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Coupled fixed point theorems for Zamfirescu type operators in ordered generalized Kasahara spaces

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Dedicated to Professor Ioan A. Rus on the occasion of his 80th anniversary

Abstract. In this paper we give some coupled fixed point theorems for Zamfirescu type operators in ordered generalized Kasahara spaces $(X, \to, d, \leq)$, where $d : X \times X \to \mathbb{R}^m_+$ is a premetric. An application concerning the existence and uniqueness of solutions for systems of functional-integral equations is also given.

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1. Introduction and preliminaries

Many coupled fixed point results were given in the context of complete generalized metric spaces, for generalized contraction mappings. If we carefully examine their proofs by the iteration method, we can see that in some cases, not all of the metric properties are essentials. We give here some coupled fixed point theorems and applications in a more general setting, the so called generalized Kasahara space.

We recall first the notion of L-space, given by M. Fréchet in [4].

Definition 1.1. *Let X be a nonempty set. Let*

$$s(X) := \{(x_n)_{n\in\mathbb{N}} \mid x_n \in X,\ n \in \mathbb{N}\}.$$

Let $c(X)$ be a subset of $s(x)$ and $Lim : c(X) \to X$ be an operator. By definition the triple $(X, c(X), Lim)$ is called an L-space (denoted by $(X, \to)$) if the following conditions are satisfied:

(i) if $x_n = x$, for all $n \in \mathbb{N}$, then $(x_n)_{n\in\mathbb{N}} \in c(X)$ and $Lim(x_n)_{n\in\mathbb{N}} = x$.

(ii) if $(x_n)_{n\in\mathbb{N}} \in c(X)$ and $Lim(x_n)_{n\in\mathbb{N}} = x$, then for all subsequences $(x_{n_i})_{i\in\mathbb{N}}$ of $(x_n)_{n\in\mathbb{N}}$ we have that $(x_{n_i})_{i\in\mathbb{N}} \in c(X)$ and

$$Lim(x_{n_i})_{i\in\mathbb{N}} = x.$$

Remark 1.2. For examples and more considerations on L-spaces, see I.A. Rus, A. Petruşel and G. Petruşel [10, pp.77-80].

The notion of generalized Kasahara space was introduced by I.A. Rus in [9] as follows:

Definition 1.3. *Let $(X, \to)$ be an L-space, $(G, +, \leq, \xrightarrow{G})$ be an L-space ordered semigroup with unity, 0 be the least element in $(G, \leq)$ and $d_G : X \times X \to G$ be an operator. The triple $(X, \to, d_G)$ is called a generalized Kasahara space if and only if the following compatibility condition between $\to$ and d_G holds:*

$$\text{for all } (x_n)_{n\in\mathbb{N}} \subset X \text{ with } \sum_{n\in\mathbb{N}} d_G(x_n, x_{n+1}) < +\infty$$
$$\Rightarrow (x_n)_{n\in\mathbb{N}} \text{ is convergent in } (X, \to). \tag{1.1}$$

Remark 1.4. Notice that by the inequality with the symbol $+\infty$ in the compatibility condition (1.1), we understand that the series $\sum_{n\in\mathbb{N}} d_G(x_n, x_{n+1})$ is bounded in $(G, \leq)$.

Remark 1.5. In the context of generalized Kasahara spaces, fixed point results for self generalized contractions were already given by S. Kasahara in [5], for the case when $G = \mathbb{R}_+ \cup \{+\infty\}$ and by I.A. Rus in [9], for the case when $G = \mathbb{R}^m_+$.

An example of generalized Kasahara space is the following one:

Example 1.6 (I.A. Rus, [9]**).** Let $\rho : X \times X \to \mathbb{R}^m_+$ be a generalized complete metric on a set X. Let $x_0 \in X$ and $\lambda \in \mathbb{R}^m_+$ with $\lambda \neq 0$. Let $d_\lambda : X \times X \to \mathbb{R}^m_+$ be defined by

$$d_\lambda(x, y) = \begin{cases} \rho(x, y) & , \text{ if } x \neq x_0 \text{ and } y \neq x_0, \\ \lambda & , \text{ if } x = x_0 \text{ or } y = x_0. \end{cases}$$

Then $(X, \xrightarrow{\rho}, d_\lambda)$ is a generalized Kasahara space.

We recall also a very useful tool which helps us to prove the uniqueness of the fixed point for operators defined on generalized Kasahara spaces.

Lemma 1.7 (Kasahara's lemma [5]**).** *Let $(X, \to, d_G)$ be a generalized Kasahara space. Then $d_G(x, y) = d_G(y, x) = 0$ implies $x = y$, for all $x, y \in X$.*

Remark 1.8. For more considerations on Kasahara spaces, see [3] and [9].

We introduce now the notion of ordered generalized Kasahara space.

Definition 1.9. *Let $(X, \to, d_G)$ be a generalized Kasahara space. Then $(X, \to, d_G, \leq)$ is an ordered generalized Kasahara space if and only if $(X, \leq)$ is a partially ordered set.*

Example 1.10. Let $X := C([a, b], \mathbb{R}^m) = \{x : [a, b] \to \mathbb{R}^m \mid x \text{ is continuous on } [a, b]\}$ be endowed with the partial order relation

$$x \leq_C y \Leftrightarrow x(t) \leq y(t) \Leftrightarrow x_i(t) \leq y_i(t), \text{ for all } t \in [a, b],\ i = \overline{1, m}.$$

We consider $\xrightarrow{\rho}$, the convergence structure induced by the Cebîşev norm

$$\rho : C([a, b], \mathbb{R}^m) \times C([a, b], \mathbb{R}^m) \to \mathbb{R}^m_+,$$

defined by

$$\rho(x,y)=\|x-y\|_C=\max_{t\in[a,b]}|x(t)-y(t)|=\begin{pmatrix}\max\limits_{t\in[a,b]}|x_1(t)-y_1(t)|\\ \vdots\\ \max\limits_{t\in[a,b]}|x_m(t)-y_m(t)|\end{pmatrix}.$$

