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# CSABA VARGA – In Memoriam

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**Abstract.** This note is devoted to present the scientific work of Professor Csaba Varga (1959-2021), who had contributions in Calculus of Variations and its applications in the theory of Partial Differential Equations and Finsler Geometry.

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## 1. Introduction

Csaba György Varga passed away on 16 August 2021, after a long illness period. He was 62 years old.

Csaba was born on 5 February 1959 in Gyulakuta (Fântânele, Romania). He finished his university studies in 1983 at the Faculty of Mathematics of the Babeş-Bolyai University, Cluj-Napoca.

After being a highschool teacher for seven years in Bistriţa-Năsăud (Romania), he started his academic career in 1990. According to him, after "seven years of darkness", he had the opportunity to restart to work again in advanced mathematics together with his former students M. Crainic and G. Farkas. In that time, they learned and investigated together algebraic topology, Ljusternik-Schnirelmann category, density and condensation problems, see the early papers [28, 30, 31, 32, 33].

These papers have proved to be influential in the coming years when Csaba has got in contact with D. Motreanu. They started together to explore topological and variational phenomena in the context of elliptic problems. Due to this fruitful collaboration, Csaba defended his doctoral dissertation in 1996, entitled *Topological Methods in Optimizations*, under the supervision of J. Kolumbán. The central theme of his doctoral thesis is the non-smooth critical point theory (for locally Lipschitz functions) with applications in the theory of differential inclusions.

In the sequel, I invite the reader on a quick tour of Csaba's mathematical interests and contributions, placing them in the main research directions of *critical point theory* and *Finsler geometry.*

## 2. Critical point theory: from smooth to nonsmooth

From the mid of the 20th century, variational principles have been subject to relevant developments, when – among others – the modern critical point theory appeared. To be more precise, let $X$ be a real Banach space, $E : X \to \mathbb{R}$ be a differentiable function; $x_0 \in X$ is said to be a *critical point* of $E$, if the derivative of $E$ at $x_0$ vanishes, i.e., $dE(x_0) = 0$. This class of problems includes important chapters from modern mathematics:

- *weak solutions* of elliptic PDEs and related problems (weak solutions of differential equations are critical points of the energy functional associated to the original equation);
- *geodesic lines* in Riemannian/Finsler manifolds (these geometric objects occur as the critical points of the natural energy functionals defined on the space of curves with further particular properties).

A basic tool to guarantee critical points of energy functionals is the celebrated *Mountain Pass Theorem*, developed by A. Ambrosetti and P. Rabinowitz [3]. The proof of this result is based on a *deformation lemma*, which requires the existence of a suitable gradient vector field, coming from the high regularity of the functional. The Mountain Pass Theorem is applied to solve various elliptic problems; for simplicity, we consider the model problem

$$\begin{cases} -\Delta u(x) = f(u(x)) & x \in \Omega, \\ u(x) = 0 & x \in \partial\Omega, \end{cases} \tag{$P$}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded domain with $C^1$-boundary, $\Delta$ is the Laplace operator, while $f : \mathbb{R} \to \mathbb{R}$ is a continuous functions verifying certain growth conditions at the origin and at infinity. In such cases, we associate to problem $(P)$ its natural energy functional $E : W_0^{1,2}(\Omega) \to \mathbb{R}$, defined by

$$E(u) = \frac{1}{2}\int_\Omega |\nabla u(x)|^2 dx - \int_\Omega \int_0^{u(x)} f(t)dtdx,\ u \in W_0^{1,2}(\Omega),$$

where $W_0^{1,2}(\Omega) = H_0^1(\Omega)$ is the usual Sobolev space. If $f$ has appropriate properties, it follows that $E$ is a $C^2$-class functional and

$$dE(u) = 0 \Longleftrightarrow u \text{ is a weak solution of } (P).$$

A highly nontrivial problem occurs when $E$ is *not* differentiable, which requires a deep analysis; in this framework, Csaba has some relevant contributions, which are presented roughly in the next two subsections.

### 2.1. Critical points for locally Lipschitz functionals

In the early eighties, K.-C. Chang [7] proposed to study the problem $(P)$ whenever $f$ is not necessarily continuous, being only locally essentially bounded. Such phenomena arise in mathematical physics, engineering, etc.

