Stud. Univ. Babeş-Bolyai Math. 69(2024), No. 1, 223–232
DOI: 10.24193/subbmath.2024.1.14

The Pólya f-curvature of plane curves

Mircea Crasmareanu and Gabriel-Teodor Pripoae

Abstract. We introduce and study a new curvature function for plane curves inspired by the weighted mean curvature of M. Gromov. We call it *Pólya*, being the difference between the usual curvature and the inner product of the normal vector field with the Pólya vector field of a given planar function f. We computed it for several examples, since the general problem of vanishing or constant values of this new curvature involves the general expression of f.

Mathematics Subject Classification (2010): 53A04, 53A45, 53A55.

Keywords: Plane curve, Pólya vector field, f-curvature, reverse potential.

1. Introduction

The last forty years known an intensive research in the area of geometric flows. The most simple of them is the *curve shortening flow* and already the excellent survey [4] is almost twenty years old. Recall that the main geometric tool in this last flow is the well-known curvature of plane curves. Hence, to give a re-start to this problem seams to search for variants of the curvature or in terms of [11], deformations of the usual curvature. The goal of this short note is to propose such a deformation using a type of planar vector fields introduced by George Pólya (1887-1985). The life and research of this brilliant mathematician is exposed in the book [1].

The contents of this paper is as follows. In the following section we introduce our new curvature, using an idea of Mikhael Gromov. This curvature function, denoted k_f, is defined with respect to a given planar function $f : \Omega \subseteq \mathbb{R}^2 \to \mathbb{R}^2$ through its associated Pólya vector field. Starting from the given curve C we compute k_f in some examples in order to determine the complexity of computation. At this level, due to the generality of function f, it is impossible to determine cases when k_f is zero or another real constant. For the examples of this section we choose in particular a

Received 03 May 2023; Accepted 17 January 2024.

holomorphic function, namely the square function $f(z) = z^2$ and hence we denote the corresponding curvature as k_{square}. At the end of the section we use the Fermi-Walker derivative to express k_f.

In the third section we start from the given f and define a notion of *reverse potential* F which involves the paracomplex structure of $\mathbb{R}^2$; hence we change the notation of our introduced curvature in k_F. Now, we can point out cases when k_F is zero or another constant and an interesting example is provided by the harmonic radial function $F(x, y) = \frac{1}{2}\ln(x^2 + y^2)$.

2. The Pólya f-curvature for a plane curve

Fix $I \subseteq \mathbb{R}$ an open interval and $C \subset \mathbb{R}^2$ a regular parametrized curve of equation:

$$C : r(t) = (x(t), y(t)), \quad \|r'(t)\| > 0, \quad t \in I. \tag{2.1}$$

The ambient setting, namely $\mathbb{R}^2$, is an Euclidean vector space with respect to the canonical inner product:

$$\langle u, v\rangle = u^1v^1 + u^2v^2, \quad u = (u^1, u^2), \quad v = (v^1, v^2) \in \mathbb{R}^2, \quad 0 \leq \|u\|^2 = \langle u, u\rangle. \tag{2.2}$$

The infinitesimal generator of the rotations in $\mathbb{R}^2$ is the linear vector field, called *angular*:

$$\xi(u) := -u^2\frac{\partial}{\partial u^1} + u^1\frac{\partial}{\partial u^2}, \quad \xi(u) = i \cdot u = i \cdot (u^1 + iu^2). \tag{2.3}$$

It is a complete vector field with integral curves the circles $\mathcal{C}(O, R)$:

$$\begin{cases} \gamma^{\xi}_{u_0}(t) = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix} \cdot \begin{pmatrix} u^1_0 \\ u^2_0 \end{pmatrix} = SO(2) \cdot u_0, \\ R = \|u_0\| = \|(u^1_0, u^2_0)\|, t \in \mathbb{R}, R(t) := \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix} \in SO(2) = S^1 \end{cases} \tag{2.4}$$

and since the rotations are isometries of the Riemannian metric $g_{can} = dx^2 + dy^2$ it follows that ξ is a Killing vector field of the Riemannian manifold $(\mathbb{R}^2, g_{can})$. The first integrals of ξ are the Gaussian functions i.e. multiples of the square norm:

$$f_C(x, y) = C(x^2 + y^2), \ C \in \mathbb{R}.$$

For an arbitrary vector field $X = A(x, y)\frac{\partial}{\partial x} + B(x, y)\frac{\partial}{\partial y}$ its Lie bracket with ξ is:

$$[X, \xi] = (yA_x - xA_y - B)\frac{\partial}{\partial x} + (A + yB_x - xB_y)\frac{\partial}{\partial y}$$

where the subscript denotes the variable corresponding to the partial derivative. For example, ξ commutes with *the radial* (or Euler) vector field:

$$E(x, y) = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y},$$

which is also a complete vector field having as integral curves the homotheties $\gamma^E_{u_0}(t) = e^tu_0$ for all $t \in \mathbb{R}$. The vector field E is the basis of the 1-dimensional annihilator of the Liouville (or tautological) 1-form $\lambda = \frac{1}{2}(-ydx + xdy)$ whose exterior derivative is the area 2-form $dx \wedge dy$. We point out also that the opposite vector field $W = -E$ is exactly the wind in the Zermelo navigation problem corresponding to the Funk metric in the unit disk of $\mathbb{R}^2$, [5]. For an arbitrary Euclidean space $\mathbb{R}^n$ with $n \geq 2$ the radial vector field $E = x^i \frac{\partial}{\partial x^i}$ defines the notion of *horizontal* 1*-form* ρ as satisfying $i_E \rho = 0$ with i_E the interior product.

The Frenet apparatus of the curve C is provided by:

$$\begin{cases} T(t) = \dfrac{r'(t)}{\|r'(t)\|} \in S^1, \\ N(t) = i \cdot T(t) = \dfrac{1}{\|r'(t)\|}(-y'(t), x'(t)) \in S^1 \\ k(t) = \dfrac{1}{\|r'(t)\|}\langle T'(t), N(t)\rangle = \dfrac{1}{\|r'(t)\|^3}[x'(t)y''(t) - y'(t)x''(t)]. \end{cases} \tag{2.5}$$

Hence, if C is naturally parametrized (or parametrized by arc-length) i.e. $\|r'(s)\| = 1$ for all $s \in I$ then $r''(s) = k(s)ir'(s)$. In a complex approach based on

$$z(t) = x(t) + iy(t) \in \mathbb{C} = \mathbb{R}^2$$

we have $2\lambda = Im(\bar{z}dz)$ and

$$\begin{cases} k(t) = \dfrac{1}{|z'(t)|^3} Im(\bar{z}'(t) \cdot z''(t)) = \dfrac{1}{|z'(t)|} Im\left(\frac{z''(t)}{z'(t)}\right), \\ Re(\bar{z}'(t) \cdot z''(t)) = \dfrac{1}{2}\dfrac{d}{dt}\|r'(t)\|^2, \quad f_C(z) = C|z|^2. \end{cases} \tag{2.6}$$

This note defines a new curvature function for C inspired by a notion introduced by M. Gromov in [8, p. 213] and concerning with hypersurfaces M^n in a weighted Riemannian manifold $(\tilde{M}, g, f \in C^\infty_+(\tilde{M}))$. More precisely, the *weighted mean curvature* of M is the difference:

$$H^f := H - \langle \tilde{N}, \tilde{\nabla} f\rangle_g \tag{2.7}$$

where H is the usual mean curvature of M and $\tilde{N}$ is the unit normal to M. This curvature was studied in several papers; for example if H^f is the constant $\lambda \in \mathbb{R}$ then M is called λ*-hypersurface* and the influence of a shrinking Ricci soliton on the geometry of such a hypersurface is studied in [2].

