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Coincidence point theorems in some generalized metric spaces

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Dedicated to Prof. Adrian Petruşel on the occasion of his 60^{th} anniversary

Abstract. Let (X, d) be a complete dislocated metric space, (Y, ρ) be a semimetric space and $f, g : X \to Y$ be two mappings. We give some metric conditions which imply that the coincidence point set,

$$C(f, g) := \{x \in X \mid f(x) = g(x)\} \neq \varnothing.$$

Several coincidence point results are obtained for singlevalued and multivalued mappings.

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1. Introduction and preliminaries

Let X be a nonempty set and $d : X \times X \to \mathbb{R}_+$ be a functional. Then the pair (X, d) is called (see [3], [5], [6], ...):

(i) semimetric space, if the following assumptions on d hold:
- (i_1) $d(x, y) = 0 \Leftrightarrow x = y$;
- (i_2) $d(x, y) = d(y, x)$, $\forall\ x, y \in X$.

(ii) dislocated metric space, if the following assumptions on d hold:
- (ii_1) $d(x, y) = d(y, x) = 0 \Rightarrow x = y$;
- (ii_2) $d(x, y) = d(y, x)$, $\forall\ x, y \in X$;

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(ii_3) $d(x,y) \le d(x,z) + d(z,y)$, $\forall\, x,y,z \in X$.

Let (X,d) be a dislocated metric space. By definition (for the standard metric space, see [10]), a mapping $f : X \to X$ is a *pre-weakly Picard mapping* (*pre-WPM*) if the sequence of successive approximations $\{f^n(x)\}_{n\in\mathbb{N}}$ is a convergent sequence, for all $x \in X$.

If $f : X \to X$ is pre-WPM, then we consider the mapping $f^\infty : X \to X$, defined by $f^\infty(x) := \lim\limits_{n\to\infty} f^n(x)$.

By definition, if $f : X \to X$ is pre-WPM with

$$f^\infty(x) \in F_f := \{x \in X \mid f(x) = x\},\ \forall\, x \in X,$$

then f is a *weakly Picard mapping* (*WPM*).

In the paper [10] the author gives some coincidence point results in a metric space. The aim of our paper is to extend some of these results in the case of dislocated metric spaces.

Throughout the paper we shall use the notations and the terminology from [2], [6] and [11].

2. Main results

We start this section with the following notions given in [10].

Let $M \in]0,+\infty]$. A function $\varphi : [0,M[\to [0,M[$ is called *comparison function on* $[0,M[$ if φ is increasing on $[0,M[$ and $\varphi^n(t) \to 0$ as $n \to \infty$, $\forall\, t \in [0,M[$.

Let $\varphi : [0,M[\to [0,M[$ and $\psi : [0,M[\to \mathbb{R}_+$ be two functions. By definition, the pair (φ,ψ) is a *comparison pair on* $[0,M[$ if:

(1) φ is a comparison function on $[0,M[$;
(2) ψ is increasing, $\psi(0) = 0$ and ψ is continuous in 0;
(3) $\sum\limits_{i=0}^{\infty} \psi(\varphi^i(t)) < +\infty$, $\forall\, t \in [0,M[$.

Our main result is the following.

Theorem 2.1. *Let (X,d) be a complete dislocated metric space, (Y,ρ) be a semimetric space, $f,g : X \to Y$ be two mappings, $M \in]0,+\infty]$. We suppose that:*

(1) $X_M := \{x \in X \mid \rho(f(x),g(x)) < M\} \neq \varnothing$;
(2) *The coincidence point displacement functional,*
$$\rho_{f,g} : X_M \to \mathbb{R}_+,\ \rho_{f,g}(x) := \rho(f(x),g(x)), \forall\, x \in X_M,$$
is lower semi-continuous (l.s.c.) on X_M;
(3) *There exists a comparison pair, (φ,ψ), on $[0,M[$ with respect to which, for each $x \in X_M$, there exists $x_1 \in X_M$ such that:*
 (a) $\rho(f(x_1),g(x_1)) \le \varphi(\rho(f(x),g(x)))$;
 (b) $d(x,x_1) \le \psi(\rho(f(x),g(x)))$.

Then there exists a pre-WPM, $h : X_M \to X_M$ such that:

(i) $h^\infty(x) \in C(f,g)$, $\forall\, x \in X_M$;

(*ii*) $d(x, h^\infty(x)) \le \sum_{i=0}^{\infty} \psi(\varphi^i(\rho(f(x), g(x))))$, $\forall\ x \in X_M$.

