Landes and Mustonen [19]. For that we choose a sequence $\{w_1, w_2, w_3, \cdots\}$ in $\mathcal{D}(\Omega)$ such that $\bigcup_{n=1}^{\infty} V_n$ with

$$V_n = span\{w_1, w_2, w_3, \cdots\}$$

is dense in $H_0^m(\Omega)$with m large enough such that $H_0^m(\Omega)$ is continuously embedded in $C^1(\Omega)$. For any $v \in H_0^m(\Omega)$, there exists a sequence $(v_k) \subset \bigcup_{n=1}^{\infty} V_n$ such that $v_k \to v$ in $H_0^m(\Omega)$and in $\mathcal{C}^1(\overline{\Omega})$ too.

We denote further $\mathcal{V}_n = C([0,T], V_n)$. We have that the closure of $\bigcup_{n=1}^{\infty} \mathcal{V}_n$ with respect to the norm:

$$\|v\|_{C^{1,0}(Q)} = \sup_{|\alpha| \le 1} \{|D_x^{|\alpha|} v(x,t)| : (x,t) \in Q\}$$

contains $\mathcal{D}(Q)$, for more detail see [11] and [18]).

This implies that, for any $f \in W^{-1,x}E_{\overline{M}}(Q)$, there exists a sequence $(f_k) \subset \bigcup_{n=1}^{\infty} \mathcal{V}_n$ such that $f_k \to f$ strongly in $W^{-1,x}E_{\overline{M}}(Q)$. Indeed, let $\varepsilon > 0$ be given. Writing

$$f = \sum_{|\alpha| \le 1} D_x^{\alpha} f^{\alpha}$$

for all $|\alpha| \leq 1$, there exists $g^\alpha \in \mathcal{D}(Q)$ such that, $\|f^\alpha - g^\alpha\|_{\overline{M},Q} \leq \frac{\varepsilon}{2N+2}$. Moreover, by setting $g = \sum_{|\alpha|\leq 1} D_x^\alpha g^\alpha$, we see that for any $g \in \mathcal{D}(Q)$, and so there exists $\varphi \in \bigcup_{n=1}^\infty \mathcal{V}_n$ such that $\|g - \varphi\|_{\infty,Q} \leq \frac{\varepsilon}{2meas(Q)}$. We deduce then

$$\|f^\alpha - g^\alpha\|_{W^{-1,x}E_{\overline{M}}(Q)} \leq \sum_{|\alpha|\leq 1} \|f^\alpha - g^\alpha\|_{\overline{M},Q} + \|g - \varphi\|_{\infty,Q}$$

Now, let us consider the following approximate problem:

$$\begin{cases} u_n \in \mathcal{V}_n; \frac{\partial u_n}{\partial t} \in L^1(0,T,V_n); \\ u_n(x,0) = u_n(x,T)\ ; \\ \text{and for all } \varphi \in \mathcal{V}_n \\ \int_Q \frac{\partial u_n}{\partial t}\varphi dxdt + \int_Q a(x,t,\nabla u_n)\nabla\varphi dxdt = \int_Q f_n\varphi dxdt \end{cases} \tag{4.2}$$

See the appendix for the prove of the existence of $u_n \in \mathcal{V}_n$.

Step 3: a priori estimates

Let as prove that:

$$\|u_n\|_{W_0^{1,x}L_M(Q)} \leq C \quad ; \int_Q a\left(x,t,\nabla u_n\right)\nabla u_n \ \mathrm{d}x \leq C' \tag{4.3}$$

and

$$a\left(x,t,\nabla u_n\right) \text{ is bounded in } \left(L_{\overline{M}}(Q)\right)^N \tag{4.4}$$

where here C, C' are a positives constants not depending on n.