Let $d: C([a,b],\mathbb{R}^m)\times C([a,b],\mathbb{R}^m)\to\mathbb{R}^m_+$, defined by

$$d(x,y)=\|x-y\|_C+\|(x-y)^p\|_C=\max_{t\in[a,b]}|x(t)-y(t)|+\max_{t\in[a,b]}|x(t)-y(t)|^p$$

$$=\begin{pmatrix}\max\limits_{t\in[a,b]}|x_1(t)-y_1(t)|+\max\limits_{t\in[a,b]}|x_1(t)-y_1(t)|^p\\ \vdots\\ \max\limits_{t\in[a,b]}|x_m(t)-y_m(t)|+\max\limits_{t\in[a,b]}|x_m(t)-y_m(t)|^p\end{pmatrix},$$

where $p\in\mathbb{N}$, $p\geq 2$.
Since $\rho(x,y)\leq d(x,y)$, for all $x,y\in C([a,b],\mathbb{R}^m)$ we get that $(C([a,b],\mathbb{R}^m),\xrightarrow{\rho},d,\leq_C)$ is an ordered generalized Kasahara space. (See also I.A. Rus, [9]).

Let $(X,\to,d_G,\leq)$ be an ordered generalized Kasahara space. Then we define

$$X_{\leq}:=\{(x_1,x_2)\in X\times X \mid x_1\leq x_2 \text{ or } x_2\leq x_1\}.$$

In the above setting, if $f: X\to X$ is an operator, then the Cartesian product of f with itself is

$$f\times f: X\times X\to X\times X, \text{ given by } (f\times f)(x_1,x_2):=(f(x_1),f(x_2)).$$

In this paper, we consider the ordered generalized Kasahara space $(X,\to,d,\leq)$, where $d: X\times X\to\mathbb{R}^m_+$ is a premetric, i.e., $d(x,x)=0$, for all $x\in X$ and $d(x,y)\leq d(x,z)+d(z,y)$, for all $x,y,z\in X$.

We mention that if $\alpha,\beta\in\mathbb{R}^m$, $\alpha=(\alpha_1,\alpha_2,\ldots,\alpha_m)$, $\beta=(\beta_1,\beta_2,\ldots,\beta_m)$ and $c\in\mathbb{R}$, then by $\alpha\leq\beta$ (respectively $\alpha<\beta$), we mean that $\alpha_i\leq\beta_i$ (respectively $\alpha_i<\beta_i$), for all $i=\overline{1,m}$ and by $\alpha\leq c$ we mean that $\alpha_i\leq c$, for all $i=\overline{1,m}$.

We denote by $\mathcal{M}_{m,m}(\mathbb{R}_+)$ the set of all $m\times m$ matrices with positive elements, by O_m the zero $m\times m$ matrix and by I_m the identity $m\times m$ matrix. If $A=(a_{ij})_{i,j=\overline{1,m}}$, $B=(b_{ij})_{i,j=\overline{1,m}}\in\mathcal{M}_{m,m}(\mathbb{R}_+)$, then by $A\leq B$ we understand $a_{ij}\leq b_{ij}$, for all $i,j=\overline{1,m}$. The symbol A^τ stands for the transpose of the matrix A. Notice also that, for the sake of simplicity, we will make an identification between row and column vectors in $\mathbb{R}^m$.

A matrix $A\in\mathcal{M}_{m,m}(\mathbb{R}_+)$ is said to be convergent to zero if and only if $A^n\to O_m$ as $n\to\infty$ (see [10]). Regarding this class of matrices we have the following classical result in matrix analysis (see [1, Lemma 3.3.1, page 55], [11], [8, page 37], [13, page 12].

Theorem 1.11. *Let $A\in\mathcal{M}_{m,m}(\mathbb{R}_+)$. The following statements are equivalent:*

(i) A is convergent to zero;
(ii) $A^n\to O_m$ as $n\to\infty$;

(*iii*) *the eigenvalues of* A *lies in the open unit disc, i.e.,* $|\lambda| < 1$, *for all* $\lambda \in \mathbb{C}$ *with* $det(A - \lambda I_m) = 0$;

(*iv*) *the matrix* $I_m - A$ *is non-singular and*

$$(I_m - A)^{-1} = I_m + A + A^2 + \ldots + A^n + \ldots ;$$

(*v*) *the matrix* $(I_m - A)$ *is non-singular and* $(I_m - A)^{-1}$ *has nonnegative elements;*

(*vi*) $A^n q \to 0 \in \mathbb{R}^m$ *and* $q^\tau A^n \to 0 \in \mathbb{R}^m$ *as* $n \to \infty$, *for all* $q \in \mathbb{R}^m$.

Remark 1.12. Some examples of matrices which converge to zero are:

a) any matrix $A := \begin{pmatrix} a & a \\ b & b \end{pmatrix}$, where $a, b \in \mathbb{R}_+$ and $a + b < 1$;

b) any matrix $A := \begin{pmatrix} a & b \\ a & b \end{pmatrix}$, where $a, b \in \mathbb{R}_+$ and $a + b < 1$;

c) any matrix $A := \begin{pmatrix} a & b \\ 0 & c \end{pmatrix}$, where $a, b, c \in \mathbb{R}_+$ and $\max\{a, c\} < 1$.

We consider now the following particular matrix set:

$$\mathcal{M}^{\Delta}_{m,m}(\mathbb{R}_+) := \left\{ \begin{pmatrix} a_{11} & a_{12} & a_{13} & \ldots & a_{1m} \\ 0 & a_{22} & a_{23} & \ldots & a_{2m} \\ 0 & 0 & a_{33} & \ldots & a_{3m} \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \ldots & a_{mm} \end{pmatrix} \in \mathcal{M}_{m,m}(\mathbb{R}_+) \;\middle|\; \max_{i=\overline{1,m}} a_{ii} < \frac{1}{2} \right\}.$$

Lemma 1.13. *Let* $A \in \mathcal{M}^{\Delta}_{m,m}(\mathbb{R}_+)$. *Then the matrices* A *and* $(I_m - A)^{-1}A$ *are convergent to zero.*

Proof. Since the eigenvalues of A and $(I_m - A)^{-1}A$ are in the open unit disk, the conclusion follows from Theorem 1.11. □

Remark 1.14. For more considerations on matrices which converge to zero, see [6], [8] and [12].

