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Univalence conditions of an integral operator on the exterior unit disk

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Abstract. Into this article, we consider the subclasses $V_j, V_{j,\mu}$ and $\sum_j(p)$, with $j = 2, 3, ...$, and generalize univalence conditions for the integral operator $G_{\alpha_i,\beta}$ of the analytic functions g in the exterior unit disk. We want to see if some univalent conditions for analytic functions obtained on the interior unit disk can be extended on the exterior unit disk, so we make use of the usual transformation

$$g(z) = \frac{1}{f(\frac{1}{z})}.$$

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1. Introduction

Let O be the class of analytical functions g defined in the exterior of the unit disk $W = \{z \in \mathbb{C} | 1 < |z| < \infty\}$.

Let $\sum$ be the subclass of O which contains the univalent functions of W.

Let O_j be the subclass of O which contains the meromorphic, normalized and injective functions $g : W \longrightarrow \mathbb{C}_\infty$, that looks like [4]:

$$g(z) = z + \sum_{k=j+1}^{\infty} \frac{b_k}{z^k}, 1 < |z| < \infty. \quad (1.1)$$

$$(j \in \mathbb{N}_1^* := \mathbb{N} - \{0, 1\} = \{2, 3, ...\})$$

With $g(\infty) = \infty$, $g'(\infty) = 1$.

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Let V be the subclass of univalent functions from O such that:

$$\left|\frac{g'(z)}{z^2}+1\right| > 1, z \in W. \tag{1.2}$$

Let V_j be the subclass of V, for which $g^{(k)}(\infty) = 0$, $(k = 2, 3, ..., j)$. Let $V_{j,\mu}$ be the subclass of V_j which contains the functions of the form (1.1) and satisfies the condition:

$$\left|\frac{g'(z)}{z^2}+1\right| > \mu, z \in W,$$

for $\mu > 1$ and we denote $V_{j,1} \equiv V_j$.

Let $p \in \mathbb{R}$, with $1 < p \leq 2$, let $\sum(p)$ be the subclass of O with all the functions $g \in O_j$ such that:

$$\left|\left(\frac{g(z)}{z}\right)''\right| \geq p, z \in W,$$

$$\left|\frac{g'(z)}{z^2}+1\right| \geq \frac{p}{|z|^j}, z \in W, j \in \mathbb{N}_1^*,$$

and we denote $\sum_2(p) \equiv \sum(p)$.

Let A be the class of analytic functions f defined in the open unit disk

$$U := \{z \in \mathbb{C} : |z| < 1\}$$

and normalized by the conditions $f(0) = 0 = f'(0) - 1$.

Let S be the subclass of A consisting of univalent functions in U, of the form [3]:

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k.$$

It is known that between the S class and the $\sum$ class there are the following links:

Proposition 1.1. [4]
(i) Let $f \in S$ and $g(\varsigma) = 1/f(1/\varsigma)$, $\varsigma \in W$. Then $g \in \sum$ and $g(\varsigma) \neq 0$, $\varsigma \in W$.
(ii) If $g \in \sum$ and $g(\varsigma) \neq 0$, $\varsigma \in W$, then $f \in S$ where $f(z) = 1/g(1/z)$, $z \in U$.

Let $F_{\alpha_1,\alpha_2,...,\alpha_n,\beta}$ be the integral operator introduced by Daniel Breaz and Narayanasamy Seenivasagan [10]:

$$F_{\alpha_1,\alpha_2,...,\alpha_n,\beta}(z) = \left\{\beta \int_0^z t^{\beta-1} \prod_{i=1}^{n} \left[\frac{f_i(t)}{t}\right]^{\frac{1}{\alpha_i}} dt\right\}^{\frac{1}{\beta}} \in S,$$

and we take into account that $f_i(t) \in S$.

When $\alpha_i = \alpha$ for all $i = 1, 2, ..., n$, $F_{\alpha_i,\beta}(z)$ becomes the integral operator $F_{\alpha,\beta}(z)$ considered in [1].

We may say that between A and O_1 there is a bijection.

We start from equation (1.2), and we apply the following transformations:

$$
\begin{aligned}
t &\to \frac{1}{t}|()', \\
dt &\to \frac{-1}{t^2}dt, \\
g_i(t) &= \frac{1}{f_i(\frac{1}{t})} \in O_1.
\end{aligned} \tag{1.3}
$$

With $g_i(t) \neq 0$; $t \in O_1$.