Proof. Taking u_n as a test function in (4.2), then using periodicity condition and young inequality we have

$$\int_Q a(x,t,\nabla u_n)\nabla u_n dxdt \leq \frac{1}{\epsilon}\|f_n\|_{\overline{M},Q} + \epsilon\|u_n\|_{M,Q}.$$

By using (3.2) and applying Poinccare inequality there exist $C_1 > 0$ such that

$$\alpha\int_Q M\left(\frac{|\nabla u_n|}{\lambda}\right)dxdt \leq \|f_n\|_{\overline{M},Q} + \epsilon C_1\int_Q M\left(\frac{|\nabla u_n|}{\lambda}\right)dxdt.$$

By a choice of ϵ and the fact that $\|f_n\|_{\overline{M},Q} \leq C$ we obtain

$$\int_Q M\left(\frac{|\nabla u_n|}{\lambda}\right)dxdt \leq C. \tag{4.5}$$

This implies that (u_n) is bounded in $W_0^{1,x}L_M(Q)$ and so in $L^2(Q)$. By using (3.1) and (4.5) we can conclude that there exist a constant $C' > 0$ such that

$$\int_Q a(x,t,\nabla u_n)\nabla u_n dxdt \leq C'; \tag{4.6}$$

To prove that $a(x,t,\nabla u_n)$ is bounded in $(L_{\overline{M}}(Q))^N$, let $\varphi \in (E_{\overline{M}}(Q))^N$ with $\|\varphi\|_{M,Q} = 1$. By (3.2) we have

$$\int_Q (a(x,t,\nabla u_n) - a(x,t,\varphi))(\nabla u_n, \varphi)dxdt > 0$$

which gives

$$\int_Q a(x,t,\nabla u_n)\varphi < \int_Q a(x,t,\nabla u_n)\nabla u_n dxdt - \int_Q a(x,t,\varphi)(\nabla u_n - \varphi)dxdt$$

Using (3.1) and (4.3) we can easily see that

$$\int_Q a(x,t,\nabla u_n)\varphi < C$$

and so $a(x,t,\nabla u_n)$ is bounded in $(L_{\overline{M}}(Q))^N$.
Thus for a subsequence still denote u_n and for some $h \in (L_{\overline{M}}(Q))^N$:

$$u_n \rightharpoonup u \text{ weakly in } W_0^{1,x}L_M(Q) \text{ for } \sigma\left(\Pi L_M, \Pi E_{\overline{M}}\right), \quad (4.7)$$

$$\textit{and} \text{ weakly in } L^2(Q).$$

$$a(x,t,\nabla u_n) \rightharpoonup h \text{ weakly in } (L_{\overline{M}}(Q))^N \text{ for } \sigma\left(\Pi L_{\overline{M}}, \Pi E_{\overline{M}}\right) \quad (4.8)$$

□

Step 4: Almost everywhere convergence of the gradient.
For all $\varphi \in C^1(0,T,\mathcal{D}(\Omega))$, we get by (4.2) and (4.8) that

$$-\int_Q u\frac{\partial \varphi}{\partial t} + \int_Q h\nabla\varphi dxdt = \int_Q f\nabla\varphi dxdt. \quad (4.9)$$

We can see by taking φ arbitrary in $\mathcal{D}(Q)$ that $\frac{\partial u}{\partial t} \in W^{-1,x}L_{\overline{M}}(Q)$, then by theorem (2.4) there exist a subsequence denote $v_k \in \mathcal{D}(Q)$ such that:

$$v_k \to u \text{ in } W_0^{1,x}L_M(Q) \cap L^2(Q) \text{ and } \frac{\partial v_k}{\partial t} \to \frac{\partial u}{\partial t} \text{ in } W^{-1,x}L_{\overline{M}}(Q) + L^2(Q)$$

for the modular convergence, then by lemma (2.6), we have $v_k \to u$ in $C([0,T], L^2(\Omega))$ and so $u \in C([0,T], L^2(\Omega))$.
From (4.2), (3.7) we have

$$\begin{aligned}
&\limsup_{n\to\infty} \int_Q a(x,t,\nabla u_n)\nabla u_n - h\nabla v_k dxdt \\
&\leq \limsup_{n\to\infty} \Big(-\int_Q \frac{\partial u_n}{\partial t} u_n dxdt\Big) + \int_Q \frac{\partial v_k}{\partial t} u dxdt \\
&\quad + \limsup_{n\to\infty} \int_Q \big(f_n u_n dxdt - \int_Q f_n v_k\big) dxdt \\
&= \limsup_{n\to\infty} \Big(\int_Q \frac{\partial u_n}{\partial t}(v_k - u_n)dxdt\Big) + \int_Q f(u - v_k)dxdt
\end{aligned}$$

where we have used the fact that

$$\begin{aligned}
-\int_Q \frac{\partial v_k}{\partial t} u dxdt &= \lim_{n\to\infty} -\int_Q \frac{\partial v_k}{\partial t} u_n dxdt \\
&= \lim_{n\to\infty} -\int_Q \frac{\partial u_n}{\partial t} v_k dxdt + \int_\Omega \big[u_n(t)v_k(t)\big]_0^T dx
\end{aligned}$$

then the periodicity condition imply

$$-\int_Q \frac{\partial v_k}{\partial t} u dxdt = \lim_{n\to\infty} -\int_Q \frac{\partial u_n}{\partial t} v_k dxdt.$$