Let $(X, \to)$ be an L-space and $f : X \to X$ be an operator. The following notations and notions will be needed in the sequel of this paper:

- $Fix(f) := \{x \in X \mid x = f(x)\}$ the set of all fixed points for f.
- $I(f) := \{Y \subset X \mid f(Y) \subset Y\}$ - the set of all invariant subsets of X with respect to f.
- $Graph(f) := \{(x, y) \in X \times X \mid y = f(x)\}$ the graph of f. We say that f has closed graph with respect to $\to$ or $Graph(f)$ is closed in $X \times X$ with respect to $\to$ if and only if for any sequences $(x_n)_{n\in\mathbb{N}} \subset X$, $(y_n)_{n\in\mathbb{N}} \subset X$ with $y_n = f(x_n)$ for all $n \in \mathbb{N}$ and $x_n \to x \in X$, $y_n \to y \in X$, as $n \to \infty$, we have that $y = f(x)$.
- A sequence $(x_n)_{n\in\mathbb{N}} \subset X$ is called sequence of successive approximations for f starting from a given point $x_0 \in X$ if $x_{n+1} = f(x_n)$, for all $n \in \mathbb{N}$. Notice that $x_n = f^n(x_0)$, for all $n \in \mathbb{N}$.

2. Main results

Our first main result is the following one:

Theorem 2.1. *Let $(X, \to, d, \leq)$ be an ordered generalized Kasahara space, where $d : X \times X \to \mathbb{R}^m_+$ is a premetric, i.e., $d(x,x) = 0$ and $d(x,y) \leq d(x,z) + d(z,y)$ for all $x, y, z \in X$. Let $f : X \to X$ be an operator. We assume that:*

(i) *for each $(x,y) \in X_\leq$, there exists $z_{(x,y)} := z \in X$ such that (x,z),$(y,z) \in X_\leq$;*
(ii) *for each $(x,y) \in X_\leq$, we have $(x, f(x))$, $(y, f(y)) \in X_\leq$;*
(iii) *$X_\leq \in I(f \times f)$;*
(iv) *$f : (X, \to) \to (X, \to)$ has closed graph;*
(v) *f is a Zamfirescu type operator, i.e., at least one of the following conditions holds:*
  (v_1) there exists $A \in \mathcal{M}_{m,m}(\mathbb{R}_+)$ which converges to zero such that
$$d(f(x), f(y)) \leq Ad(x,y), \text{ for all } (x,y) \in X_\leq$$
  (v_2) there exists $B \in \mathcal{M}^{\Delta}_{m,m}(\mathbb{R}_+)$ such that
$$d(f(x), f(y)) \leq B[d(x, f(x)) + d(y, f(y))], \text{ for all } (x,y) \in X_\leq$$
  (v_3) there exists $C \in \mathcal{M}^{\Delta}_{m,m}(\mathbb{R}_+)$ such that
$$d(f(x), f(y)) \leq C[d(x, f(y)) + d(y, f(x))], \text{ for all } (x,y) \in X_\leq$$
(vi) *there exists $x_0 \in X$ such that $(x_0, f(x_0)) \in X_\leq$.*

Then $f : (X, \to) \to (X, \to)$ is a Picard operator.

Proof. Let $x \in X$ be arbitrary.

Since $(x_0, f(x_0)) \in X_\leq$, by (iii) we have $(f(x_0), f^2(x_0)) \in X_\leq$.

If f satisfies (v_1) then
$$d(f(x_0), f^2(x_0)) \leq Ad(x_0, f(x_0)).$$

If f satisfies (v_2) then
$$\begin{aligned} d(f(x_0), f^2(x_0)) &\leq B[d(x_0, f(x_0)) + d(f(x_0), f^2(x_0))], \\ \text{i.e., } d(f(x_0), f^2(x_0)) &\leq (I_m - B)^{-1} Bd(x_0, f(x_0)). \end{aligned}$$

If f satisfies (v_3) then
$$\begin{aligned} d(f(x_0), f^2(x_0)) &\leq C[d(x_0, f^2(x_0)) + d(f(x_0), f(x_0))] \\ &\leq C[d(x_0, f(x_0)) + d(f(x_0), f^2(x_0))], \\ \text{i.e., } d(f(x_0), f^2(x_0)) &\leq (I_m - C)^{-1} Cd(x_0, f(x_0)). \end{aligned}$$

Let $\Omega := \{A, (I_m - B)^{-1}B, (I_m - C)^{-1}C\}$. For any matrix $M \in \Omega$, we have $M \in \mathcal{M}_{m,m}(\mathbb{R}_+)$ and by Lemma 1.13, it follows that M is a matrix that converges to zero. In addition, we have
$$d(f(x_0), f^2(x_0)) \leq Md(x_0, f(x_0)), \text{ for all } (x_0, f(x_0)) \in X_\leq \text{ and all } M \in \Omega.$$

Now, since $(f(x_0), f^2(x_0)) \in X_\leq$, by (iii) it follows that $(f^2(x_0), f^3(x_0)) \in X_\leq$.

If f satisfies (v_1) then

$$d(f^2(x_0), f^3(x_0)) \leq Ad(f(x_0), f^2(x_0)) \leq A^2 d(x_0, f(x_0)).$$

If f satisfies (v_2) then

$$\begin{aligned} d(f^2(x_0), f^3(x_0)) &\leq B[d(f(x_0), f^2(x_0)) + d(f^2(x_0), f^3(x_0))], \\ \text{i.e., } d(f^2(x_0), f^3(x_0)) &\leq (I_m - B)^{-1} Bd(f(x_0), f^2(x_0)) \\ &\leq [(I_m - B)^{-1}B]^2 d(x_0, f(x_0)). \end{aligned}$$

If f satisfies (v_3) then

$$\begin{aligned} d(f^2(x_0), f^3(x_0)) &\leq C[d(f(x_0), f^3(x_0)) + d(f^2(x_0), f^2(x_0))] \\ &\leq C[d(f(x_0), f^2(x_0)) + d(f^2(x_0), f^3(x_0))], \\ \text{i.e., } d(f^2(x_0), f^3(x_0)) &\leq (I_m - C)^{-1} Cd(f(x_0), f^2(x_0)) \\ &\leq [(I_m - C)^{-1}C]^2 d(x_0, f(x_0)). \end{aligned}$$

In all three cases presented above, we conclude that

$$d(f^2(x_0), f^3(x_0)) \leq M^2 d(x_0, f(x_0))$$

for all $(x_0, f(x_0)) \in X_{\leq}$ and all $M \in \Omega$.