Since in the new situation the nonlinear term $f$ is only locally essentially bounded, it is possible to have the unlikely situation that problem $(P)$ has only the zero solution, in spite of the fact that one could expect the presence of nontrivial solutions from practical point of view. For this reason, one usually substitutes the value $f(t)$ by the interval $[\underline{f}(t), \overline{f}(t)]$, where

$$\underline{f}(t) = \lim_{\delta\to 0^+} \mathrm{essinf}_{|s-t|<\delta} f(s), \quad \overline{f}(t) = \lim_{\delta\to 0^+} \mathrm{esssup}_{|s-t|<\delta} f(s),$$

while $\mathrm{essinf}_A f = \sup\{a \in \mathbb{R} : f(x) \geq a \text{ for a.e. } x \in A\}$ and $\mathrm{esssup}_A f = -\mathrm{essinf}_A(-f)$, $A \neq \emptyset$. In this way, instead of problem $(P)$ we consider the *differential inclusion*

$$\begin{cases} -\Delta u(x) \in \partial F(u(x)) & x \in \Omega, \\ u(x) = 0 & x \in \partial\Omega, \end{cases} \qquad (DI)$$

where $F(t) = \int_0^t f(s)ds$ is a *locally Lipschitz* function[1], whose Clarke subgradient is

$$\partial F(t) = [\underline{f}(t), \overline{f}(t)], \; t \in \mathbb{R}.$$

The energy functional $E : W_0^{1,2}(\Omega) \to \mathbb{R}$ associated to $(DI)$ is not of class $C^1$, being only locally Lipschitz on the Sobolev space $W_0^{1,2}(\Omega)$, while its critical point in the sense of Chang, i.e., $0 \in \partial E(u)$, is a solution of the differential inclusion $(DI)$.

In general, if $E : X \to \mathbb{R}$ is a locally Lipschitz function in a given Banach space $X$, its *Clarke subgradient* at $u \in X$ is defined by

$$\partial E(u) = \{\xi \in X^* : E^o(u; v) \geq \langle \xi, v\rangle, \; \forall v \in X\},$$

see F. H. Clarke [8], where $X^*$ is the dual of $X$, $\langle \cdot, \cdot \rangle$ is the duality mapping, and

$$E^o(u; v) = \limsup_{w\to v, t\to 0^+} \frac{E(w + tv) - E(w)}{t}$$

stands for the *Clarke directional derivative* of $E$ at the point $u \in X$ and direction $v \in X$.

In a joint work with D. Motreanu, Csaba provided the *first extension of the celebrated Mountain Pass Theorem to locally Lipschitz functions*, see [27]. Moreover, they provided the so-called 'zero altitude' version of the result, which was new even in the smooth setting. The main tool they used is a *non-smooth deformation lemma*, where the key idea is the introduction of the so-called *pseuso-gradient vector field* for locally Lipschitz functions. Their non-smooth deformation lemma implies further non-smooth minimax results (saddle point, linking theorems).

The results from [27] has several applications and extensions, see e.g. C.O. Alves and J.A. Santos [1], or C.O. Alves, R.C. Duarte and M.A.S. Souto [2]. Moreover, various applications of the non-smooth Mountain Pass Theorem have been developed, both in the theory of *differential inclusions* and *hemivariational inequalities*. Moreover, spectacular arguments were provided not only in *bounded domains*, but also on *unbounded domains*. While in the former case Sobolev compactness is expected, in the latter case – in order to regain some sort of compactness – either certain coercivity or symmetric structures are required on unbounded domains. Such an approach is the so-called *principle of symmetric criticality* (both for smooth and nonsmooth

[1]The function $F : X \to \mathbb{R}$ is *locally Lipschitz*, if for every $x \in X$ there exist a neighborhood $U$ and a constant $K_x > 0$ such that $|f(u) - f(v)| \leq K_x\|u - v\|$ for every $u, v \in U$, see F. H. Clarke [8].

functionals); over the years, Csaba became a worldwide expert of this principle, most of his results in this direction being influential in the literature, see e.g. [23, 24].

### 2.2. Critical points for continuous and set-valued functions

In the early nineties, M. Degiovanni and M. Marzocchi [11] have developed the theory of critical points for continuous functionals, by introducing the so-called *weak slope* of a continuous function $E : X \to \mathbb{R}$ defined on a metric space $X$. A point $u \in X$ is a *critical point* of $E$ if its weak slope vanishes at $u$. In addition, if the functional $E$ is of class $C^1$, the weak slope coincides with the norm of the usual differential of $E$.