For the first term in the right hand sand we have

$$\begin{aligned}
\limsup_{n\to\infty} \int_Q \frac{\partial u_n}{\partial t}(v_k - u_n)dxdt &= \limsup_{n\to\infty} \big(-\frac{1}{2}\frac{d}{dt}\int_Q (u_n(t) - v_k(t))^2 dxdt\big) \\
&+ \limsup_{n\to\infty} \int_Q \frac{\partial v_k}{\partial t}(v_k - u_n)dxdt \\
&= \limsup_{n\to\infty} \big(-\frac{1}{2}\int_\Omega \big[u_n(t) - v_k(t))^2\big]_0^T dx\big) \\
&+ \limsup_{n\to\infty} \int_Q \frac{\partial v_k}{\partial t}(v_k - u_n)dxdt
\end{aligned}$$

the fact that $\frac{\partial v_k}{\partial t} \in E_{\overline{M}}(Q)$ and $v_k \to u$ gives

$$\limsup_{k\to\infty} \limsup_{n\to\infty} \int_Q \frac{\partial v_k}{\partial t}(v_k - u_n)dxdt = 0.$$

By periodicity condition we have

$$\limsup_{k\to\infty} \limsup_{n\to\infty} \int_Q \frac{\partial u_n}{\partial t}(v_k - u_n)dxdt = 0.$$

Then we obtain

$$\limsup_{n\to\infty} \int_Q a(x,t,\nabla u_n)\nabla u_n dxdt = \int_Q h\nabla v_k dxdt + \int_Q f(u - v_k)dxdt$$

Having in mind that v_k converge strongly to u in $W_0^{1,x}L_M(Q)$ for the modular convergence, we can pass to the limit sup in k, to deduce

$$\limsup_{k\to\infty} \limsup_{n\to\infty} \int_Q a(x,t,\nabla u_n)\nabla u_n = \int_Q h\nabla v dxdt. \tag{4.10}$$

Fix a real number $r > 0$ and any $k \in \mathbb{N}$, we denote by χ_k^r and χ^r the characteristic functions of $Q_k^r = \{(x,t) \in Q : |\nabla v_k| \leqslant r\}$ and $Q^r = \{(x,t) \in Q\colon |\nabla u| \leqslant r\}$, respectively. We also denote by $\varepsilon(n,k,s)$ all quantities (possibly different)such that

$$\lim_{s\to\infty} \lim_{k\to\infty} \lim_{n\to\infty} \varepsilon(n,k,s) = 0,$$

and this will be the order in which the parameters we use will tend to infinity, that is, first n, then k, and finally s. Similarly, we will write only $\varepsilon(n)$, or $\varepsilon(n,k), \ldots$ to mean that the limits are only on the specified parameters.

Taking $s \geq r$ one has

$$\begin{aligned} 0 &\leq \int_{Q^r} \big(a(x,t,\nabla u_n) - a(x,t,\nabla u)\big)\big(\nabla u_n - \nabla u\big)dxdt \\ &\leq \int_{Q^s} \big(a(x,t,\nabla u_n) - a(x,t,\nabla u)\big)\big(\nabla u_n - \nabla u\big)dxdt \\ &= \int_{Q^s} \big(a(x,t,\nabla u_n) - a(x,t,\nabla u\chi^s)\big)\big(\nabla u_n - \nabla u\chi^s\big)dxdt \\ &\leq \int_{Q} \big(a(x,t,\nabla u_n) - a(x,t,\nabla u\chi^s)\big)\big(\nabla u_n - \nabla u\chi^s\big)dxdt. \end{aligned}$$