We can form the integral operator from the definition below:

Definition 1.1. (see [9]) Let $g_i \in O_1$ with $i = 1, 2, ..., n$ and $\alpha_1, \alpha_2, ..., \alpha_n, \beta \in \mathbb{C}$ we define the integral operator $G_{\alpha_1,\alpha_2,...,\alpha_n,\beta} : O_1^n \longrightarrow O_1$, considering $|z| > 1$:

$$
G_{\alpha_i,\beta}(z) = \left[\beta \int_1^z t^{-1-\beta} \prod_{i=1}^{n} \left(\frac{t}{g_i(t)}\right)^{\frac{1}{\alpha_i}} dt\right]^{\frac{1}{\beta}}. \tag{1.4}
$$

Theorem 1.1. *(see [7]) Let $\alpha \in \mathbb{C}$, $\Re(\alpha) > 0$ and $f \in A$. If the function f satisfies:*

$$
\frac{1 - |z|^{2\Re(\alpha)}}{\Re(\alpha)} \cdot \left|\frac{z \cdot f''(z)}{f'(z)}\right| \leq 1, (z \in U),
$$

then, for any complex number β with $\Re(\beta) \geq \Re(\alpha)$, the integral operator:

$$
F_\beta(z) = \left\{\beta \int_0^z t^{\beta-1} \cdot f'(t) dt\right\}^{\frac{1}{\beta}},
$$

is in the class S.

Lemma 1.1. (The Schwarz lemma) (*see* [2], [5], [6]) *Let the analytic function f be regular in the unit disk and let $f(0) = 0$. If $|f(z)| \leq 1$, then:*

$$
|f(z)| \leq |z|,
$$

for all $z \in U$, where the equality can hold only if $|f(z)| = K \cdot z$ and $K = 1$.

Lemma 1.2. (General Schwarz Lemma) (*see* [6]) *Let the function f be regular in the disk $U_R = \{z \in \mathbb{C} : |z| < R\}$, with $|f(z)| < M$ for fixed M. If has one zero with multiplicity order bigger than m for $z = 0$, then:*

$$
|f(z)| \leq \frac{M}{R^m} \cdot |z|^m, \; (z \in U_R).
$$

The equality can hold only if $f(z) = e^{i\theta} \cdot \frac{M}{R^m} \cdot z^m$, where θ is constant.

2. Main results

Lemma 2.1. *Let the analytic function g be regular in the exterior of the unit disk and let $g(\infty) = \infty, g'(\infty) = 1$. If $|g(z)| \geq 1$, then:*

$$\left|f\left(\frac{1}{z}\right)\right| \leq \left|\frac{1}{z}\right|,$$

$$\frac{1}{|g(z)|} \leq \frac{1}{|z|} \quad \Big|()^{-1},$$

$$|g(z)| \geq |z|,$$

for all $z \in W$, where the equality can hold only if $|g(z)| = K \cdot z$ and $K = 1$.

In Lemma 1.2 we apply the transformation from equation (1.3), and we get the following Lemma:

Lemma 2.2. *Let the function g be regular in the exterior unit disk*

$$W_R = \{z \in \mathbb{C} : |z| > R\},$$

with $|f(z)| > M$ for fixed M. If has one zero with multiplicity order bigger than m for $z = \infty$, then:

$$\left|f\left(\frac{1}{z}\right)\right| \leq \frac{M}{R^m} \cdot \left|\frac{1}{z}\right|^m,$$

$$\left|\frac{1}{g(z)}\right| \leq \frac{M}{R^m} \cdot \frac{1}{|z|^m} \Big|()^{-1},$$

$$|g(z)| \geq \frac{R^m}{M} \cdot |z|^m,$$

for all $z \in W$, where the equality can hold only if $f(z) = e^{i\theta} \cdot \frac{R^m}{M} \cdot z^m$, where θ is constant.

Theorem 2.1. *Let $\alpha \in \mathbb{C}$, $\Re(\alpha) > 0$ and $k \in O$. If k satisfies:*

$$\frac{|z|^{2\Re(\alpha)} - 1}{\Re(\alpha) \cdot |z|^{2\Re(\alpha)}} \cdot \left|\frac{k''(z)}{z \cdot k'(z)}\right| > 1, (z \in W),$$

and

$$\left|\frac{k''(z)}{zk'(z)}\right| > \Re(\alpha) \cdot |z|, \tag{2.1}$$

then, for any complex number β with $\Re(\beta) \leq \Re(\alpha)$, the integral operator:

$$G_\beta(z) = \left\{\beta \int_1^z t^{-\beta-1} \cdot k'(t)dt\right\}^{\frac{1}{\beta}},$$

is in the class $\sum$.