Being an expert of the critical point theory for locally Lipschitz functions, Csaba obtained several important results also in the context of weak slopes. More precisely, Csaba and his co-authors obtained quantitative versions of the *deformation lemma* (without using pseudo gradient vector fields, which is not defined in such non-smooth settings), minimax results, see e.g. [22].

In addition, inspired by the work of M. Frigon [13], Csaba and his co-authors provided quantitative deformation lemmas and minimax results for *set-valued maps*, see [21]. The Mountain Pass Theorem for set-valued maps from [21] has a central place in the monograph of Y. Jabri [15].

## 3. Finsler geometry: from synthetic aspects to PDEs

In general, Finsler geometry is viewed as an extension of Riemannian geometry. S.-S. Chern claimed that Finsler geometry is just Riemannian geometry without the quadratic restriction. In certain sense, Chern's statement is confirmed, since many classical results can be easily extended from Riemannian to Finsler structures, as Hopf-Rinow, Cartan-Hadamard and Bonnet-Myers theorems, Rauch and Bishop-Gromov comparison principles, see D. Bao, S.-S. Chern and Z. Shen [4]. In spite of these facts, deep differences appear between the two geometries. Csaba was also extremely motivated to identify such nontrivial differences. In the sequel, we focus to the following two topics, both of them being his favorite research directions:

- *Busemann inequalities* and the existence of 'orthogonal' geodesic segments between Finsler submanifolds;
- *Sobolev spaces over Finsler manifolds* and their applications in the theory of PDEs.

To be more precise, let us give some basic notions from Finsler geometry. Let $M$ be an $n(\geq 2)$-dimensional differentiable manifold and its tangent bundle $TM = \bigcup_{x\in M} T_xM$. The pair $(M, F)$ is called a *Finsler manifold*, if the continuous function $F : TM \to [0, \infty)$ verifies the assumptions:

(a) $F \in C^\infty(TM \setminus \{0\})$;
(b) $F(x, \lambda y) = \lambda F(x, y)$ for every $\lambda \geq 0$ and $(x, y) \in TM$;
(c) $g_{ij}(x, y) = [\frac{1}{2}F^2]_{y^iy^j}(x, y)$ is positive definite for every $(x, y) \in TM \setminus \{0\}$, where $F(x, y) = F(y^i \frac{\partial}{\partial x^i}|_x)$.

$(M, F)$ is *reversible*, if instead of (b) one has:

(b') $F(x, \lambda y) = |\lambda| F(x, y)$ for every $\lambda \in \mathbb{R}$ and $(x, y) \in TM$.

Unlike the Levi-Civita connection in Riemannian geometry, there is no unique natural connection in the Finsler setting; either the metric compatibility or the torsion-free property fails for a generic Finsler connection. Among these objects, the Chern connection has appropriate properties to provide qualitative results on Finsler manifolds, see D. Bao, S.-S. Chern and Z. Shen [4]. By means of this connection, one can introduce Jacobi fields, geodesics, flag curvature (replacing the sectional curvature), etc.

### 3.1. Busemann inequalities and 'orthogonal' geodesics on Finsler manifolds

In the forties, parallel to A.D. Alexandrov's theory, H. Busemann [5] developed a synthetic geometry on non-smooth metric spaces. Among others, H. Busemann elaborated axiomatically the theory of non-positively curved metric spaces, where no differential structure is needed. This notion of non-positive curvature requires that in small geodesic triangles the length of a side is at least the twice of the geodesic distance of the midpoints of the other two sides, see H. Busemann [5, p. 237]; if this property is valid in every small geodesic triangle, the space is called *Busemann NPC space*. By making a connection between smooth and synthetic objects, H. Busemann proved that a Riemannian manifold $(M, g)$ is a Busemann NPC space if and only if its sectional curvature is non-positive. At the same time, he formulated the open question for non-Riemannian manifolds asking if non-positively curved Finsler manifolds are Busemann NPC space. It turned out that the picture for non-Riemannian Finsler spaces is totally different with respect to Riemannian manifolds. Indeed, P. Kelly and E. Straus [16] proved that a convex closed planar domain endowed with the standard Hilbert distance (providing a Finsler structure with constant flag curvature -1) is a Busemann NPC space if and only if the curve is an ellipse, thus the geometry reduces to the Riemannian one. After this result, nothing relevant happened till the early 2000s concerning Busemann's question on Finsler manifolds.