On the other hand

$$\int_Q \left[a\left(x,t,\nabla u_n\right) - a\left(x,t,\nabla u\chi^s\right)\right]\left[\nabla u_n - \nabla u\chi^s\right]dxdt$$

$$\begin{aligned} &= \int_Q \left[a\left(x,t,\nabla u_n\right) - a\left(x,t,\nabla v_k\chi_k^s\right)\right] \\ &\qquad \times \left[\nabla u_n - \nabla v_k\chi_k^s\right]dxdt \\ &+ \int_Q a\left(x,t,\nabla v_k\chi_k^s\right)\left[\nabla u_n - \nabla v_k\chi_k^s\right]dxdt \\ &+ \int_Q a\left(x,t,\nabla u_n\right)\left[\nabla v_k\chi_k^s - \nabla u\chi^s\right]dxdt \\ &+ \int_Q a\left(x,t,\nabla u\chi^s\right)\left[\nabla u\chi^s - \nabla u_n\right]dxdt \\ &= I_1 + I_2 + I_3 + I_4. \end{aligned}$$

We shall go to the limit in all integrals I_i (for i=1, 2, 3, 4) as first n, then k, and finally s tend to infinity.
Starting with I_2 and letting $n \to \infty$, since $\nabla u_n \rightharpoonup \nabla u$ in $L_{\overline{M}}(Q)^N$ by Lebesgue theorem we get that

$$I_2 = \int_Q a\left(x,t,\nabla v_k\chi_k^s\right)\left[\nabla u - \nabla v_k\chi_k^s\right]dxdt + \varepsilon(n).$$

Letting then $k \to \infty$ this imply

$$I_2 = \int_{\{|\nabla u|>s\}} a\left(x,t,0\right)\nabla u dxdt + \varepsilon(n,k).$$

Finally we deduce when s tends to infinity that

$$I_2 = \varepsilon(n,k,s). \tag{4.11}$$

For I_3 we have by letting $n \to \infty$ and using (4.8) that

$$I_3 = \int_Q h(\nabla v_k\chi_k^s - \nabla u\chi^s)dxdt$$

and so, by letting $k \to \infty$ in the integral of the last side and using the fact that $\nabla v_k \chi_k^s \to \nabla u \chi^s$ strongly in $(E_M(Q))^N$, we deduce that $I_2 = \varepsilon(n,k)$. For the fourth term I_4, we have, by letting $n \to \infty$,

$$I_4 = -\int_{\{|\nabla u|>s\}} a(x,t,0)\nabla u dxdt + \varepsilon(n),$$

and since the first term of the last side tends to zero as $s \to \infty$, we obtain $I_4 = \varepsilon(n,k,s)$. We have then proved that

$$\begin{aligned}&\int_Q \left[a\left(x,t,\nabla u_n\right) - a\left(x,t,\nabla u \chi^s\right)\right]\left[\nabla u_n - \nabla u \chi^s\right] dxdt \\ &= \int_Q \left[a\left(x,t,\nabla u_n\right) - a\left(x,t,\nabla v_k \chi_k^s\right)\right]\left[\nabla u_n - \nabla v_k \chi_k^s\right] dxdt \\ &\quad + \varepsilon(n,k,s).\end{aligned}$$

Finally we can deduce that

$$\begin{aligned}0 &\le \int_{Q^r} \big(a(x,t,\nabla u_n) - a(x,t,\nabla u\big)\big(\nabla u_n - \nabla u\big) dxdt \qquad (4.12)\\ &\le \int_Q \left[a\left(x,t,\nabla u_n\right) - a\left(x,t,\nabla v_k \chi_k^s\right)\right]\left[\nabla u_n - \nabla v_k \chi_k^s\right] dxdt + \varepsilon(n,k,s)\end{aligned}$$

we can write

$$\begin{aligned}\int_Q \left[a\left(x,t,\nabla u_n\right) - a\left(x,t,\nabla v_k \chi_k^s\right)\right]\left[\nabla u_n - \nabla v_k \chi_k^s\right] dxdt &= \int_Q a(x,t,\nabla u_n)\nabla u_n dxdt \\ - \int_Q (a(x,t,\nabla u_n) - a(x,t,\nabla v_k \chi_k^s)\nabla v_k \chi_k^s dxdt \\ = J_1 + J_2 + J_3. \qquad (4.13)\end{aligned}$$

First all we have by using (4.10) that

$$\limsup_{k\to\infty} \limsup_{n\to\infty} J_1 = \int_Q h\nabla u dxdt. \qquad (4.14)$$