Proof. From the assumption (3), we can define an operator $h : X_M \to X_M$, by $h(x) = x_1$ such that

$$\rho(f(h(x)), g(h(x))) \le \varphi(\rho(f(x), g(x))),\ \forall\ x \in X_M \tag{2.1}$$

and

$$d(x, h(x)) \le \psi(\rho(f(x), g(x))),\ \forall\ x \in X_M. \tag{2.2}$$

From (2.1), we have

$$\begin{aligned}
\rho(f(h(x)), g(h(x))) &\le \varphi(\rho(f(x), g(x)))\\
\rho(f(h^2(x)), g(h^2(x))) &\le \varphi(\rho(f(h(x)), g(h(x)))) \le \varphi^2(\rho(f(x), g(x)))\\
&\vdots\\
\rho(f(h^n(x)), g(h^n(x))) &\le \varphi(\rho(f(h^{n-1}(x)), g(h^{n-1}(x))))\\
&\le \varphi(\varphi(\rho(f(h^{n-2}(x)), g(h^{n-2}(x)))))\\
&\le \ldots \le \varphi^n(\rho(f(x), g(x))),\ \text{for all } n \in \mathbb{N}^*.
\end{aligned}$$

Notice that since φ is a comparison function, it follows that

$$\rho(f(h^n(x)), g(h^n(x))) \le \varphi^n(\rho(f(x), g(x))) \to 0 \text{ as } n \to \infty,\ \forall\ x \in X_M. \tag{2.3}$$

From (2.2), we have

$$\begin{aligned}
d(x, h(x)) &\le \psi(\rho(f(x), g(x)))\\
d(h(x), h^2(x)) &\le \psi(\rho(f(h(x)), g(h(x)))) \le \psi(\varphi(\rho(f(x), g(x))))\\
&\vdots\\
d(h^n(x), h^{n+1}(x)) &\le \psi(\rho(f(h^n(x)), g(h^n(x))))\\
&\le \psi(\varphi^n(\rho(f(x), g(x)))),\ \text{for all } n \in \mathbb{N}^*.
\end{aligned}$$

Since (φ, ψ) is a comparison pair on $[0, M[$, we have

$$\sum_{n\in\mathbb{N}} d(h^n(x), h^{n+1}(x)) \le \sum_{n\in\mathbb{N}} \psi(\varphi^n(\rho(f(x), g(x)))) < +\infty. \tag{2.4}$$

This implies that $\{h^n(x)\}_{n\in\mathbb{N}}$ is a Cauchy sequence in (X, d). Since (X, d) is a complete dislocated metric space, it follows that $\{h^n(x)\}_{n\in\mathbb{N}}$ is convergent in (X, d), for all $x \in X_M$. So, h is a pre-WPM. Thus, $h^\infty(x) := \lim_{n\to\infty} h^n(x)$.

On the other hand, from the assumption (2) and by (2.3), we have

$$0 \le \rho(f(h^\infty(x)), g(h^\infty(x))) \le \underline{\lim}_{n\to\infty} \rho(f(h^n(x)), g(h^n(x))) = 0.$$

Since ρ is a semimetric, we get $f(h^\infty(x)) = g(h^\infty(x))$, i.e., $h^\infty(x) \in C(f, g), \forall\ x \in X_M$. Since d satisfies the triangle inequality and taking into account the first inequality of

(2.4), we have

$$d(x,h^{\infty}(x)) \leq \sum_{i=0}^{\infty} d(h^i(x),h^{i+1}(x)) \leq \sum_{i=0}^{\infty} \psi(\varphi^i(\rho(f(x),g(x)))),\ \forall\ x \in X_M. \quad \square$$

Remark 2.2. In general, a semimetric is not continuous (see L.M. Blumenthal [3, p. 9]). That is why we have considered the assumption (2) in the above theorem. It would be of great interest to find conditions that imply the lower semi-continuity of the coincidence point displacement functional $\rho_{f,g}$.

Remark 2.3. In Theorem 2.1, if we consider $Y := X$, $g := 1_X$, $\varphi(t) := lt$, where $0 < l < 1$ and $\psi(t) := kt$, with $k > 0$, for all $t \in [0, M[$, we obtain the following result:

Theorem 2.4. *Let (X,d) be a complete dislocated metric space, ρ be a semimetric on X and $f : X \to X$ be a mapping. We suppose that:*

(2′) *The coincidence point displacement functional,*
$\rho_f : (X,d) \to \mathbb{R}_+$, $\rho_f(x) := \rho(x, f(x))$, $\forall\ x \in X$, *is l.s.c. on* X*;*

(3′) *There exists $0 < l < 1$ and $k > 0$ w.r.t. which, for each $x \in X$, there exists $x_1 \in X$ such that:*
- (a′) $\rho(x_1, f(x_1)) \leq l\rho(x, f(x))$;
- (b′) $d(x, x_1) \leq k\rho(x, f(x))$.

Then there exists a pre-WPM, $h : (X,d) \to (X,d)$ such that:

(i′) $h^{\infty}(x) \in F_f$, $\forall\ x \in X$, *i.e.*, $F_f \neq \varnothing$;

(ii′) $d(x, h^{\infty}(x)) \leq \frac{k}{1-l}\rho(x, f(x))$, $\forall\ x \in X$.

Remark 2.5. In the context of Theorem 2.4, the triple $(X, \stackrel{d}{\to}, \rho)$ is a Kasahara space. Several results given in [5] can be proved using this theorem.

