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Multiple solutions for eigenvalue problems involving the (p, q)–Laplacian

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Dedicated to the memory of Professor Csaba Varga with high feelings of admiration for his notable contributions in Mathematics and great affection

Abstract. This paper is devoted to a subject that Professor Csaba Varga suggested during his frequent visits to the University of Perugia and in my regular stays at the "Babeş-Bolyai" University. More specifically, continuing the work started in [7] jointly with Professor Varga, here we establish the existence of two nontrivial (weak) solutions of some one parameter eigenvalue (p, q)–Laplacian problems under homogeneous Dirichlet boundary conditions in bounded domains of $\mathbb{R}^N$.

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1. Introduction

The paper concerns certain nonlinear eigenvalue homogeneous Dirichlet boundary condition problems in bounded domains Ω of $\mathbb{R}^N$, involving the (p, q)–Laplacian. Hence the subject is strongly connected with the paper [7], we wrote jointly with Professor Csaba Varga during his frequent visits to the University of Perugia and in my regular stays at the "Babeş-Bolyai" University. More specifically, continuing the work started in [7] for problems involving a general elliptic operator in divergence form with p growth, we extend the existence theorems of two nontrivial (weak) solutions of [7] to eigenvalue (p, q)–Laplacian problems. More specifically, we consider for a nonnegative real parameter λ the problem

$$\begin{cases} -\Delta_p u - \Delta_q u = \lambda\{a(x)|u|^{q-2}u + f(x,u)\} & \text{in } \Omega, \\ \quad u = 0 & \text{on } \partial\Omega, \end{cases} \tag{$\mathscr{P}_\lambda$}$$

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in a bounded domain Ω of $\mathbb{R}^N$ and we assume for simplicity that the exponents p, q are such that $1 < p < q < N$. The operator $\Delta_\wp$, with $\wp \in \{p, q\}$, appearing in problem $(\mathscr{P}_\lambda)$, is the well known $\wp$*–Laplacian*, which is defined as

$$\Delta_\wp \varphi = \operatorname{div}(|\nabla\varphi|_H^{\wp-2}\nabla\varphi) \quad \text{for all } \varphi \in C^2(\mathbb{R}^N).$$

Throughout the paper, *the weight* a *in* $(\mathscr{P}_\lambda)$ *is required to be positive a.e. in* Ω *and of class* $L^\alpha(\Omega)$*, with* $\alpha > N/q$. The nonlinear perturbation $f : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function, which satisfies the natural growth conditions $(\mathscr{F})$ from (a) to (c) given in Section 3, with part (c) of $(\mathscr{F})$ fairly technical, due to the complexity in handling the nonhomogeneous (p, q)– Laplacian.

In Section 4 we find up the exact intervals of λ's for which problem $(\mathscr{P}_\lambda)$ admits only the trivial solution and for which $(\mathscr{P}_\lambda)$ has at least two nontrivial solutions. More precisely, following the strategies introduced in [7], we prove the existence theorems for problem $(\mathscr{P}_\lambda)$, using as a crucial tool Theorem 2.1 of [7], which is a differentiable version and a variant of Theorem 3.4 in [1] due to Arcoya and Carmona.

For further previous contributions in the subject, beside [7], we mention the papers [11, 13] due to Varga, the latter related articles [6, 19], written at the University of Perugia. For noteworthy comments and an extensive bibliography as well as for applications of the well known three critical points theorems we refer to the monumental monograph [12] of Kristály, Rădulescu and Varga.

In Section 5 we treat the different nonlinear eigenvalue problem

$$\begin{cases} -\Delta_p u - \Delta_q u = \lambda f(x, u) & \text{in } \Omega, \\ \quad u = 0 & \text{on } \partial\Omega, \end{cases} \tag{$\mathcal{P}_\lambda$}$$

for which the technical assumption $(\mathscr{F})$–(c) is replaced by the more direct transparent request $(\mathscr{F})$–(c'), which is much easier to verify. This straight approach first started in [7] and we show here that the technique can be extended to cover the nonhomogeneous case of the (p, q)–Laplacian as well.