In 2003, Csaba and his co-authors proved in [17] that *non-positively curved Berwald manifolds*[2] *are Busemann NPC spaces.* In this way, Berwald manifolds became the first non-Riemannian Finsler spaces where H. Busemann's original question has been affirmatively answered. This result has been extended to further synthetic properties in [19], where the authors conjectured that non-positively curved Berwald manifolds are the largest Finsler objects which are Busemann NPC spaces. This question has been confirmed recently by S. Ivanov and A. Lytchak [14].

Since Busemann's inequality can be reformulated in terms of convexity, several applications can be found of the main results of [17, 19] by treating optimization problems, as Weber-type transportation phenomena on curved spaces; the reader may consult the monograph [20] for further applications in Economics and Geometry, written by Csaba and his co-authors.

Another important aspect of Finsler manifolds is to determine the number of geodesic segments perpendicular to certain submanifolds. Since the notion of perpendicularity as well as the behavior of the energy functional defined on the space of

[2]Special Finsler structures, where the coefficients of the Chern connection are not directional-dependent.

curves are delicate issues on Finsler manifolds, a comprehensive study of this problem was completed by Csaba and his co-authors in [18]; the most challenging part of the proof is the validity of the Palais-Smale compactness condition of the energy functional defined on the space of curves. The result from [18] has been extended by E. Caponio, M.Á. Javaloyes and A. Masiello [6] to stationary spacetimes over Finsler structures.

### 3.2. Sobolev spaces versus Finsler manifolds

Within the class of reversible Finsler manifolds (including in particular the class of Riemannian manifolds), the synthetic notion of Sobolev spaces on metric measure spaces and the analytic notion of Sobolev spaces coincide. However, the case when the Finsler manifold is *not* reversible (modeling e.g. Randers spaces, the Matsumoto mountain slope metric, or the Finsler-Poincaré ball), it turns out that surprising phenomena arise, which was described in the paper [12] of Csaba and his co-authors. To be more precise, let

$$W^{1,2}(M,F,\mathsf{m}) = \left\{ u \in W^{1,2}_{\mathrm{loc}}(M) : \int_M F^{*2}(x, Du(x))\mathrm{d}\mathsf{m}(x) < +\infty \right\},$$

and $W_0^{1,2}(M,F,\mathsf{m})$ be the closure of $C_0^\infty(M)$ with respect to the (asymmetric) norm

$$\|u\|_F = \left( \int_M F^{*2}(x, Du(x))\mathrm{d}\mathsf{m}(x) + \int_M u^2(x)\mathrm{d}\mathsf{m}(x) \right)^{1/2}, \tag{3.1}$$

where $\mathsf{m}$ is the usual measure on $(M,F)$. Let

$$r_F = \sup_{x\in M} \sup_{y \in T_xM\setminus\{0\}} \frac{F(x,y)}{F(x,-y)}$$

be the *reversibility constant* on $(M,F)$. Clearly, $r_F \geq 1$ and $r_F = 1$ if and only if $(M,F)$ is reversible. Let

$$F_s(x,y) = \left( \frac{F^2(x,y) + F^2(x,-y)}{2} \right)^{1/2}, \ (x,y) \in TM.$$

It is clear that $(M,F_s)$ is a reversible Finsler manifold, $F_s$ being the *symmetrized* Finsler metric associated with $F$.

In [12] the authors proved that if $r_F < +\infty$, then $(W_0^{1,2}(M,F,\mathsf{m}), \|\cdot\|_{F_s})$ is a reflexive Banach space, while the norm $\|\cdot\|_{F_s}$ and the asymmetric norm $\|\cdot\|_F$ are equivalent; in particular,

$$\left( \frac{1+r_F^2}{2} \right)^{-1/2} \|u\|_F \leq \|u\|_{F_s} \leq \left( \frac{1+r_F^{-2}}{2} \right)^{-1/2} \|u\|_F, \ \forall u \in W_0^{1,2}(M,F,\mathsf{m}).$$

A more surprising fact – which shows the genuine difference between Riemannian and Finsler geometry – is that the authors of [12] constructed a function $u$ on the Finsler-Poincaré ball (having the reversibility constant $+\infty$) such that $u \in W_0^{1,2}(M,F,\mathsf{m})$ but $-u \notin W_0^{1,2}(M,F,\mathsf{m})$. In this way, the Sobolev space over a non-compact Finsler manifold $(M,F)$ with $r_F = +\infty$ need not be even a vector space.