Proof. We apply in Theorem 1.1, the transformation $z \to \frac{1}{z}|(\ )'$.
We use $k(z) = \frac{1}{h(\frac{1}{z})}$, $\left|\frac{k''(z)}{z \cdot k'(z)}\right| > 1$ (see [9]) and $|k(z)| > 1$.

We multiply (2.1) with $\frac{|z|^{2\Re(\alpha)}-1}{\Re(\alpha)\cdot|z|^{2\Re(\alpha)}}$, and we get:

$$\frac{|z|^{2\Re(\alpha)}-1}{\Re(\alpha)\cdot|z|^{2\Re(\alpha)}}\cdot\left|\frac{k''(z)}{z\cdot k'(z)}\right|\geq\frac{|z|^{2\Re(\alpha)}-1}{|z|^{2\Re(\alpha)-1}}$$
$$\geq\frac{|z|^{2\Re(\alpha)-1}+|z|^{2\Re(\alpha)-2}+...+|z|+1}{|z|^{2\Re(\alpha)-1}}>1.$$

We obtain that, for any complex number β with $\Re(\beta)\leq\Re(\alpha)$, the integral operator:

$$G_\beta(z)=\left\{\beta\int_1^z t^{-\beta-1}\cdot k'(t)dt\right\}^{\frac{1}{\beta}},$$

is in the class $\sum$. □

Theorem 2.2. *Let g_i defined by:*

$$g_i(z)=z+\sum_{k=j+1}^{\infty}\frac{b_k^i}{z^k},|z|>1, \tag{2.2}$$

be in the class V_j for $i\in\{1,2,...,n\},n\in\mathbb{N}^*$, $j\in\mathbb{N}_1^*$.

If $|g_i(z)|\geq M_i(M_i\geq 1,\ z\in W)$.

Then the integral operator $G_{\alpha_i,\beta}$ defined by (1.4) is in the class $\sum$, with $\alpha,\beta\in\mathbb{C}$,

$$\Re(\alpha)\leq\sum_{i=1}^{n}\frac{1}{M_i|\alpha_i|}, \tag{2.3}$$

and $\Re(\beta)\leq\Re(\alpha)$.

Proof. We define a function:

$$k(z)=\int_0^z\prod_{i=1}^{n}\left(\frac{t}{g_i(t)}\right)^{\frac{1}{\alpha_i}}dt,$$

then we consider that $k(\infty)=\infty, k'(\infty)=1$.
After computation (see [8]) we obtain:

$$\frac{k''(z)}{z\cdot k'(z)}=\sum_{i=1}^{n}\frac{1}{\alpha_i}\cdot\left(\frac{1}{z^2}-\frac{g_i'(z)}{z\cdot g_i(z)}\right)$$

$$\left|\frac{k''(z)}{z\cdot k'(z)}\right|\geq\sum_{i=1}^{n}\frac{1}{M_i|\alpha_i|}.$$

We apply Theorem 2.1 and we consider (2.3), so we get:

$$\frac{|z|^{2\Re(\alpha)}-1}{\Re(\alpha)\cdot|z|^{2\Re(\alpha)}}\cdot\left|\frac{k''(z)}{z\cdot k'(z)}\right|\geq\frac{|z|^{2\Re(\alpha)}-1}{\Re(\alpha)\cdot|z|^{2\Re(\alpha)}}\cdot\sum_{i=1}^{n}\frac{1}{M_i|\alpha_i|}\geq$$
$$\geq\frac{1}{\Re(\alpha)}\cdot\sum_{i=1}^{n}\frac{1}{M_i|\alpha_i|}>1.$$

Applying Theorem 2.1, we obtain that $G_{\alpha_i,\beta}$ is univalent. □

Corollary 2.1. *Let g_i defined by (2.2) be in the class V_j for $i \in \{1, 2, ..., n\}$, $n \in \mathbb{N}^*$, $j \in \mathbb{N}_1^*$. If $|g_i(z)| \geq M$ $(M \geq 1,\ z \in W)$.*
Then the integral operator $G_{\alpha_i,\beta}$ defined by (1.4) is in the class $\sum$, where $\alpha, \beta \in \mathbb{C}$,

$$\Re(\alpha) \leq \sum_{i=1}^{n} \frac{1}{M|\alpha_i|},$$

and $\Re(\beta) \leq \Re(\alpha)$.