By induction, for $n \in \mathbb{N}$, we get

$$d(f^n(x_0), f^{n+1}(x_0)) \leq M^n d(x_0, f(x_0))$$

for all $(x_0, f(x_0)) \in X_{\leq}$ and all $M \in \Omega$.

Next, we obtain

$$\begin{aligned} \sum_{n \in \mathbb{N}} d(f^n(x_0), f^{n+1}(x_0)) &\leq \sum_{n \in \mathbb{N}} M^n d(x_0, f(x_0)) \\ &= (I_m - M)^{-1} d(x_0, f(x_0)) < +\infty \end{aligned}$$

for all $(x_0, f(x_0)) \in X_{\leq}$ and all $M \in \Omega$.

Since $(X, \to, d)$ is a generalized Kasahara space, we get that the sequence of successive approximations for f, starting from x_0, is convergent in $(X, \to)$. So, there exists $x^* \in X$ such that $f^n(x_0) \to x^*$ as $n \to \infty$. By (iv) we get that $x^* \in Fix(f)$.

Notice also that:

• If $(x, x_0) \in X_{\leq}$ then by (iii) we have $(f^n(x), f^n(x_0)) \in X_{\leq}$ and by (ii) that $(x, f(x))$, $(y, f(y)) \in X_{\leq}$.

If f satisfies (v_1) then

$$\begin{aligned} 0 &\leq d(f^n(x), f^n(x_0)) + d(f^n(x_0), f^n(x)) \\ &\leq Ad(f^{n-1}(x), f^{n-1}(x_0)) + Ad(f^{n-1}(x_0), f^{n-1}(x)) \\ &\leq \ldots \leq A^n d(x, x_0) + A^n d(x_0, x) \overset{\mathbb{R}^m_+}{\to} 0 \text{ as } n \to \infty. \end{aligned}$$

If f satisfies (v_2) then

$$\begin{aligned}
0 &\le d(f^n(x), f^n(x_0)) + d(f^n(x_0), f^n(x)) \\
&\le 2B[d(f^{n-1}(x), f^n(x)) + d(f^{n-1}(x_0), f^n(x_0))] \\
&\le 2B[(I_m - B)^{-1}B]^{n-1}[d(x, f(x)) + d(x_0, f(x_0))] \\
&\le 2(I_m + B + B^2 + \ldots)B[(I_m - B)^{-1}B]^{n-1}[d(x, f(x)) + d(x_0, f(x_0))] \\
&= 2[(I_m - B)^{-1}B]^n[d(x, f(x)) + d(x_0, f(x_0))] \overset{\mathbb{R}^m_+}{\to} 0 \text{ as } n \to \infty.
\end{aligned}$$

If f satisfies (v_3) then

$$\begin{aligned}
0 &\le d(f^n(x), f^n(x_0)) + d(f^n(x_0), f^n(x)) \\
&\le 2C[d(f^{n-1}(x), f^n(x_0)) + d(f^{n-1}(x_0), f^n(x))] \\
&\le 2C[d(f^{n-1}(x), f^n(x)) + d(f^n(x), f^n(x_0)) \\
&\quad + d(f^{n-1}(x_0), f^n(x_0)) + d(f^n(x_0), f^n(x))],
\end{aligned}$$

$$\begin{aligned}
\text{i.e., } 0 &\le d(f^n(x), f^n(x_0)) + d(f^n(x_0), f^n(x)) \\
&\le (I_m - 2C)^{-1}2C[d(f^{n-1}(x), f^n(x)) + d(f^{n-1}(x_0), f^n(x_0))] \\
&\le (I_m - 2C)^{-1}2C[(I_m - C)^{-1}C]^{n-1}[d(x, f(x)) + d(x_0, f(x_0))] \\
&\le (I_m - 2C)^{-1}2[(I_m - C)^{-1}C]^n[d(x, f(x)) + d(x_0, f(x_0))] \overset{\mathbb{R}^m_+}{\to} 0 \\
&\quad \text{as } n \to \infty.
\end{aligned}$$

In all three cases we get that $d(f^n(x), f^n(x_0)) = d(f^n(x_0), f^n(x)) = 0$. By Kasahara's lemma 1.7, it follows that $f^n(x) = f^n(x_0)$, for all $n \in \mathbb{N}$.

• If $(x, x_0) \notin X_\le$, then by (i), there exists $z_{(x,x_0)} := z \in X$ such that (x, z), $(x_0, z) \in X_\le$. Since $(x, z) \in X_\le$, by (iii) we have $(f^n(x), f^n(z)) \in X_\le$ and by (ii) that $(x, f(x))$, $(z, f(z)) \in X_\le$. In a similar way as presented above, we obtain $f^n(x) = f^n(z)$, for all $n \in \mathbb{N}$. On the other hand, since $(x_0, z) \in X_\le$ we get that $f^n(x_0) = f^n(z)$, for all $n \in \mathbb{N}$. Hence $f^n(x) = f^n(x_0) \to x^*$ as $n \to \infty$.

We show next the uniqueness of the fixed point x^*.

Let $y^* \in Fix(f)$ such that $y^* \neq x^*$.

If $(x^*, y^*) \in X_\le$, then by (iii) we have $(f^n(x^*), f^n(y^*)) \in X_\le$ and by (ii) that $(x^*, f(x^*))$, $(y^*, f(y^*)) \in X_\le$.