Csaba was also interested to study elliptic PDEs involving the Finsler-Laplace operator. Such kind of problems were discussed in [25], where the authors established Hardy-type inequalities on Finsler manifolds with some applications. Further results of Csaba and his co-authors, involving elliptic operators on different domains can be found in [10, 26, 29].

## 4. Concluding part

Csaba's most important contributions to applied mathematics have been published in internationally recognized journals such as *Calculus of Variations and Partial Differential Equations*, *Nonlinear Differential Equations and Applications*, *Discrete and Continuous Dynamical Systems-A*, *Advances in Differential Equations*, *Nonlinear Analysis Real World Applications*, etc. A summary of these results has been published in two monographs by *Cambridge University Press* in 2010 (see [20]) and *Springer* in 2021 (see [9]).

Csaba was invited to various research institutes and universities, as *Universita di Perugia, Eötvös Lóránd University, Alfréd Rényi Institute of Mathematics, Universita di Catania, Technical University of Athens*, etc. He collaborated with dozens of national and international mathematicians, resulting joint publications. He has more than 90 research papers, being cited in prestigious journals such as *Mathematische Annalen, Journal of Functional Analysis, Journal of Differential Equations* and others.

In addition to his scientific achievements, one of Csaba's greatest merits lies in discovering and educating young mathematical talents. Many of his former students became world-renowned mathematicians, working at prestigious European and American universities such as *Humboldt University, Utrecht University, Virginia Polytechnic Institute and State University.* As a doctoral supervisor, he advised numerous students, who became outstanding researchers and lecturers at the Babeş-Bolyai University and Sapientia University of Transylvania.

Csaba's absence remains an unfilled void in our soul.

## References

[1] Alves, C.O., Santos, J.A., *Multivalued elliptic equation with exponential critical growth in* $\mathbb{R}^2$, J. Differential Equations, **261**(2016), no. 9, 4758–4788.

[2] Alves, C.O., Duarte, R.C., Souto, M.A.S., *A Berestycki-Lions type result and applications*, Rev. Mat. Iberoam., **35**(2019), no. 6, 1859–1884.

[3] Ambrosetti, A., Rabinowitz, P.H., *Dual variational methods in critical point theory and applications*, J. Funct. Anal., **14**(1973), 349–381.

[4] Bao, D., Chern, S.-S., Shen, Z., *Introduction to Riemann-Finsler Geometry*, Graduate Texts in Mathematics, 200, Springer-Verlag, 2000.

[5] Busemann, H., *The Geometry of Geodesics*, Academic Press, 1955.

[6] Caponio, E., Javaloyes, M.Á., Masiello, A., *On the energy functional on Finsler manifolds and applications to stationary spacetimes*, Math. Ann., **351**(2011), no. 2, 365–392.

[7] Chang, K.-C., *Variational methods for nondifferentiable functionals and their applications to partial differential equations*, J. Math. Anal. Appl., **80**(1981), no. 1, 102–129.

[8] Clarke, F.H., *Optimization and Nonsmooth Analysis*, Wiley, 1983.

[9] Costea, N., Kristály, A., Varga, Cs., *Variational and Monotonicity Methods in Nonsmooth Analysis*, Springer International Publishing, 2021.

[10] Costea, N., Moroşanu, Gh., Varga, Cs., *Weak solvability for Dirichlet partial differential inclusions in Orlicz-Sobolev spaces*, Adv. Differential Equations, **23**(2018), no. 7-8, 523–54.

[11] Degiovanni, M., Marzocchi, M., *A critical point theory for nonsmooth functionals*, Ann. Mat. Pura Appl., **167**(1994), 73–100.

[12] Farkas, Cs., Kristály, A., Varga, Cs., *Singular Poisson equations on Finsler-Hadamard manifolds*, Calc. Var. Partial Differential Equations, **54**(2015), no. 2, 1219–1241.