Suppose that the geometric image of the given curve is contained in a domain $\Omega \subseteq \mathbb{R}^2$ and we have also a given function $f : \Omega \to \mathbb{R}^2 = \mathbb{C}$, $f = (u, v) = u + iv$ for $u, v \in C^\infty(\Omega)$. This function has an associated vector field, called *Pólya*:

$$V_f := u\frac{\partial}{\partial x} - v\frac{\partial}{\partial y} \tag{2.8}$$

whose Lie bracket with ξ and E is:

$$\begin{cases} [V_f, \xi] = (yu_x - xu_y + v)\dfrac{\partial}{\partial x} + (u - yv_x + xv_y)\dfrac{\partial}{\partial y}, \\ [V_f, E] = (u - xu_x - yu_y)\dfrac{\partial}{\partial x} + (xv_x + yv_y - v)\dfrac{\partial}{\partial y}. \end{cases} \tag{2.9}$$

For details concerning this type of vector fields see [3] and [9]. Hence we follow this path and we consider:

Definition 2.1. The *Pólya f-curvature* of C is the smooth function $k_f : I \to \mathbb{R}$ given by:

$$k_f(t) := k(t) - \langle N(t), V_f(r(t))\rangle. \tag{2.10}$$

Before starting its study we point out that this work is dedicated to the memory of Academician Radu Miron (1927-2022). He was always interested in the geometry of curves and, besides its theory of *Myller configuration* ([13]), he generalizes also a type of curvature for space curves in [12]. It is worth to remark that for its meaningfully contribution to the geometry, the Romanian edition (1966) of the book ([13]) has received the "Gheorghe Ţiţeica" Prize of the Romanian Academy in 1968. Obviously, we can present on several pages the enormous contributions of Academician Radu Miron to the theory of space curves (e.g. by extensions of the celebrated Gauss-Bonnet theorem) but due to the planar character of our study we stop here our commemorative discourse.

Returning to our subject we note:

Theorem 2.2. *(i) The expression of the Pólya f-curvature is:*

$$k_f(t) = k(t) + \frac{x'(t)v(x(t), y(t)) + y'(t)u(x(t), y(t))}{\|r'(t)\|}. \tag{2.11}$$

(ii) Moreover:

$$k_f(t) \leq k(t) + \sqrt{[u(x(t), y(t))]^2 + [v(x(t), y(t))]^2} = k(t) + \|V_f(r(t))\| \tag{2.12}$$

with equality if and only if the vector field $V_f \circ r$ is parallel to N but in the opposite direction.
(iii) In particular, if C is an integral curve of V_f then k_f is exactly k.
(iv) If the normal projection of $V_f \circ r$ is invariant with respect to the orientation preserving parameter changes on C then k_f is invariant too, and conversely.
(v) If the angle made by $V_f \circ r$ with the normal is invariant w.r.t. positively oriented isometries then k_f is invariant too, and conversely.

Proof. We have directly:

$$\langle N(t), V_f(r(t))\rangle = \langle iT(t), V_f(r(t))\rangle \tag{2.13}$$

and the conclusion (2.11) follows. The inequality (2.12) is the direct application of the CBS inequality. The claimed consequence follows from the ODE system:

$$x' = u,$$

$$y' = -v. \qquad \square$$

Theorem 2.3. *With the previous notations, let $I \subseteq \mathbb{R}$ be an open subset and let $h : I \to \mathbb{R}$ be a smooth function. Fix $t_0 \in I$, $(x_0, y_0) \in \mathbb{R}^2$ and an orthonormal pair $\{T_0 \in S^1, N_0 \in S^1\}$ of $\mathbb{R}^2$. Then there exists a maximal open interval $J \subseteq I$ around t_0 and a unique parameterized curve $C : J \to \mathbb{R}^2$, such that $k_f = h$, $C(t_0) = (x_0, y_0)$ and $T(t_0) = T_0$, $N(t_0) = N_0$.*

Proof. This result is an analogue of the fundamental theorem of plane curves ([10], 1.3.6) and the proof is similar. Consider the ODEs system:

$$X'(t) = (h(t) + \langle Y(t), V_f(x(t), y(t))\rangle) \cdot Y(t)$$
$$Y'(t) = -(h(t) + \langle Y(t), V_f(x(t), y(t))\rangle) \cdot X(t)$$
$$X(t) = \frac{1}{(x'(t))^2 + (y'(t))^2} \cdot (x'(t), y'(t)),$$
$$Y(t) = \frac{1}{(x'(t))^2 + (y'(t))^2} \cdot (-y'(t), x'(t))$$

with the initial conditions $(x(t_0), y(t_0)) = (x_0, y_0)$ and $(x'(t_0), y'(t_0)) = (T_0, N_0)$. The existence and uniqueness theorem for ODEs ensures there exists a solution $C(t) = (x(t), y(t))$ on a maximal open interval $J \subseteq I$ around t_0. A short computation proves that $\{X, Y\}$ is the Frenet frame along C and the that the first two formulas of the previous system are the Frenet equations. As the function $(h(t) + \langle Y(t), V_f(x(t), y(t))\rangle)$ must be the curvature function $k = k(t)$ of C, we obtain the relation (2.10), hence the equality $k_f = h$. □