Proof. In Theorem 2.2, we consider $M_1 = M_2 = ... = M_n = M$. □

Corollary 2.2. *Let g_i defined by (2.2) be in the class V_j for $i \in \{1, 2, ..., n\}$, $n \in \mathbb{N}^*$, $j \in \mathbb{N}_1^*$. If $|g_i(z)| \geq M$ $(M \geq 1, z \in W)$.*
Then the integral operator $G_{\alpha,\beta}$ defined by (1.4) is in the class $\sum$, where $\alpha, \beta \in \mathbb{C}$,

$$\Re(\alpha) \leq \frac{n}{M|\alpha|},$$

and $\Re(\beta) \leq \Re(\alpha)$.

Proof. In Corollary 2.1, we consider $\alpha_1 = \alpha_2 = ... = \alpha_n = \alpha$. □

Corollary 2.3. *Let g_i defined by (2.2) be in the class V_2, for $i \in \{1, 2, ..., n\}$, $n \in \mathbb{N}^*$, If $|g_i(z)| \geq M$, $(M \geq 1, z \in W)$.*
Then the integral operator $G_{\alpha,\beta}$ defined by (1.4) is in the class $\sum$, where $\alpha, \beta \in \mathbb{C}$,

$$\Re(\alpha) \leq \frac{n}{M|\alpha|}$$

and $\Re(\beta) \leq \Re(\alpha)$.

Proof. In Corollary 2.2, we consider $j = 2$. □

Corollary 2.4. *Let g_i defined by (2.2) be in the class V_2 for $i \in \{1, 2, ..., n\}, n \in \mathbb{N}^*$. If $|g_i(z)| \geq 1$ $(z \in W)$.*
Then the integral operator $G_{\alpha,\beta}$ defined by (1.4) is in the class $\sum$, where $\alpha, \beta \in \mathbb{C}$,

$$\Re(\alpha) \leq \frac{n}{|\alpha|},$$

and $\Re(\beta) \leq \Re(\alpha)$.

Proof. In Corollary 2.2, we consider $j = 2$ and $M = 1$. □

Theorem 2.3. *Let g_i defined by (2.2) be in the class V_{j,μ_i} for $i \in \{1, 2, ..., n\}$, $n \in \mathbb{N}^*$, $j \in \mathbb{N}_1^*$. If $|g_i(z)| \geq M_i$ $(M_i \geq 1, z \in W)$.*
Then the integral operator $G_{\alpha_i,\beta}$ defined by (1.4) is in the class $\sum$, where $\alpha, \beta \in \mathbb{C}$,

$$\Re(\alpha) \leq \sum_{i=1}^{n} \frac{1}{(1+\mu_i)M_i|\alpha_i|},$$

and $\Re(\beta) \leq \Re(\alpha)$.

Proof. The proof of this theorem is very similar with the proof of Theorem 2.2. □

Corollary 2.5. *Let g_i defined by (2.2) be in the class V_{j,μ_i} for $i \in \{1, 2, ..., n\}$, $n \in \mathbb{N}^*$, $j \in \mathbb{N}_1^*$. If $|g_i(z)| \geq M$ $(M \geq 1, z \in W)$.*
Then the integral operator $G_{\alpha_i,\beta}$ defined by (1.4) is in the class $\sum$, where $\alpha, \beta \in \mathbb{C}$,

$$\Re(\alpha) \leq \sum_{i=1}^{n} \frac{1}{(1 + \mu_i)M|\alpha_i|},$$

and $\Re(\beta) \leq \Re(\alpha)$.

Proof. In Theorem 2.3, we consider $M_1 = M_2 = ... = M_n = M$. □

Corollary 2.6. *Let g_i defined by (2.2) be in the class V_{j,μ_i} for $i \in \{1, 2, ..., n\}$, $n \in \mathbb{N}^*$, $j \in \mathbb{N}_1^*$. If $|g_i(z)| \geq M(M \geq 1, z \in W)$.*
Then the integral operator $G_{\alpha,\beta}$ defined by (1.4) is in the class $\sum$, where $\alpha, \beta \in \mathbb{C}$,

$$\Re(\alpha) \leq \sum_{i=1}^{n} \frac{1}{(1 + \mu_i)M|\alpha|},$$

and $\Re(\beta) \leq \Re(\alpha)$.