The importance of studying problems involving operators with non standard growth conditions, or (p, q) operators, begins independently with the pioneering papers of Zhikov in 1986 and Marcellini in 1991. The (p, q) operators were introduced in order to describe the behavior of highly anisotropic materials, that is, materials whose properties change drastically from point to point. Since then, increasing attention has been focused on the study of existence, regularity and qualitative properties of the solutions of problems of this type. For a detailed historical presentation and for a wide list of contributions on the subject we refer to the recent paper [17] due to Mingione and Rădulescu, editors of the Special Issue *New developments in non–uniformly elliptic and nonstandard growth problems*.

Concerning PDEs applications, the (p, q)–Laplacian $\Delta_p + \Delta_q$ arises from the study of general reaction–diffusion equations with nonhomogeneous diffusion and transport aspects. These nonhomogeneous operators have applications in biophysics, plasma physics and chemical reactions, with double phase features, where the function u corresponds to the concentration term, and the differential operator represents the diffusion coefficient. For further details we mention [14] as well as [17] and references therein. Different eigenvalue problems for the (p, q)–Laplacian have been extensively studied in recent years. In the context of Dirichlet boundary conditions we refer to the papers [4] by Bobkov and Tanaka, [8] by Colasuonno and Squassina, [14] by Marano and Mosconi, [15, 16] by Marano, Mosconi and Papageorgiou, to the recent paper [20] due to Tanaka and finally to the references therein.

For (p,q)–Laplacian eigenvalue problems under various boundary conditions (Robin, Steklov, etc.) we quote the recent papers [2] by Barbu and Moroşanu and [18] by Papageorgiou, Qin and Rădulescu, as well as their wide bibliography.

Let us end the comments by noting that the results of this note can be extended to the equations of problems $(\mathscr{P}_\lambda)$ and $(\mathcal{P}_\lambda)$ under Robin boundary conditions, as obtained in [7] via a delicate consequence of the three critical points Theorem 2.1 of [7].

2. Preliminaries and auxiliary results for $(\mathscr{P}_\lambda)$

In this section we introduce the main notation and assumptions for $(\mathscr{P}_\lambda)$. Throughout the paper, $\cdot$ denotes the Euclidean inner product and $|\cdot|$ the corresponding Euclidean norm in any space $\mathbb{R}^n$, $n = 1, 2, \dots$.

Let $1 < p < q < N$ and let $\mathscr{A}_{p,q} : \mathbb{R}^N \to \mathbb{R}$ be the potential

$$\mathscr{A}_{p,q}(\xi) = \frac{1}{p}|\xi|^p + \frac{1}{q}|\xi|^q \quad \text{of} \quad \mathbf{A}_{p,q}(\xi) = |\xi|^{p-2}\xi + |\xi|^{q-2}\xi. \tag{2.1}$$

Then both $\mathscr{A}_{p,q}$ and $\mathbf{A}_{p,q}$ are continuous in $\mathbb{R}^N$, $\mathscr{A}_{p,q}$ is even and strictly convex in $\mathbb{R}^N$. Clearly, $\mathbf{A}_{p,q}(\xi)\cdot\xi \geq \mathscr{A}_{p,q}(\xi)$ for all $\xi \in \mathbb{R}^N$.

Lemma 3 of [10] can also be generalized in this framework and we use the proof of Lemma 2.4 of [7], adopting the notation in (2.1).

Lemma 2.1. *Let ξ, $(\xi_n)_n$ be in $\mathbb{R}^N$ such that*

$$(\mathbf{A}_{p,q}(\xi_n) - \mathbf{A}_{p,q}(\xi)) \cdot (\xi_n - \xi) \to 0 \quad \text{as } n \to \infty. \tag{2.2}$$

Then $(\xi_n)_n$ converges to ξ.

Proof. First we assert that $(\xi_n)_n$ is bounded. Otherwise, up to a subsequence, still denoted by $(\xi_n)_n$, we would have $|\xi_n| \to \infty$. Hence, as $n \to \infty$

$$(\mathbf{A}_{p,q}(\xi_n) - \mathbf{A}_{p,q}(\xi)) \cdot (\xi_n - \xi) \sim \mathbf{A}_{p,q}(\xi_n)\xi_n = |\xi_n|^p + |\xi_n|^q \to \infty,$$

which is impossible by (2.2). Therefore, $(\xi_n)_n$ is bounded and possesses a subsequence, still denoted by $(\xi_n)_n$, which converges to some $\eta \in \mathbb{R}^N$. Thus $(\mathbf{A}_{p,q}(\eta) - \mathbf{A}_{p,q}(\xi)) \cdot (\eta - \xi) = 0$ by (2.2) and the strict convexity of $\mathscr{A}_{p,q}$ implies that $\eta = \xi$. This also shows that actually the entire sequence $(\xi_n)_n$ converges to ξ. $\square$