Example 2.4. i) If C is the line $r_0 + tU, t \in \mathbb{R}$ with the vector $U = (U^1, U^2) \neq \bar{0} = (0, 0)$ then k_f is the constant:

$$k_f(t) = \frac{U^1 v(x_0 + tU^1, y_0 + tU^2) + U^2 u(x_0 + tU^1, y_0 + tU^2)}{\|U\|}. \tag{2.14}$$

In particular, if $O \in C$ then

$$k_f(t) = \frac{U^1 v(tU^1, tU^2) + U^2 u(tU^1, tU^2)}{\sqrt{(U^1)^2 + (U^2)^2}}$$

and for $f(z) = z^2$ we have:

$$k_{square}(t) = \frac{U^2[3(U^1)^2 - (U^2)^2]}{\sqrt{(U^1)^2 + (U^2)^2}} t^2. \tag{2.15}$$

ii) If C is the circle $\mathcal{C}(O, R) : r(t) = Re^{it}$ then:

$$k_f(t) = \frac{1}{R} - v(R\cos t, R\sin t)\sin t + u(R\cos t, R\sin t)\cos t. \tag{2.16}$$

For $f(z) = z^2$ we have:

$$k_{square}(t) = \frac{1}{R} + R^2 \cos 3t \in \left[\frac{1}{R} - R^2, \frac{1}{R} + R^2\right]. \tag{2.17}$$

iii) For the case of logarithmic spiral expressed in polar coordinates as $\rho_{R,\alpha}(t) = Re^{\alpha t}$, $R, \alpha > 0$ and $t \in \mathbb{R}$ we have the f-curvature:

$$\sqrt{\alpha^2 + 1} k_f(t) = R^{-1} e^{-\alpha t} + (\alpha \cos t - \sin t) v(Re^{\alpha t} \cos t, Re^{\alpha t} \sin t)$$
$$+ (\alpha \sin t + \cos t) u(Re^{\alpha t} \cos t, Re^{\alpha t} \sin t) \tag{2.18}$$

and for $\alpha \to 0$ we re-obtain the f-curvature of the circle $\mathcal{C}(O, R)$. Again for $f(z) = z^2$ we have:

$$\sqrt{\alpha^2 + 1} k_{square}(t) = R^{-1} e^{-\alpha t} + R^2 e^{2\alpha t}[\cos 3t + \alpha \sin 3t]. \tag{2.19}$$

In the following since the problem of vanishing or of constant values for k_f can not be treated due to the generality of f we continue to present some concrete examples in order to remark the computational aspects of our approach.

Example 2.5. We study completely a curve with non-constant rotational curve. Namely, the involute of the unit circle S^1 is:

$$C : r(t) = (\cos t + t\sin t, \sin t - t\cos t) = (1 - it)e^{it}, \quad t \in (0, +\infty). \tag{2.20}$$

A direct computation gives:

$$r'(t) = (t\cos t, t\sin t) = te^{it}, \quad k(t) = \frac{1}{t} > 0, \quad \|r'(t)\| = t \tag{2.21}$$

and then the f-curvature is:

$$k_f(t) = \frac{1}{t} + v(\cos t + t\sin t, \sin t - t\cos t)\cos t + u(\cos t + t\sin t, \sin t - t\cos t)\sin t \tag{2.22}$$

which for $f(z) = z^2$ becomes:

$$k_{square}(t) = \frac{1}{t} + 3(1 - t^2)\sin t\cos^2 t - 2t\cos^3 t - \sin^3 t + 6t\sin^2 t\cos t. \tag{2.23}$$