Proof. In Corollary 2.5, we consider $\alpha_1 = \alpha_2 = ... = \alpha_n = \alpha$. □

Corollary 2.7. *Let g_i defined by (2.2) be in the class $V_{j,\mu}$ for $n \in \mathbb{N}^*$, $j \in \mathbb{N}_1^*$. If $|g_i(z)| \geq M(M \geq 1, z \in W)$.*
Then the integral operator $G_{\alpha,\beta}$ defined by (1.4) is in the class $\sum$, where $\alpha, \beta \in \mathbb{C}$,

$$\Re(\alpha) \leq \frac{n}{(1 + \mu)M|\alpha|},$$

and $\Re(\beta) \leq \Re(\alpha)$.

Proof. In Corollary 2.5, we consider $\mu_1 = \mu_2 = ... = \mu_n = \mu$. □

Corollary 2.8. *Let g_i defined by (2.2) be in the class V_{2,μ_i} for $i \in \{1, 2, ..., n\}$, $n \in \mathbb{N}^*$. If $|g_i(z)| \geq M(M \geq 1, z \in W)$.*
Then the integral operator $G_{\alpha_i,\beta}$ defined by (1.4) is in the class $\sum$, where $\alpha, \beta \in \mathbb{C}$,

$$\Re(\alpha) \leq \sum_{i=1}^{n} \frac{1}{(1 + \mu_i)M|\alpha_i|},$$

and $\Re(\beta) \leq \Re(\alpha)$.

Proof. In Corollary 2.5, we set $j = 2$. □

Corollary 2.9. *Let g_i defined by (2.2) be in the class $V_{2,\mu}$ for $n \in \mathbb{N}^*$. If $|g_i(z)| \geq M(M \geq 1, z \in W)$.*
Then the integral operator $G_{\alpha,\beta}$ defined by (1.4) is in the class $\sum$, where $\alpha, \beta \in \mathbb{C}$,

$$\Re(\alpha) \leq \frac{n}{(1 + \mu)M|\alpha|},$$

and $\Re(\beta) \leq \Re(\alpha)$.

Proof. In Corollary 2.7, we set $j = 2$. □

Corollary 2.10. *Let g_i defined by (2.2) be in the class $V_{2,\mu}$ for $n \in \mathbb{N}^*$. If $|g_i(z)| \geq$ $(z \in W)$.*
Then the integral operator $G_{\alpha,\beta}$ defined by (1.4) is in the class $\sum$, where $\alpha, \beta \in \mathbb{C}$,

$$\Re(\alpha) \leq \frac{n}{(1+\mu)|\alpha|},$$

and $\Re(\beta) \leq \Re(\alpha)$.

Proof. In Corollary 2.7, we set $j = 2$ and $M = 1$. □

Theorem 2.4. *Let g_i defined by (2.2) be in the class $\sum_j(p_i)$ for $i \in \{1, 2, ..., n\}$, $n \in \mathbb{N}^*$, $j \in \mathbb{N}_1^*$. If $|g_i(z)| \geq M_i(M_i \geq 1, z \in W)$.*
Then the integral operator $G_{\alpha_i,\beta}$ defined by (1.4) is in the class $\sum$, where $\alpha, \beta \in \mathbb{C}$,

$$\Re(\alpha) \leq \sum_{i=1}^{n} \frac{1}{(1+p_i)M_i|\alpha_i|},$$

and $\Re(\beta) \leq \Re(\alpha)$.

Proof. The proof of this theorem is very similar with the proof of Theorem 2.2. □

Corollary 2.11. *Let g_i defined by (2.2) be in the class $\sum_j(p_i)$ for $i \in \{1, 2, ..., n\}$, $n \in \mathbb{N}^*$, $j \in \mathbb{N}_1^*$. If $|g_i(z)| \geq M(M \geq 1, z \in W)$.*
Then the integral operator $G_{\alpha_i,\beta}$ defined by (1.4) is in the class $\sum$, where $\alpha, \beta \in \mathbb{C}$,

$$\Re(\alpha) \leq \sum_{i=1}^{n} \frac{1}{(1+p_i)M|\alpha_i|},$$

and $\Re(\beta) \leq \Re(\alpha)$.

Proof. In Theorem 2.4, we consider $M_1 = M_2 = ... = M_n = M$. □

Corollary 2.12. *Let g_i defined by (2.2) be in the class $\sum_j(p_i)$ for $i \in \{1, 2, ..., n\}$, $n \in \mathbb{N}^*$, $j \in \mathbb{N}_1^*$. If $|g_i(z)| \geq M(M \geq 1, z \in W)$.*
Then the integral operator $G_{\alpha,\beta}$ defined by (1.4) is in the class $\sum$, where $\alpha, \beta \in \mathbb{C}$,

$$\Re(\alpha) \leq \sum_{i=1}^{n} \frac{1}{(1+p_i)M|\alpha|},$$

and $\Re(\beta) \leq \Re(\alpha)$.