If f satisfies (v_1) then we have:

$$\begin{aligned}
&0 \le d(f(x^*), f(y^*)) + d(f(y^*), f(x^*)) \le Ad(x^*, y^*) + Ad(y^*, x^*), \\
&\text{i.e., } 0 \le d(x^*, y^*) + d(y^*, x^*) \le (I_m - A)^{-1}0 = 0.
\end{aligned}$$

If f satisfies (v_2) then we have:

$$\begin{aligned}
&0 \le d(f(x^*), f(y^*)) + d(f(y^*), f(x^*)) \le 2B[d(x^*, f(x^*)) + d(y^*, f(y^*))], \\
&\text{i.e., } 0 \le d(x^*, y^*) + d(y^*, x^*) \le 2B[d(x^*, x^*) + d(y^*, y^*)] = 0.
\end{aligned}$$

If f satisfies (v_3) then we have:

$$\begin{aligned}0 &\leq d(f(x^*), f(y^*)) + d(f(y^*), f(x^*))\\ &\leq 2C[d(x^*, f(y^*)) + d(y^*, f(x^*))] = 2C[d(x^*, y^*) + d(y^*, x^*)],\end{aligned}$$

i.e., $0 \leq d(x^*, y^*) + d(y^*, x^*) \leq (I_m - 2C)^{-1}0 = 0$.

So, in all three cases, we conclude that $d(x^*, y^*) = d(y^*, x^*) = 0$. By Kasahara's lemma 1.7, it follows that $x^* = y^*$.

If $(x^*, y^*) \notin X_{\leq}$, then by (i), there exists $z_{(x^*,y^*)} := z \in X$ such that (x^*, z), $(y^*, z) \in X_{\leq}$. Since $(x^*, z) \in X_{\leq}$, by following the same way of proof as presented above, replacing y^* with z, we get that $x^* = z$. On the other hand, since $(y^*, z) \in X_{\leq}$, we get in a similar way that $y^* = z$. Hence $x^* = y^*$. □

In the sequel, we will apply the above result to the coupled fixed point problem generated by an operator.

Let X be a nonempty set, endowed with a partial order relation denoted by $\leq$. If we consider two arbitrary elements $z := (x, y)$, $w = (u, v)$ of $X \times X$, then, we can introduce a partial ordering relation on $X \times X$, denoted by $\preceq$ and defined as follows:

$$z \preceq w \text{ if and only if } (x \geq u \text{ and } y \leq v).$$

Theorem 2.2. *Let $(X, \to, d, \leq)$ be an ordered Kasahara space, where $d : X \times X \to \mathbb{R}_+$ is a functional, satisfying the following conditions: $d(x, x) = 0$, for all $x \in X$ and $d(x, y) \leq d(x, z) + d(z, y)$, for all $x, y, z \in X$.*

Let $S : X \times X \to X$ be an operator. We suppose that:

(i) *for each $z = (x, y)$, $w = (u, v) \in X \times X$, which are not comparable with respect to the partial ordering $\preceq$ in $X \times X$, there exists $t := (t_1, t_2) \in X \times X$, which may depend on (x, y) and (u, v), such that t is comparable with respect to the partial ordering $\preceq$, with both z and w;*

(ii) *for each $x = (x_1, x_2)$, $y = (y_1, y_2) \in X \times X$, with $(x_1 \geq y_1$ and $x_2 \leq y_2)$ or $(y_1 \geq x_1$ and $y_2 \leq x_2)$ we have*

$$\left(\begin{cases} x_1 \geq S(x_1, x_2) \\ x_2 \leq S(x_2, x_1) \end{cases} \text{ or } \begin{cases} S(x_1, x_2) \geq x_1 \\ S(x_2, x_1) \leq x_2 \end{cases}\right)$$

and

$$\left(\begin{cases} y_1 \geq S(y_1, y_2) \\ y_2 \leq S(y_2, y_1) \end{cases} \text{ or } \begin{cases} S(y_1, y_2) \geq y_1 \\ S(y_2, y_1) \leq y_2 \end{cases}\right)$$

(iii) *for all $(x \geq u$ and $y \leq v)$ or $(u \geq x$ and $v \leq y)$, we have*

$$\begin{cases} S(x, y) \geq S(u, v) \\ S(y, x) \leq S(v, u) \end{cases} \text{ or } \begin{cases} S(u, v) \geq S(x, y) \\ S(v, u) \leq S(y, x) \end{cases}$$

i.e., S has the generalized mixed monotone property;

(iv) *$S : X \times X \to X$ has closed graph with respect to $\to$;*

(v) *at least one of the following conditions holds:*

(v_1) *there exists* $k_1, k_2 \in \mathbb{R}_+$, $k_1 + k_2 < 1$ *such that*

$$d(S(x,y), S(u,v)) \leq k_1 d(x,u) + k_2 d(y,v)$$

(v_2) *there exists* $k \in [0, \frac{1}{2}[$ *such that*

$$d(S(x,y), S(u,v)) \leq k[d(x, S(x,y)) + d(u, S(u,v))]$$

(v_3) *there exists* $k \in [0, \frac{1}{2}[$ *such that*

$$d(S(x,y), S(u,v)) \leq k[d(x, S(u,v)) + d(u, S(x,y))]$$

(vi) *there exists* $z_0 := (z_0^1, z_0^2) \in X \times X$ *such that*

$$\begin{cases} z_0^1 \geq S(z_0^1, z_0^2) \\ z_0^2 \leq S(z_0^2, z_0^1) \end{cases} \quad or \quad \begin{cases} S(z_0^1, z_0^2) \geq z_0^1 \\ S(z_0^2, z_0^1) \leq z_0^2 \end{cases}.$$

Then there exists a unique element $(x^*, y^*) \in X \times X$ *such that* $x^* = S(x^*, y^*)$ *and* $y^* = S(y^*, x^*)$ *and the sequence of successive approximations* $(S^n(w_0^1, w_0^2),\ S^n(w_0^2, w_0^1))$ *converges to* (x^*, y^*) *as* $n \to \infty$, *for all* $w_0 = (w_0^1, w_0^2) \in X \times X$.

Proof. Let $Z := X \times X$ and consider $\preceq$, the partial order relation on Z, defined as follows: for all $z := (x,y)$, $w := (u,v) \in Z$, $z \preceq w$ if and only if ($x \geq u$ and $y \leq v$).

Let $Z_{\preceq} := \{(z,w) := ((x,y),(u,v)) \in Z \times Z \mid z \preceq w \text{ or } w \preceq z\}$.