Example 2.6. For the square function $f(z) = z^2$ the integral curves of its Pólya vector field are the solutions of the ODE system:

$$\dot{x} = x^2 - y^2, \quad \dot{y} = -2xy \tag{2.24}$$

having the first integral:

$$F_{square}(z = x + iy) = 3x^2y - y^3 = Im(z^3). \tag{2.25}$$

Fix then a arbitrary real number $a \neq 0$; the implicit plane curve:

$$C(a) : F(z) = a \tag{2.26}$$

has the usual curvature:

$$k(C(a)) = -\frac{a}{27(x^2 + y^2)^2}. \tag{2.27}$$

We end this section with an approach in terms of Fermi-Walker derivative. Let $\mathcal{X}_\gamma$ be the set of vector fields along the curve γ. Then the Fermi-Walker derivative is the map ([7]) $\nabla^{FW}_\gamma : \mathcal{X}_\gamma \to \mathcal{X}_\gamma$:

$$\nabla^{FW}_\gamma(X) := \frac{d}{dt}X + k\|r'(\cdot)\|[\langle X, N\rangle T - \langle X, T\rangle N] = \frac{d}{dt}X + k[X^\flat(N)T - X^\flat(T)N] \tag{2.28}$$

with $X^\flat$ the differential 1-form dual to X with respect to the Euclidean metric. For $X = V_f \circ r$ we have:

$$\nabla^{FW}_\gamma(V_f \circ r)(t) = \frac{d}{dt}V_f(r(t)) + \|r'(t)\|k(t)[\langle V_f \circ r(t), N(t)\rangle T(t) - \langle V_f \circ r(t), T(t)\rangle N(t)] \tag{2.29}$$

and then we restrict to the tangential component of this equation:

$$\langle (\nabla^{FW}_\gamma V_f \circ r)(t) - \frac{d}{dt}V_f(r(t)), T(t)\rangle = \|r'(t)\|k(t)\langle V_f \circ r(t), N(t)\rangle. \tag{2.30}$$

Hence, if C is not a line we have:

$$k_f(t) = k(t) - \frac{\langle (\nabla_\gamma^{FW} V_f \circ r)(t) - \frac{d}{dt} V_f(r(t)), T(t)\rangle}{\|r'(t)\| k(t)}. \tag{2.31}$$

3. A reverse potential for f and the corresponding Pólya curvature

Usually, the smooth function $F \in C^\infty(\Omega)$ is called *a potential of f* if the gradient relation holds $f = \nabla F$ which means $u = F_x$ and $v = F_y$. But for our formulae (2.11) another object seems more naturally:

Definition 3.1. F is a *reverse-potential* of f if $u = F_y$ and $v = F_x$.

In a matrix form we express this condition as:

$$\begin{pmatrix} u \\ v \end{pmatrix} = \Gamma \cdot \nabla F, \quad \Gamma := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \in Sym(2). \tag{3.1}$$

We point out that since $\Gamma^2 = I_2$ and $\dim Ker(I_2 + \Gamma) = \dim Ker(I_2 - \Gamma) = 1$ the endomorphism Γ is exactly the paracomplex structure of the plane $\mathbb{R}^2$, [6]. The kernel of $I_2 + \Gamma$ is the second bisectrix $B_2 : x + y = 0$ while the kernel of $I_2 - \Gamma$ is the first bisectrix $B_1 : x - y = 0$. The paracomplex structure Γ and the complex structure $J := R\left(\frac{\pi}{2}\right)$ of the plane commute:

$$\Gamma \cdot J = J \cdot \Gamma = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = diag(1, -1). \tag{3.2}$$

In fact, in [9, p. 5] there is another vector field associated to f, namely

$$V_f^\perp := -V_{if} = v\frac{\partial}{\partial x} + u\frac{\partial}{\partial y}$$

and hence if F is a reverse potential of f then its gradient is exactly $V_f^\perp$.

It results immediately that our considered curvature, denoted now k_F, is:

$$k_F(t) = k(t) + \frac{1}{\|r'(t)\|}\frac{d}{dt}F(r(t)), \quad k_F(t) \leq k(t) + \|\nabla F(r(t))\| \tag{3.3}$$

since $\|V_f\| = \|\nabla F\|$.