Remark 2.6. If in Theorem 2.4 we take, $\rho := d$ and f an l-graphic contraction, then we have:

Theorem 2.7. *Let (X,d) be a complete dislocated metric space and $f : X \to X$ be an l-graphic contraction. If the coincidence point displacement functional, $d_f : X \to \mathbb{R}_+$, $x \mapsto d(x, f(x))$ is l.s.c. on X, then f is a WPM and*

$$d(x, f^{\infty}(x)) \leq \frac{1}{1-l} d(x, f(x)),\ \forall\ x \in X.$$

Proof. We apply Theorem 2.4, by considering $h(x) := f(x)$, for all $x \in X$. $\square$

Remark 2.8. If in Theorem 2.4 we take, $\rho := d$ and f an l-contraction, then we have the following variant of contraction principle:

Theorem 2.9. *Let (X,d) be a complete dislocated metric space and $f : X \to X$ be an l-contraction. Then we have that:*

(i) $F_f = F_{f^n} = \{x^*\}$, $\forall\ n \in \mathbb{N}^*$;

(ii) $f^n(x) \to x^*$ *as* $n \to \infty$, $\forall\ x \in X$;

(iii) $d(x, x^*) \leq \frac{1}{1-l} d(x, f(x))$, $\forall\ x \in X$.

Remark 2.10. For similar results given in a metric space, see: [4], [10], [11], [1], [7].

3. The case of multivalued mappings

Throughout this section we follow the notations and terminology given in [9] and [10]. We will use in our result the gap functional between two sets, recalled bellow.

Let (X, d) be a dislocated metric space.

The functional $D : P_{cl}(X) \times P_{cl}(X) \to \mathbb{R}_+ \cup \{+\infty\}$, defined by

$$D(A, B) := inf\{d(a, b) \mid a \in A,\ b \in B\}$$

for all $A, B \in P_{cl}(X)$, is called the gap functional between the sets A and B.

For this operator we have the following property:

If $A \in P_{cl}(X)$ and $x \in X$ then $D(x, A) = 0 \Leftrightarrow x \in A$.

The basic result of this section is the following:

Theorem 3.1. *Let (X, d) be a complete dislocated metric space, (Y, ρ) be a semimetric space, $T, S : X \to P_{cl}(Y)$ be two multivalued mappings, $M \in]0, +\infty]$, (φ, ψ) be a comparison pair on $[0, M[$. We suppose that:*

(1) $X_M := \{x \in X \mid D(T(x), S(x)) < M\} \neq \varnothing$;

(2) *The D-coincidence point displacement functional,* $D_{T,S} : X_M \to \mathbb{R}_+$,

$$D_{T,S}(x) := D(T(x), S(x)),\ \forall\ x \in X_M,$$

is l.s.c. on X_M;

(3) *For each $x \in X_M$ there exists $x_1 \in X_M$ such that:*

(a) $D(T(x_1), S(x_1)) \leq \varphi(D(T(x), S(x)))$;

(b) $d(x, x_1) \leq \psi(D(T(x), S(x)))$.

Then there exists a pre-WPM, $h : X_M \to X_M$ such that:

(i) $D(T(h^\infty(x)), S(h^\infty(x))) = 0$, $\forall\ x \in X_M$;

(ii) $d(x, h^\infty(x)) \leq \sum_{i=0}^{\infty} \psi(\varphi^i(D(T(x), S(x))))$, $\forall\ x \in X_M$;

(iii) *If in addition, for $A, B \in P_{cl}(Y)$, $D(A, B) = 0$ implies that:*

(iii_1) $A \cap B \neq \varnothing$,

then, $C(T, S) := \{x \in X \mid T(x) \cap S(x) \neq \varnothing\} \neq \varnothing$;

(iii_2) $A = B$,

then, $C(T, S) \neq \varnothing$ *and* $T(h^\infty(x)) = S(h^\infty(x))$, $\forall\ x \in X_M$;

(iii_3) $A = B = \{y^*\}$,

then $C(T, S) \neq \varnothing$ *and* $T(h^\infty(x)) = S(h^\infty(x)) = \{y^*_x\}$.

Proof. If we take, $h(x) := x_1$, then we have that:

$$D(T(h(x)), S(h(x))) \leq \varphi(D(T(x), S(x))),\ \forall\ x \in X_M,$$

and

$$d(x, h(x)) \leq \psi(D(T(x), S(x))),\ \forall\ x \in X_M.$$

These imply that,

$$D(T(h^n(x)), S(h^n(x))) \to 0 \text{ as } n \to \infty,$$

and h is a pre-WPM, and

$$d(x, h^{\infty}(x)) \leq \sum_{i=0}^{\infty} \psi(\varphi^i(D(T(x), S(x)))),\ \forall\, x \in X_M.$$

Since, $D_{T,S}$ is l.s.c., it follows that,

$$\begin{aligned} 0 &\leq D(T(h^{\infty}(x)), S(h^{\infty}(x))) \leq \underline{\lim}_{n\to\infty} D(T(h^n(x)), S(h^n(x))) \\ &= \lim_{n\to\infty} D(T(h^n(x)), S(h^n(x))) = 0. \end{aligned}$$

So, we have the conclusions (i), (ii) and (iii). □

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