Let $F : Z \to Z$ be an operator defined by

$$F(x,y) := \begin{pmatrix} S(x,y) \\ S(y,x) \end{pmatrix} = (S(x,y), S(y,x)).$$

We show that all of the assumptions of Theorem 2.1 are satisfied.

By (i) and (iv) it follows that the assumptions (i) and (iv) of Theorem 2.1 are satisfied.

By (ii), since $x = (x_1, x_2) \in X \times X$ with

$$\begin{cases} x_1 \geq S(x_1, x_2) \\ x_2 \leq S(x_2, x_1) \end{cases} \quad \text{or} \quad \begin{cases} S(x_1, x_2) \geq x_1 \\ S(x_2, x_1) \leq x_2 \end{cases}$$

we have $(x_1, x_2) \preceq (S(x_1, x_2), S(x_2, x_1))$ and so, $x \preceq F(x)$. By a similar approach we get $F(x) \preceq x$. So, $(x, F(x)) \in Z_{\preceq}$. By following the same way of proof, we get $(y, F(y)) \in Z_{\preceq}$. Hence, the assumption (ii) of Theorem 2.1 holds.

By (iii), we have $Z_{\preceq} \in I(F \times F)$.

Indeed, let $z = (x,y)$, $w = (u,v) \in Z_{\preceq}$ be two arbitrary elements, where ($x \geq u$ and $y \leq v$) or ($u \geq x$ and $v \leq y$) such that

$$(1) \begin{cases} S(x,y) \geq S(u,v) \\ S(y,x) \leq S(v,u) \end{cases} \quad \text{or} \quad (2) \begin{cases} S(u,v) \geq S(x,y) \\ S(v,u) \leq S(y,x) \end{cases}$$

From (1) and (2) we have that $(S(x,y), S(y,x)) \preceq (S(u,v), S(v,u))$, i.e., $F(x,y) \preceq F(u,v)$ or $F(z) \preceq F(w)$. Similarly, we get $F(w) \preceq F(z)$. Hence, $(F(z), F(w)) \in Z_{\preceq}$, for all $(z,w) \in Z_{\preceq}$. So, $(F \times F)(Z_{\preceq}) \subset Z_{\preceq}$, i.e., $Z_{\preceq} \in I(F \times F)$. Thus, the assumption (iii) holds.

By (vi), since $(z_0^1, z_0^2) \in X \times X$ such that

$$\begin{cases} z_0^1 \geq S(z_0^1, z_0^2) \\ z_0^2 \leq S(z_0^2, z_0^1) \end{cases} \quad \text{or} \quad \begin{cases} S(z_0^1, z_0^2) \geq z_0^1 \\ S(z_0^2, z_0^1) \leq z_0^2 \end{cases}$$

we get that $(z_0^1, z_0^2) \preceq (S(z_0^1, z_0^2), S(z_0^2, z_0^1))$ and thus, $z_0 \preceq F(z_0)$. By a similar approach we get $F(z_0) \preceq z_0$. Hence, there exists $z_0 \in Z$ such that $(z_0, F(z_0)) \in Z_{\preceq}$, so, the assumption (vi) of Theorem 2.1 holds.

Finally, we prove the assumption (v) of Theorem 2.1.

Let $\tilde{d} : Z \times Z \to \mathbb{R}^2_+$, defined by $\tilde{d}((x,y),(u,v)) := \begin{pmatrix} d(x,u) \\ d(y,v) \end{pmatrix}$.

Since $(X, \to, d, \leq)$ is an ordered Kasahara space, it follows that $(X, \to, \tilde{d}, \leq)$ is an ordered generalized Kasahara space.

• If (v_1) holds, then we have

$$\begin{aligned} \tilde{d}(F(x,y), F(u,v)) &= \tilde{d}((S(x,y), S(y,x)), (S(u,v), S(v,u))) \\ &= \begin{pmatrix} d(S(x,y), S(u,v)) \\ d(S(y,x), S(v,u)) \end{pmatrix} \leq \begin{pmatrix} k_1 d(x,u) + k_2 d(y,v) \\ k_1 d(y,v) + k_2 d(x,u) \end{pmatrix} \\ &= \begin{pmatrix} k_1 & k_2 \\ k_2 & k_1 \end{pmatrix} \begin{pmatrix} d(x,u) \\ d(y,v) \end{pmatrix} = A\tilde{d}((x,y),(u,v)). \end{aligned}$$

Since $k_1 + k_2 < 1$, we get that the matrix $A := \begin{pmatrix} k_1 & k_2 \\ k_2 & k_1 \end{pmatrix}$ is convergent to zero.

• If (v_2) holds, then we have

$$\begin{aligned} \tilde{d}(F(x,y), F(u,v)) &= \tilde{d}((S(x,y), S(y,x)), (S(u,v), S(v,u))) \\ &= \begin{pmatrix} d(S(x,y), S(u,v)) \\ d(S(y,x), S(v,u)) \end{pmatrix} \leq \begin{pmatrix} k[d(x, S(x,y)) + d(u, S(u,v))] \\ k[d(y, S(y,x)) + d(v, S(v,u))] \end{pmatrix} \\ &= \begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix} \begin{pmatrix} d(x, S(x,y)) + d(u, S(u,v)) \\ d(y, S(y,x)) + d(v, S(v,u)) \end{pmatrix} \\ &= B[\tilde{d}((x,y),(S(x,y), S(y,x))) + \tilde{d}((u,v),(S(u,v), S(v,u)))] \\ &= B[\tilde{d}((x,y), F(x,y)) + \tilde{d}((u,v), F(u,v))]. \end{aligned}$$

Since $0 \leq k < \frac{1}{2}$, we get that the matrix $B := \begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix} \in \mathcal{M}^{\Delta}_{2,2}(\mathbb{R}_+)$.