For J_2, letting first $n \to \infty$ then k, and using Lebesgue theorem hence $\nabla v_k \chi_k^s \to \nabla u \chi^s$ strongly in $(E_M(Q))^N$ we get

$$J_2 = -\int_Q (h - a(x,t,\nabla u \chi^s))\nabla u \chi^s dxdt + \varepsilon(n,k).$$

We can easily see that

$$J_2 = -\int_{Q^s} (h - a(x,t,\nabla u))\nabla u dxdt + \varepsilon(n,k). \qquad (4.15)$$

Letting $n \to \infty$ on J_3 we have

$$J_3 = -\int_{Q^s} a(x,t,\nabla u)\nabla u dxdt - \int_{\{|\nabla u|>s\}} a(x,t,0)\nabla u dxdt. \qquad (4.16)$$

Finally by combining (4.13), (4.14), (4.15), (4.16) we conclude that

$$\int_Q \left[a\left(x,t,\nabla u_n\right) - a\left(x,t,\nabla v_k \chi_k^s\right)\right]\left[\nabla u_n - \nabla v_k \chi_k^s\right] dxdt = \tag{4.17}$$

$$= -\int_{\{|\nabla u|>s\}} a(x,t,0)\nabla u dxdt + \varepsilon(n,k)$$

So, when s tend to infinity (4.12) and(4.17) gives

$$\lim_{n\to\infty}\int_{Q^r}\big(a(x,t,\nabla u_n) - a(x,t,\nabla u)\big)\big(\nabla u_n - \nabla u\big)dxdt = 0$$

and thus, as in the elliptic case see [3], we deduce that, for a subsequence still denoted by u_n,

$$\nabla u_n \to \nabla u \quad \text{ a.e. in } Q \tag{4.18}$$

Since $a(x,t,.)$ is continuous then

$$a(x,t,\nabla u_n) \to a(x,t,\nabla u) \text{ a.e in } Q$$

If we take in consideration that $a(x,t,\nabla u_n)$ is bounded in $(L_{\overline{M}}(Q))^N$ we have by lemma (4.4) of [19] that

$$a(x,t,\nabla u_n) \rightharpoonup a(x,t,\nabla u) \text{ weakly in } (L_{\overline{M}}(Q))^N.$$

Therefore, we get for all $\varphi \in C^1([0,T],\mathcal{D}(\Omega))$,

$$-\int_Q u\frac{\partial\varphi}{\partial t}dxdt + \int_Q a(x,t,\nabla u)\nabla\varphi dxdt =< f,\varphi >_Q \tag{4.19}$$

Step 5: Passage to the limit

Going back to the approximating equations (4.2), then we obtain in the sense of distribution when n tend to infinity that

$$\frac{\partial u}{\partial t} - div(a(x,t,\nabla u)) = f(x,t) \text{ and } u(x,t) = 0$$

Furthermore, by the fact that $\frac{\partial u_n}{\partial t} \to \frac{\partial u}{\partial t}$ in $W^{-1,x}L_{\overline{M}}(Q) + L^2(Q)$ for the modular convergence and we have already that $u_n \to u$ in $W_0^{1,x}L_M(Q)\cap L^2(Q)$ for the modular convergence, then by lemma (2.6) we get $u_n \to u$ in $C([0,T],L^2(\Omega)$, so using the periodicity condition, since

$$< \frac{\partial u}{\partial t}, u >= \lim_{n\to\infty} < \frac{\partial u_n}{\partial t}, u_n >= \frac{1}{2}[u_n(T)^2 - u_n(0)^2] = 0$$

we deduce finally

$$u(x,0) = u(x,T) \text{ in } \Omega.$$

Then the proof of theorem (3.1) is completed.