Since $1 < p < q < N$, the natural solution space of $(\mathscr{P}_\lambda)$ is the separable uniformly convex Sobolev space $W_0^{1,q}(\Omega)$, endowed with the usual norm $\|u\| = \|\nabla u\|_q$, being Ω a bounded domain of $\mathbb{R}^N$. From here on, any Lebesgue space $L^\wp(\Omega)$, $\wp \geq 1$, is equipped with the canonical norm $\|\cdot\|_\wp$, while $\wp'$ is the conjugate exponent of $\wp$. It is clear that $W^{-1,q'}(\Omega)$ is the dual space of $W_0^{1,q}(\Omega)$ and that $q^\star = Nq/(N-q)$ is the Sobolev critical exponent of $W_0^{1,q}(\Omega)$.

Lemma 2.2. *Let $\mathscr{A}_{p,q}$ be as in* (2.1)*. Then the functional*

$$\Phi_{p,q}(u) = \int_\Omega \mathscr{A}_{p,q}(\nabla u(x))dx = \frac{\|\nabla u\|_p^p}{p} + \frac{\|u\|^q}{q}, \qquad \Phi_{p,q} : W_0^{1,q}(\Omega) \to \mathbb{R},$$

is convex, weakly lower semicontinuous and of class $C^1(W_0^{1,q}(\Omega))$.

Moreover, $\Phi'_{p,q} : W_0^{1,q}(\Omega) \to W^{-1,q'}(\Omega)$ verifies the $(\mathscr{S}_+)$ condition, i.e., for every sequence $(u_n)_n \subset W_0^{1,q}(\Omega)$ such that $u_n \rightharpoonup u$ weakly in $W_0^{1,q}(\Omega)$ and

$$\limsup_{n\to\infty} \int_\Omega \mathbf{A}_{p,q}(\nabla u_n) \cdot (\nabla u_n - \nabla u)dx \le 0, \tag{2.3}$$

then $u_n \to u$ strongly in $W_0^{1,q}(\Omega)$.

Proof. A simple calculation shows that the functional $\Phi_{p,q}$ is convex and of class $C^1(W_0^{1,q}(\Omega))$. Hence, in particular $\Phi_{p,q}$ is weakly lower semicontinuous in $W_0^{1,q}(\Omega)$ by Corollary 3.9 of [5].

Let $(u_n)_n$ be a sequence in $W_0^{1,q}(\Omega)$ as in the statement. Then

$$\Phi_{p,q}(u) \le \liminf_n \Phi_{p,q}(u_n),$$

since $\Phi_{p,q}$ is weakly lower semicontinuous on $W_0^{1,q}(\Omega)$.

We claim that $\int_\Omega \mathbf{A}_{p,q}(\nabla u) \cdot (\nabla u_n - \nabla u)dx \to 0$ as $n \to \infty$. Indeed, since $u_n \rightharpoonup u$ in $W_0^{1,q}(\Omega)$ as $n \to \infty$, in particular $\nabla u_n \rightharpoonup \nabla u$ in $[L^q(\Omega)]^N$ and $\nabla u_n \rightharpoonup \nabla u$ in $[L^p(\Omega)]^N$ as $n \to \infty$. Moreover, (2.1) implies that $|\mathbf{A}_{p,q}(\nabla u)| \le |\nabla u|^{p-1} + |\nabla u|^{q-1}$, with clearly $|\nabla u|^{p-1} \in L^{p'}(\Omega)$ and $|\nabla u|^{q-1} \in L^{q'}(\Omega)$. This gives at once that

$$\begin{aligned}\int_\Omega \mathbf{A}_{p,q}(\nabla u) \cdot (\nabla u_n - \nabla u)dx &= \int_\Omega |\nabla u|^{p-2}\nabla u \cdot (\nabla u_n - \nabla u)dx \\ &\quad + \int_\Omega |\nabla u|^{q-2}\nabla u \cdot (\nabla u_n - \nabla u)dx\end{aligned}$$

tends to 0 as $n \to \infty$, as claimed.