• If (v_3) holds, then we have

$$\begin{aligned} \tilde{d}(F(x,y), F(u,v)) &= \tilde{d}((S(x,y), S(y,x)), (S(u,v), S(v,u))) \\ &= \begin{pmatrix} d(S(x,y), S(u,v)) \\ d(S(y,x), S(v,u)) \end{pmatrix} \leq \begin{pmatrix} k[d(x, S(u,v)) + d(u, S(x,y))] \\ k[d(y, S(v,u)) + d(v, S(y,x))] \end{pmatrix} \\ &= \begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix} \begin{pmatrix} d(x, S(u,v)) + d(u, S(x,y)) \\ d(y, S(v,u)) + d(v, S(y,x)) \end{pmatrix} \\ &= C[\tilde{d}((x,y),(S(u,v), S(v,u))) + \tilde{d}((u,v),(S(x,y), S(y,x)))] \\ &= C[\tilde{d}((x,y), F(u,v)) + \tilde{d}((u,v), F(x,y))]. \end{aligned}$$

Since $0 \le k < \frac{1}{2}$, we get that the matrix $C := \begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix} \in \mathcal{M}^{\Delta}_{2,2}(\mathbb{R}_+)$.

We apply next Theorem 2.1 and the conclusion follows. □

3. Application

Let us consider the following system of functional-integral equations

$$(\mathbb{S}) \quad \begin{cases} x(t) = f(t, x(t), \int_a^b \Phi(t,s,x(s),y(s))ds) \\ y(t) = f(t, y(t), \int_a^b \Phi(t,s,y(s),x(s))ds) \end{cases}, \text{ for all } t \in [a,b] \subset \mathbb{R}_+.$$

By a solution of the system $(\mathbb{S})$ we understand a couple $(x,y) \in C[a,b] \times C[a,b]$, which satisfies the system for all $t \in [a,b] \subset \mathbb{R}_+$.

Let $X = C[a,b]$ be endowed with the partial order relation

$$x \le_C y \Leftrightarrow x(t) \le y(t), \text{ for all } t \in [a,b].$$

We consider $\overset{\rho}{\to}$, the convergence structure induced by the Cebîşev norm

$$\rho : C[a,b] \times C[a,b] \to \mathbb{R}_+, \ \rho(x,y) = \|x-y\|_C = \max_{t\in[a,b]} |x(t)-y(t)|.$$

Let $d : C[a,b] \times C[a,b] \to \mathbb{R}_+$, defined by

$$d(x,y) = \|(x-y)\|_C + \|(x-y)^2\|_C = \max_{t\in[a,b]} |x(t)-y(t)| + \max_{t\in[a,b]} (x(t)-y(t))^2.$$

Since $\rho(x,y) \le d(x,y)$, for all $x,y \in C[a,b]$ we get that $(C[a,b], \overset{\rho}{\to}, d, \le_C)$ is an ordered Kasahara space.

Theorem 3.1. *Let $\Phi : [a,b] \times [a,b] \times \mathbb{R}^2 \to \mathbb{R}$ and $f : [a,b] \times \mathbb{R}^2 \to \mathbb{R}$ be two continuous mappings and consider the system $(\mathbb{S})$. We suppose that:*

(i) there exists $z_0 := (z_0^1, z_0^2) \in C[a,b] \times C[a,b]$ such that

$$\begin{cases} z_0^1(t) \ge f(t, z_0^1(t), \int_a^b \Phi(t,s,z_0^1(t),z_0^2(t))ds) \\ z_0^2(t) \le f(t, z_0^2(t), \int_a^b \Phi(t,s,z_0^2(t),z_0^1(t))ds) \end{cases}$$

or

$$\begin{cases} z_0^1(t) \le f(t, z_0^1(t), \int_a^b \Phi(t,s,z_0^1(t),z_0^2(t))ds) \\ z_0^2(t) \ge f(t, z_0^2(t), \int_a^b \Phi(t,s,z_0^2(t),z_0^1(t))ds) \end{cases};$$

(ii) $f(t,\cdot,z)$ is increasing for all $t \in [a,b]$, $z \in \mathbb{R}$ and $\Phi(t,s,\cdot,z)$ is increasing, $\Phi(t,s,w,\cdot)$ is decreasing and $f(t,w,\cdot)$ is increasing for all $t,s \in [a,b]$, $w,z \in \mathbb{R}$, or, $f(t,\cdot,z)$ is decreasing for all $t \in [a,b]$, $z \in \mathbb{R}$ and $\Phi(t,s,\cdot,z)$ is decreasing, $\Phi(t,s,w,\cdot)$ is increasing and $f(t,w,\cdot)$ is decreasing for all $t,s \in [a,b]$, $w,z \in \mathbb{R}$

(iii) there exists $k_1, k_2 \in [0, \frac{\sqrt{5}-1}{4}[$ such that

$$|f(t,w_1,z_1) - f(t,w_2,z_2)| \le k_1|w_1 - f(t,w_1,z_1)| + k_2|w_2 - f(t,w_2,z_2)|$$

for all $t \in [a,b]$ and $w_1, w_2, z_1, z_2 \in \mathbb{R}$.

(iv) *for all* $x = (x_1, x_2)$, $y = (y_1, y_2) \in C[a,b] \times C[a,b]$, *with* ($x_1(t) \geq y_1(t)$ *and* $x_2(t) \leq y_2(t)$) *or* ($y_1(t) \geq x_1(t)$ *and* $y_2(t) \leq x_2(t)$) *we have*

$$\left(\begin{cases} x_1(t) \geq f(t, x_1(t), \int_a^b \Phi(t,s,x_1(t),x_2(t))ds) \\ x_2(t) \leq f(t, x_2(t), \int_a^b \Phi(t,s,x_2(t),x_1(t))ds) \end{cases} \quad or \right.$$

$$\left. \begin{cases} f(t, x_1(t), \int_a^b \Phi(t,s,x_1(t),x_2(t))ds) \geq x_1(t) \\ f(t, x_2(t), \int_a^b \Phi(t,s,x_2(t),x_1(t))ds) \leq x_2(t) \end{cases} \right)$$

and

$$\left(\begin{cases} y_1(t) \geq f(t, y_1(t), \int_a^b \Phi(t,s,y_1(t),y_2(t))ds) \\ y_2(t) \leq f(t, y_2(t), \int_a^b \Phi(t,s,y_2(t),y_1(t))ds) \end{cases} \quad or \right.$$

$$\left. \begin{cases} f(t, y_1(t), \int_a^b \Phi(t,s,y_1(t),y_2(t))ds) \geq y_1(t) \\ f(t, y_2(t), \int_a^b \Phi(t,s,y_2(t),y_1(t))ds) \leq y_2(t) \end{cases} \right)$$

for all $t \in [a,b]$.