Proof. In Corollary 2.11, we consider $\alpha_1 = \alpha_2 = ... = \alpha_n = \alpha$. □

Corollary 2.13. *Let g_i defined by (2.2) be in the class $\sum_j(p)$ for $i \in \{1, 2, ..., n\}$, $n \in \mathbb{N}^*$, $j \in \mathbb{N}_1^*$. If $|g_i(z)| \geq M(M \geq 1, z \in W)$.*
Then the integral operator $G_{\alpha_i,\beta}$ defined by (1.4) is in the class $\sum$, where $\alpha, \beta \in \mathbb{C}$,

$$\Re(\alpha) \leq \sum_{i=1}^{n} \frac{1}{(1+p)M|\alpha_i|},$$

and $\Re(\beta) \leq \Re(\alpha)$.

Proof. In Corollary 2.11, we consider $p_1 = p_2 = ... = p_n = p$. □

Corollary 2.14. *Let g_i defined by (2.2) be in the class $\sum_j(p)$ for $i \in \{1, 2, ..., n\}$, $n \in \mathbb{N}^*$, $j \in \mathbb{N}_1^*$. If $|g_i(z)| \geq M(M \geq 1, z \in W)$.*
Then the integral operator $G_{\alpha,\beta}$ defined by (1.4) is in the class $\sum$, where $\alpha, \beta \in \mathbb{C}$,

$$\Re(\alpha) \leq \frac{n}{(1+p)M|\alpha|},$$

and $\Re(\beta) \leq \Re(\alpha)$.

Proof. If in Corollary 2.12, we consider $p_1 = p_2 = ... = p_n = p$ or in Corollary 2.13 we consider $\alpha_1 = \alpha_2 = ... = \alpha_n = \alpha$, we get the same result. □

Corollary 2.15. *Let g_i defined by (2.2) be in the class $\sum_2(p)$ for $i \in \{1, 2, ..., n\}$, $n \in \mathbb{N}^*$. If $|g_i(z)| \geq M(M \geq 1, z \in W)$.*
Then the integral operator $G_{\alpha_i,\beta}$ defined by (1.4) is in the class $\sum$, where $\alpha, \beta \in \mathbb{C}$,

$$\Re(\alpha) \leq \sum_{i=1}^{n} \frac{1}{(1+p)M|\alpha_i|},$$

and $\Re(\beta) \leq \Re(\alpha)$.

Proof. In Corollary 2.13, we set $j = 2$. □

Corollary 2.16. *Let g_i defined by (2.2) be in the class $\sum_2(p)$ for $i \in \{1, 2, ..., n\}$, $n \in \mathbb{N}^*$. If $|g_i(z)| \geq M(M \geq 1, z \in W)$.*
Then the integral operator $G_{\alpha,\beta}$ defined by (1.4) is in the class $\sum$, where $\alpha, \beta \in \mathbb{C}$,

$$\Re(\alpha) \leq \frac{n}{(1+p)M|\alpha|},$$

and $\Re(\beta) \leq \Re(\alpha)$.

Proof. In Corollary 2.14, we set $j = 2$. □

Corollary 2.17. *Let g_i defined by (2.2) be in the class $\sum_2(p)$ for $i \in \{1, 2, ..., n\}$, $n \in \mathbb{N}^*$. If $|g_i(z)| \geq 1(z \in W)$.*
Then the integral operator $G_{\alpha,\beta}$ defined by (1.4) is in the class $\sum$, where $\alpha, \beta \in \mathbb{C}$,

$$\Re(\alpha) \leq \frac{n}{(1+p)|\alpha|},$$

and $\Re(\beta) \leq \Re(\alpha)$.

Proof. In Corollary 2.14, we set $j = 2$ and $M = 1$. □

3. Final remarks

The main issue of the class of analytic functions defined on the exterior unit disk is that there are few studies in this branch. In this article, the authors studied some univalent conditions in the subclasses V_j, $V_{j,\mu}$ and $\sum_j(p)$ for analytic functions of an integral operator defined on the exterior of the unit disk, in order to find out if in the exterior unit disk and in the interior unit disk can be applied the same properties.

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