Therefore, by convexity and (2.3) we get that

$$0 \le \limsup_{n\to\infty} \int_\Omega (\mathbf{A}_{p,q}(\nabla u_n) - \mathbf{A}_{p,q}(\nabla u)) \cdot (\nabla u_n - \nabla u)dx \le 0.$$

In other words,

$$\lim_{n\to\infty} \int_\Omega (\mathbf{A}_{p,q}(\nabla u_n) - \mathbf{A}_{p,q}(\nabla u)) \cdot (\nabla u_n - \nabla u)dx = 0,$$

that is the sequence $n \mapsto (\mathbf{A}_{p,q}(\nabla u_n) - \mathbf{A}_{p,q}(\nabla u)) \cdot (\nabla u_n - \nabla u) \ge 0$ converges to 0 in $L^1(\Omega)$. Hence, up to a subsequence, still denoted in the same way,

$$(\mathbf{A}_{p,q}(\nabla u_n) - \mathbf{A}_{p,q}(\nabla u)) \cdot (\nabla u_n - \nabla u) \to 0 \quad \text{a.e. in } \Omega.$$

Lemma 2.1 gives that also $\nabla u_n \to \nabla u$ a.e. in Ω. In particular, the Brézis–Lieb theorem gives as $n \to \infty$

$$\begin{aligned}\|\nabla u\|_p^p &= \|\nabla u_n\|_p^p - \|\nabla u_n - \nabla u\|_p^p + o(1), \\ \|\nabla u\|_q^q &= \|\nabla u_n\|_q^q - \|\nabla u_n - \nabla u\|_q^q + o(1),\end{aligned}$$

and (2.3) holds in the stronger form

$$\lim_{n\to\infty}\int_\Omega \mathbf{A}_{p,q}(\nabla u_n)\cdot(\nabla u_n-\nabla u)dx=0.$$

Consequently, the combination of the above facts implies that as $n\to\infty$

$$\begin{aligned}
o(1)&=\int_\Omega \mathbf{A}_{p,q}(\nabla u_n)\cdot(\nabla u_n-\nabla u)dx\\
&=\|\nabla u_n\|_p^p-\int_\Omega |\nabla u_n|^{p-2}\nabla u_n\cdot\nabla u dx\\
&\qquad+\|\nabla u_n\|_q^q-\int_\Omega |\nabla u_n|^{p-2}\nabla u_n\cdot\nabla u dx\\
&=\|\nabla u\|_p^p+\|\nabla u_n-\nabla u\|_p^p-\|\nabla u\|_p^p\\
&\qquad+\|\nabla u\|_q^q+\|\nabla u_n-\nabla u\|_q^q-\|\nabla u\|_q^q+o(1)\\
&=\|\nabla u_n-\nabla u\|_p^p+\|\nabla u_n-\nabla u\|_q^q+o(1),
\end{aligned}$$

since

$$|\nabla u_n|^{p-2}\nabla u_n\rightharpoonup|\nabla u|^{p-2}\nabla u \text{ in } [L^{p'}(\Omega)]^N$$

and similarly

$$|\nabla u_n|^{q-2}\nabla u_n\rightharpoonup|\nabla u|^{q-2}\nabla u \text{ in } [L^{q'}(\Omega)]^N.$$

In particular, $\|\nabla u_n-\nabla u\|_q=o(1)$ as $n\to\infty$, that is $u_n\to u$ strongly in $W_0^{1,q}(\Omega)$, as required. $\square$

3. Formulation of the problem $(\mathscr{P}_\lambda)$

The assumptions on the coefficient a make it a good Lebesgue weight. Thus, throughout the paper, for brevity in notation, we denote by $L^\wp(\Omega;a)$, $\wp\ge1$, the weighted $\wp$–Lebesgue space equipped with the norm

$$\|u\|_{\wp,a}=\left(\int_\Omega a(x)|u(x)|^\wp dx\right)^{1/\wp}.$$

In this section, we study $(\mathscr{P}_\lambda)$, so that $1<p<q<N$, the set Ω is a bounded domain of $\mathbb{R}^N$, and the natural solution space for $(\mathscr{P}_\lambda)$ is $W_0^{1,q}(\Omega)$. Before introducing the main structural assumptions on f, let us recall some basic properties, following somehow [7].