Remark 3.2. An useful formalism is that of [14, p. 2]; if $r : S^1 \simeq [0, 2\pi) \to \mathbb{R}^2$ is naturally parametrized then there exists the smooth function $\theta : S^1 \to \mathbb{R}$, called *normal angle*, such that:

$$N(s) = e^{i\theta(s)} = (\cos\theta(s), \sin\theta(s)), \quad T(s) = -iN(s) = -ie^{i\theta(s)} = e^{i(\theta(s) - \frac{\pi}{2})} \tag{3.4}$$

and then the Frenet equations yield:

$$\frac{d\theta}{ds}(s) = k(s). \tag{3.5}$$

Then k_F is a derivative:

$$k_F(s) = \frac{d}{ds}\left(\theta(s) + F(r(s))\right) \tag{3.6}$$

and hence k_F is vanishing if and only if the function $\theta + F \circ r$ is a constant.

Example 3.3. Suppose that f is a holomorphic function i.e. its real and imaginary components satisfy the Cauchy-Riemann equations: $u_x = v_y$, $u_y = -v_x$. If f is provided by the reverse potential F then the first equation holds directly while the second equation implies the harmonicity of F i.e. the vanishing of the Euclidean Laplacian: $\Delta F = 0$. If we restrict the class of F to radial (i.e. S^1-invariant) ones $F = \tilde{F}(x^2 + y^2)$ we have the solution $F(x, y) = \frac{1}{2}\ln(x^2 + y^2) = \frac{1}{2}\ln f_1(x, y)$ for $0 \notin \Omega$ and then:

$$\begin{cases} f(z) = \dfrac{i}{z} = \frac{y}{x^2+y^2} + i\dfrac{x}{x^2+y^2} = \left(\dfrac{y}{x^2+y^2}, \dfrac{x}{x^2+y^2}\right), \\ V_f = \dfrac{y}{x^2+y^2}\dfrac{\partial}{\partial x} - \dfrac{x}{x^2+y^2}\dfrac{\partial}{\partial y} \\ k_F(t) = k(t) + \dfrac{\langle r(t), r'(t)\rangle}{\|r(t)\|^2\|r'(t)\|} \leq k(t) + \dfrac{1}{\|r(t)\|}. \end{cases} \tag{3.7}$$

The circles $\mathcal{C}(O, R) : r(t) = Re^{it}$ are exactly the integral curves of V_f and applying the last part of proposition 2.2 we get: $k_F(t) = k(\mathcal{C}(O, R)) = \frac{1}{R}$=constant. For the more general example of logarithmic spiral $r(t) = Re^{i\alpha t}$, $\alpha > 0$ we obtain:

$$k_F(t) = \frac{\alpha + 1}{Re^{\alpha t}\sqrt{\alpha^2 + 1}}, \quad \lim_{\alpha \to 0} k_F = k(\mathcal{C}(O, R)). \tag{3.8}$$

We have

$$V_f^{\perp}(x, y) = \frac{1}{\|(x, y)\|^2} E(x, y)$$

and then

$$\|V_f\| = \|V_f^{\perp}\| = \frac{1}{\sqrt{x^2 + y^2}}.$$

For a harmonic function f the Lie brackets (2.9) can be expressed only with the partial derivatives of u:

$$\begin{cases} [V_f, \xi] = (yu_x - xu_y + v)\dfrac{\partial}{\partial x} + (u + xu_x + yu_y)\dfrac{\partial}{\partial y}, \\ [V_f, E] = (u - xu_x - yu_y)\dfrac{\partial}{\partial x} + (uu_x - xu_y - v)\dfrac{\partial}{\partial y} \end{cases} \tag{3.9}$$

and then V_f commutes with ξ while $[V_f, E] = 2V_F$, equality which follows also from the (-1)-homogeneity of coefficients of f.

4. Pólya related curves

Let $f(x, y) = u(x, y) + iv(x, y)$ be an arbitrary function on the complex plane and $C : I \to \mathbb{R}^2$ be a regular parameterized curve, as in Section 2. Denote by k and k_f the curvature function and the Pólya curvature function of C, respectively. From the fundamental theorem of the theory of plane curves, we know there exists a regular parameterized curve $\tilde{C} : I \to \mathbb{R}^2$, whose curvature $\tilde{k}$ is exactly k_f; moreover, this curve is unique, up to a positively oriented isometry and an orientation preserving parameter change.