[13] Frigon, M., *On a critical point theory for multivalued functionals and application to partial differential inclusions*, Nonlinear Anal., **31**(1998), no. 5-6, 735–753.

[14] Ivanov, S., Lytchak, A., *Rigidity of Busemann convex Finsler metrics*, Comment. Math. Helv., **94**(2019), no. 4, 855–868.

[15] Jabri, Y., *The Mountain Pass Theorem. Variants, Generalizations and Some Applications*, Encyclopedia of Mathematics and its Applications, 95. Cambridge University Press, Cambridge, 2003.

[16] Kelly, P., Straus, E., *Curvature in Hilbert geometry*, Pacific J. Math., **8**(1958), 119–125.

[17] Kozma, L., Kristály, A., Varga, Cs., *The dispersing of geodesics in Berwald spaces of non-positive flag curvature*, Houston J. Math., **30**(2004), no. 2, 413–420.

[18] Kozma, L., Kristály, A., Varga, Cs., *Critical point theorems on Finsler manifolds*, Beiträge Algebra Geom., **45**(2004), no. 1, 47–59.

[19] Kristály, A., Kozma, L., *Metric characterization of Berwald spaces of non-positive flag curvature*, J. Geom. Phys., **56**(2006), no. 8, 1257–1270.

[20] Kristály, A., Rădulescu, V., Varga, Cs., *Variational Principles in Mathematical Physics, Geometry, and Economics*, Volume 136 of Encyclopedia of Mathematics and its Applications. Cambridge University Press, Cambridge, 2010. Qualitative Analysis of Nonlinear Equations and Unilateral Problems, With a foreword by Jean Mawhin.

[21] Kristály, A., Varga, Cs., *Cerami (C) condition and mountain pass theorem for multivalued mappings*, Serdica Math. J., **28**(2002), no. 2, 95–108.

[22] Kristály, A., Varga, Cs., *A note on minimax results for continuous functionals*, Studia Univ. Babeş-Bolyai Math., **43**(1998), no. 3, 35–55.

[23] Kristály, A., Varga, Cs., Varga, V., *A nonsmooth principle of symmetric criticality and variational-hemivariational inequalities*, J. Math. Anal. Appl., **325**(2007), no. 2, 975–986.

[24] Lisei, H., Varga, Cs., *Some applications to variational-hemivariational inequalities of the principle of symmetric criticality for Motreanu-Panagiotopoulos type functionals*, J. Global Optim., **36**(2006), no. 2, 283–305.

[25] Mester, Á., Peter, I.R., Varga, Cs., *Sufficient criteria for obtaining Hardy inequalities on Finsler manifolds*, Mediterr. J. Math., **18**(2021), no. 2, Paper No. 76, 22 pp.

[26] Mihăilescu, M., Stancu-Dumitru, D., Varga, Cs., *The convergence of nonnegative solutions for the family of problems* $-\Delta_p u = \lambda e^u$ *as* $p \to \infty$, ESAIM Control Optim. Calc. Var., **24**(2018), no. 2, 569–578.

[27] Motreanu, D., Varga, Cs., *Some critical point results for locally Lipschitz functionals*, Comm. Appl. Nonlinear Anal., **4(3)**(1997), 17–33.

[28] Pintea, C., Varga, Cs., *A note on homology and homotopy groups of fiber spaces*, Mathematica, **39(62)(1)**(1997), 95–101.

[29] Precup, R., Pucci, P., Varga, Cs., *Energy-based localization and multiplicity of radially symmetric states for the stationary p-Laplace diffusion*, Complex Var. Elliptic Equ., **65**(2020), no. 7, 1198–1209.

[30] Varga, Cs., Crainic, M., *A note on the denseness of complete invariant metrics*, Publ. Math. Debrecen, **5**(1997), 265–271.

[31] Varga, Cs., Farcas, G., *On completeness of metric spaces*, Studia Univ. Babeş-Bolyai Math., **37(4)**(1993), 95–101.

[32] Varga, Cs., Farcas, G., *Ljusternik-Schnirelman theory on closed subsets of $C^1$-manifolds*, Studia Univ. Babeş-Bolyai Math., **38(2)**(1993), 75–89.

[33] Varga, Cs., Farcas, G., *A multiplicity theorem in equivariant case*, Mathematica, **38(61)(1-2)**(1996), 221–226.

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