Since $a\in L^\alpha(\Omega)$ and $\alpha>N/q$, the embedding $W_0^{1,q}\hookrightarrow\hookrightarrow L^{\alpha' q}(\Omega)$ is compact. Moreover, $L^{\alpha' q}(\Omega)\hookrightarrow L^q(\Omega;a)$ is continuous, being by the Hölder inequality $\|u\|_{q,a}^q\le\|a\|_\alpha\|u\|_{\alpha' q}^q$ for all $u\in L^{\alpha' q}(\Omega)$. Hence, also the embedding $W_0^{1,q}(\Omega)\hookrightarrow\hookrightarrow L^q(\Omega;a)$ is compact.

Let λ_1 be *the first eigenvalue* of the problem

$$-\Delta_q u=\lambda\, a(x)|u|^{q-2}u$$

in $W_0^{1,q}(\Omega)$, that is λ_1 is defined by the Rayleigh quotient

$$\lambda_1 = \inf_{\substack{u \in W_0^{1,q}(\Omega) \\ u \neq 0}} \frac{\int_\Omega |\nabla u|^q dx}{\int_\Omega a(x)|u|^q dx} = \inf_{\substack{u \in W_0^{1,q}(\Omega) \\ u \neq 0}} \frac{\|u\|^q}{\|u\|_{q,a}^q}. \tag{3.1}$$

By Proposition 3.1 of [9], the infimum in (3.1) is achieved and $\lambda_1 > 0$. Denote by $u_1 \in W_0^{1,q}(\Omega)$ the normalized eigenfunction corresponding to λ_1, that is $\|u_1\|_{q,a} = 1$ and $\|u_1\|^q = \lambda_1$. In particular,

$$\lambda_1 \|u\|_{q,a}^q \leq \|u\|^q \quad \text{for every } u \in W_0^{1,q}(\Omega). \tag{3.2}$$

On f we assume the next condition.

($\mathscr{F}$) *Let $f : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function in $\mathbb{R}$, $f \not\equiv 0$, satisfying the following properties.*

(a) *There exist two measurable functions f_0, f_1 on Ω and a real exponent $m \in (1, q)$, such that $0 \leq f_0 \leq C_f a$, $0 \leq f_1 \leq C_f a$ a.e. in Ω for some appropriate constant $C_f > 0$, and*

$$|f(x,s)| \leq f_0(x) + f_1(x)|s|^{m-1} \quad \textit{for a.a. } x \in \Omega \textit{ and all } s \in \mathbb{R}.$$

(b) *There exists $\gamma \in (q, q^\star/\alpha')$ such that $\limsup_{s\to 0} \dfrac{|f(x,s)|}{a(x)|s|^{\gamma-1}} < \infty$, uniformly a.e. in Ω.*

(c) $\displaystyle\int_\Omega F(x, u_1(x))dx \geq \frac{1}{q'} + \frac{q'}{p\lambda_1}\|\nabla u_1\|_p^p$, *where u_1 is the first normalized eigenfunction defined above, $F(x,s) = \displaystyle\int_0^s f(x,t)dt$ and q' is the Hölder conjugate of q.*

Note that, in the literature, $a \in L^\infty(\Omega)$ in the more familiar and standard setting of the p–Laplacian, so that the exponent γ in ($\mathscr{F}$)–(b) belongs to the open interval $(p, p^\star)$. For further comments on p–growth problems, we refer to [7].

As shown in [7], conditions ($\mathscr{F}$)–(a) and (b) imply that $f(x, 0) = 0$ for a.a. $x \in \Omega$, that by the L'Hôpital rule

$$\limsup_{s\to 0} \frac{|F(x,s)|}{a(x)|s|^\gamma} < \infty \quad \text{uniformly a.e. in } \Omega, \tag{3.3}$$

and finally that

$$S_f = \operatorname*{ess\,sup}_{s\neq 0, x\in\Omega} \frac{|f(x,s)|}{a(x)|s|^{q-1}} \in \mathbb{R}^+ \tag{3.4}$$

is positive and finite by ($\mathscr{F}$)–(b) and the fact that $\gamma > q$. Moreover, $|f(x,s)|/a(x)|s|^{m-1} \leq 2C_f|s|^{m-q}$ for a.a. $x \in \Omega$ and all s, with $|s| \geq 1$, by ($\mathscr{F}$)–(a). Thus,

$$\lim_{s\to\infty} \frac{|f(x,s)|}{a(x)|s|^{q-1}} = 0 \quad \text{uniformly a.e. in } \Omega,$$

since $1 < m < q$ by ($\mathscr{F}$)–(a).