5. Appendix

let us consider the following approximate problem:

$$\begin{cases} u_n \in \mathcal{V}_n; \frac{\partial u_n}{\partial t} \in L^1(0,T,V_n); \\ u_n(x,0) = u_n(x,T)\ ; \\ \text{and for all } \varphi \in \mathcal{V}_n \\ \int_Q \frac{\partial u_n}{\partial t}\varphi dxdt + \int_Q a(x,t,\nabla u_n)\nabla\varphi dxdt = \int_Q f_n\varphi dxdt \end{cases} \tag{5.1}$$

we will use the point fixed theorem due to Leray-Schauder to prove the existence of solution, for that let us consider the following initial boundary value problem

$$\begin{cases} \frac{\partial u_n}{\partial t} + A(u_n) = f_n \\ u_n(x,t) = 0 \\ u_n(0) = u_{0n} \end{cases} \tag{5.2}$$

where u_{0n} in V_n. And let $\overline{B}_n(0,R)$ be a closed ball in the space V_n with the norm $\|.\|$. We define the Poincarré operator by

$$P : \overline{B}_n(0,R) \to \overline{B}_n(0,R)$$

$$u_{0n} \mapsto u_n(T)$$

We have to prove that P is continuous and relatively compact (i.e find the existence of a constant $R > 0$ such that $\|u_{0n}\| \le R \to \|u_n(T)\| \le R$.
let consider $\varphi = u_n$ in (4.2) we have

$$\int_\Omega \frac{\partial u_n}{\partial t} u_n dx + \int_\Omega a(x,t,\nabla u_n)\nabla u_n dx = \int_\Omega f_n u_n dx.$$

Using Hölder inequality to the term in the left hide sand we get

$$\int_\Omega \frac{\partial u_n}{\partial t} u_n dx + \int_\Omega a(x,t,\nabla u_n)\nabla u_n dx \le 2\|f_n\|_{\overline{M},\Omega}\|u_n\|_{M,\Omega}.$$

Then we can easily see that for $\varepsilon > 0$ there exist a constant $c(\varepsilon)$ such that

$$\frac{1}{2}\frac{d}{dt}\int_\Omega (u_n(t))^2 dx + \int_\Omega a(x,t,\nabla u_n)\nabla u_n dx \le C(\varepsilon)\|f_n\|^2_{\overline{M},\Omega} + \varepsilon\|u_n\|^2_{M,\Omega}$$

Using (3.2) we obtain

$$\frac{1}{2}\frac{d}{dt}\int_\Omega (u_n(t))^2 dx + \alpha\int_\Omega M(\frac{|\nabla u_n|}{\lambda})dx \le C(\varepsilon)\|f_n\|^2_{\overline{M},\Omega} + \varepsilon\|u_n\|^2_{M,\Omega}.$$

By lemma 5.7 of [19] there exist two positive constants δ, λ such that

$$\int_Q M(v)dxdt \le \delta\int_Q M(\lambda|\nabla v|)dxdt \quad \text{ for all } v \in W_0^{1,x}L_M(Q).$$

Then for $c_1 > 0$ we obtain

$$\frac{1}{2}\frac{d}{dt}\int_\Omega (u_n(t))^2 dx + \alpha c_1\int_\Omega M(|u_n|)dx \le C(\varepsilon)\|f_n\|^2_{\overline{M},\Omega} + \varepsilon\|u_n\|^2_{M,\Omega}$$

Using now (3.5), and by the choice of ε we can easily see that there exist $c_2 > 0$ such that

$$\frac{1}{2}\frac{d}{dt}\int_\Omega (u_n(t))^2dx + c_2\|u_n\|^2 \leq C(\varepsilon)\|f_n\|_{\overline{M},\Omega}$$

Multiplying by e^{c_2t} and integrating by part we obtain

$$e^{c_2T}\|u_n(T)\|^2 \leq 2\|f_n\|_{\overline{M},Q} + R^2$$

we choice R such that $R^2 > \frac{2e^{-c_2T}}{1-e^{-c_2T}}$ we deduce the existence of $R > 0$.

Now we pass to prove the continuity of P, for that we consider u_{0n} and ν_{0n} two sequences in $\overline{B}_n(0,R)$, by taking $\varphi = u_n - \nu_n$ such that u_n and ν_n satisfy (4.2) we get

$$\frac{1}{2}\frac{d}{dt}\int_Q (u_n(t) - \nu_n(t))^2dxdt + \int_Q (a(x,t,\nabla u_n) - a(x,t,\nabla \nu_n)(\nabla u_n - \nabla \nu_n)dx = 0$$

then using (3.2), we can write

$$\|u_n(T) - \nu_n(T)\|^2 \leq \|u_{0n} - \nu_{0n}\|^2.$$

Finally we deduce the continuity of P, hence by the point fixed argument there exist u_n solution of (4.2) satisfy $u_n(T) = u_n(0)$. □

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