Definition 4.1. We say $\tilde{C}$ is *the Pólya mate* of C w.r.t. the function f.

Example 4.2. Let again $C = \mathcal{C}(O, R) : r(t) = Re^{it}$ and consider $f(z) = \bar{z}$. Then, from the formula (2.16) it results $k_f = \frac{1}{R} + R$ and then $\tilde{C} = \mathcal{C}(O, \tilde{R})$ is the Pólya mate of C for:

$$\tilde{R} = \frac{R}{R^2 + 1} \leq \min\{\frac{1}{2}, R\}. \tag{4.1}$$

Continuing this process with the fixed f we obtain the Pólya mate of $\tilde{C}$ as being the circle $\hat{C} = \mathcal{C}(O, \hat{R})$ with:

$$\hat{R} = \frac{R(R^2 + 1)}{R^2 + (R^2 + 1)^2} \tag{4.2}$$

which proves that the "Pólya mate" relation for a fixed f is not a symmetric one in general.

References

[1] Alexanderson, G.L., *The Random Walks of George Pólya*, MAA Spectrum, Washington, DC, Mathematical Association of America (MAA), 2000.

[2] Barbosa, E., Santana, F., Upadhyay, A., *λ-Hypersurfaces on shrinking gradient Ricci solitons*, J. Math. Anal. Appl., **492**(2020), no. 2, Article ID 124470, 18 p.

[3] Braden, B., *Pólya's geometric picture of complex contour integrals*, Math. Mag., **60**(1987), 321-327.

[4] Chou, K.S., Zhu, X.P., *The Curve Shortening Problem*, Boca Raton, FL, Chapman & Hall/CRC, 2001.

[5] Crasmareanu, M., *New tools in Finsler geometry: Stretch and Ricci solitons*, Math. Rep., Bucharest, **16**(66)(2014), no. 1, 83-93.

[6] Crasmareanu, M., *Para-mixed linear spaces*, Acta Univ. Sapientiae Math., **9**(2017), no. 2, 275-282.

[7] Crasmareanu, M., Frigioiu, C., *Unitary vector fields are Fermi-Walker transported along Rytov-Legendre curves*, Int. J. Geom. Methods Mod. Phys., **12**(10)(2015), Article ID 1550111, 9 pages.

[8] Gromov, M., *Isoperimetry of waists and concentration of maps*, Geom. Funct. Anal., **13**(2003), no. 1, 178-215; erratum ibid., **18**(2008), no. 5, 1786-1786.

[9] Ionescu, L.M., Pripoae, C.L., Pripoae, G.T., *Classification of holomorphic functions as Pólya vector fields via differential geometry*, Mathematics, **9**(16)(2021), article no. 1890.

[10] Klingenberg, W., *A Course in Differential Geometry*, Springer, 1978.

[11] Mazur, B., *Perturbations, deformations, and variations in geometry, physics, and number theory*, Bull. Am. Math. Soc., New Ser., **41**(3)(2004), 307-336.

[12] Miron, R., *Une généralisation de la notion de courbure de parallélisme*, Gaz. Mat. Fiz., Bucureşti, Ser. A, **10**(63)(1958), 705-708.

[13] Miron, R., *The Geometry of Myller Configurations. Applications to Theory of Surfaces and Nonholonomic Manifolds*, Bucharest, Editura Academiei Române, 2010.

[14] Zhu, C.P., *Lectures on Mean Curvature Flows*, AMS/IP Studies in Advanced Mathematics, **32**, Providence, RI, American Mathematical Society, 2002.

Mircea Crasmareanu
Faculty of Mathematics,
University "Al. I. Cuza",
Iaşi, 700506, Romania
e-mail: mcrasm@uaic.ro

Gabriel-Teodor Pripoae
Faculty of Mathematics and Computer Science,
University of Bucharest, Academiei 14,
Bucharest, RO-010014, Romania
e-mail: gpripoae@yahoo.com