Then there exists a unique solution (x^*, y^*) *for the system* $(\mathbb{S})$.

Proof. Let us consider the operator $S : C[a,b] \times C[a,b] \to C[a,b]$, defined by

$$S(x,y)(t) := f(t, x(t), \int_a^b \Phi(t,s,x(s),y(s))ds).$$

Then the system $(\mathbb{S})$ is equivalent with $\begin{cases} x = S(x,y) \\ y = S(y,x) \end{cases}$.

Since $S(x,y)$ is a continuous operator on $(C[a,b] \times C[a,b], \xrightarrow{\rho})$, it follows that $Graph(S)$ is closed with respect to $\xrightarrow{\rho}$.

For all ($x \geq u$ and $y \leq v$) or ($u \geq x$ and $v \leq y$) we have

$$\begin{aligned} &|S(x,y)(t) - S(u,v)(t)| \\ &\quad = |f(t, x(t), \int_a^b \Phi(t,s,x(s),y(s))ds) - f(t, u(t), \int_a^b \Phi(t,s,u(s),v(s))ds)| \\ &\quad \overset{(iii)}{\leq} k_1 |x(t) - f(t, x(t), \int_a^b \Phi(t,s,x(s),y(s))ds)| \\ &\qquad + k_2 |u(t) - f(t, u(t), \int_a^b \Phi(t,s,u(s),v(s))ds)| \\ &\quad \leq k_1 \big(|x(t) - S(x,y)(t)| + |x(t) - S(x,y)(t)|^2\big) \\ &\qquad + k_2 \big(|u(t) - S(u,v)(t)| + |u(t) - S(u,v)(t)|^2\big). \end{aligned}$$

On the other hand, we have

$$\begin{aligned}
&|S(x,y)(t)-S(u,v)(t)|^2\\
&\quad\le \Big(k_1|x(t)-f(t,x(t),\int_a^b \Phi(t,s,x(s),y(s))ds)|\\
&\qquad + k_2|u(t)-f(t,u(t),\int_a^b \Phi(t,s,u(s),v(s))ds)|\Big)^2\\
&\quad= \big(k_1|x(t)-S(x,y)(t)|+k_2|u(t)-S(u,v)(t)|\big)^2\\
&\quad\le 2\big(k_1^2|x(t)-S(x,y)(t)|^2+k_2^2|u(t)-S(u,v)(t)|^2\big)\\
&\quad\le 2k_1^2\big(|x(t)-S(x,y)(t)|+|x(t)-S(x,y)(t)|^2\big)\\
&\qquad + 2k_2^2\big(|u(t)-S(u,v)(t)|+|u(t)-S(u,v)(t)|^2\big).
\end{aligned}$$

We get further that:

$$\begin{aligned}
&|S(x,y)(t)-S(u,v)(t)|+|S(x,y)(t)-S(u,v)(t)|^2\\
&\quad\le (k_1+2k_1^2)\big(|x(t)-S(x,y)(t)|+|x(t)-S(x,y)(t)|^2\big)\\
&\qquad + (k_2+2k_2^2)\big(|u(t)-S(u,v)(t)|+|u(t)-S(u,v)(t)|^2\big).
\end{aligned}$$

Hence, by taking the maximum over $t\in[a,b]$ we get

$$d(S(x,y),S(u,v))\le \mathcal{K}\big[d(x,S(x,y))+d(u,S(u,v))\big],$$

for all ($x\ge u$ and $y\le v$) or ($u\ge x$ and $v\le y$), where

$$\mathcal{K}:=\max\{k_1+2k_1^2,k_2+2k_2^2\}.$$

Since $k_1,k_2\in[0,\frac{\sqrt{5}-1}{4}[$, we get that $0\le\mathcal{K}<\frac{1}{2}$.

We see that all the assumptions of Theorem 2.2 are satisfied and the conclusion follows. □

Remark 3.2. Similar applications were given in [2] and [7].

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References

[1] Allaire, G., *Numerical Linear Algebra*, Springer, New York, 2008.

[2] Bota, M.F., Petruşel, A., Petruşel, G., Samet, B., *Coupled fixed point theorems for single-valued operators in b-metric spaces*, Fixed Point Theory Appl., (2015), DOI 10.1186/s13663-015-0482-3.

[3] Filip, A.-D., *Fixed Point Theory in Kasahara Spaces*, Casa Cărţii de Ştiinţă, Cluj-Napoca, 2015.

[4] Fréchet, M., *Les espaces abstraits*, Gauthier-Villars, Paris, 1928.

[5] Kasahara, S., *On some generalizations of the Banach contraction theorem*, Publ. RIMS, Kyoto Univ., **12**(1976), 427-437.

[6] Perov, A.I., *On Cauchy problem for a system of ordinary differential equations*, Pviblizhen, Met. Reshen. Differ. Uvavn., **2**(1964), 115-134.

[7] Petruşel, A., Petruşel, G., Samet, B., Yao, J.-C., *Coupled fixed point theorems for symmetric contractions in b-metric spaces with applications to operator equation systems*, Fixed Point Theory, **17**(2016), no. 2, 457-476.

[8] Rus, I.A., *Principles and Applications of the Fixed Point Theory*, (Romanian), Ed. Dacia, 1979.

[9] Rus, I.A., *Kasahara spaces*, Sci. Math. Jpn., **72**(2010), no. 1, 101-110.

[10] Rus, I.A., Petruşel, A., Petruşel, G., *Fixed Point Theory*, Cluj University Press, Cluj-Napoca, 2008.

[11] Precup, R., *The role of the matrices that are convergent to zero in the study of semilinear operator systems*, Math. Comput. Modelling, **49**(2009), 703-708.

[12] Turinici, M., *Finite dimensional vector contractions and their fixed points*, Studia Univ. Babeş-Bolyai Math., **35**(1990), no.1, 30-42.

[13] Varga, R.S., *Matrix Iterative Analysis*, Springer, Berlin, 2000.

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