Hence *the positive number*

$$\lambda_\star = \frac{\lambda_1}{1+S_f} \tag{3.5}$$

is well defined. Furthermore, by (3.4)

$$\operatorname*{ess\,sup}_{s\neq 0, x\in\Omega} \frac{|F(x,s)|}{a(x)|s|^q} = \frac{S_f}{q}. \tag{3.6}$$

The main result of the section is proved by using the underlying energy functional J_λ associated to the variational problem $(\mathscr{P}_\lambda)$. For later purposes, we write J_λ in the form

$$\begin{gathered} J_\lambda(u) = \Phi_{p,q}(u) + \lambda\Psi(u), \\ \Psi(u) = -\mathcal{H}(u), \quad \mathcal{H}(u) = \mathcal{H}_1(u) + \mathcal{H}_2(u), \\ \mathcal{H}_1(u) = \frac{1}{q}\|u\|_{q,a}^q, \quad \mathcal{H}_2(u) = \int_\Omega F(x,u(x))dx. \end{gathered} \tag{3.7}$$

Thanks to Lemma 2.2, $(\mathscr{F})$–(a) and (b) it is easy to see that the functional J_λ is well defined in $W_0^{1,q}(\Omega)$ and of class $C^1(W_0^{1,q}(\Omega))$. Furthermore,

$$\begin{aligned} \langle J'_\lambda(u), \varphi\rangle = &\int_\Omega \mathbf{A}_{p,q}(\nabla u(x))\cdot\nabla\varphi(x)dx \\ &- \lambda\int_\Omega \left\{a(x)|u(x)|^{q-2}u(x) + f(x,u(x))\right\}\varphi(x)dx, \end{aligned}$$

where $\langle\cdot,\cdot\rangle$ denotes the duality pairing between $W_0^{1,q}(\Omega)$ and its dual space $W^{-1,q'}(\Omega)$. Therefore, the critical points $u\in W_0^{1,q}(\Omega)$ of the functional J_λ are exactly the (weak) solutions of problem $(\mathscr{P}_\lambda)$.

By convenience, for every $r \in (\inf_{u\in W_0^{1,q}(\Omega)} \Psi(u), \sup_{u\in W_0^{1,q}(\Omega)} \Psi(u))$ let us introduce the two functions

$$\varphi_1(r) = \inf_{u\in\Psi^{-1}(I_r)} \frac{\inf_{v\in\Psi^{-1}(r)} \Phi_{p,q}(v) - \Phi_{p,q}(u)}{\Psi(u) - r}, \quad I_r = (-\infty, r), \tag{3.8}$$

$$\varphi_2(r) = \sup_{u\in\Psi^{-1}(I^r)} \frac{\inf_{v\in\Psi^{-1}(r)} \Phi_{p,q}(v) - \Phi_{p,q}(u)}{\Psi(u) - r}, \quad I^r = (r, \infty). \tag{3.9}$$

If $\Psi(v) < 0$ at some $v\in W_0^{1,q}(\Omega)$, then *the crucial positive number*

$$\lambda^\star = \varphi_1(0) = \inf_{u\in\Psi^{-1}(I_0)} -\frac{\Phi_{p,q}(u)}{\Psi(u)}, \qquad I_0 = (-\infty, 0), \tag{3.10}$$

is well defined.

The proof of the next result, as well as the proof on the main existence theorem for $(\mathscr{P}_\lambda)$, is where we use the technical assumption $(\mathscr{F})$–(c).

Lemma 3.1. *If $(\mathscr{F})$–(a), (b) and (c) hold, then $\Psi^{-1}(I_0)$ is non–empty and moreover $\lambda_\star \le \lambda^\star < \lambda_1$.*

Proof. From $(\mathscr{F})$–(c) and (3.7) it follows in particular that that $\Psi(u_1) < 0$, since

$$\mathcal{H}(u_1) > \frac{1}{q}, \quad \text{i.e. } u_1 \in \Psi^{-1}(I_0).$$

Hence, $\lambda^\star$ is well defined. Again by $(\mathscr{F})$–(c) and (3.7)

$$\begin{aligned}\lambda^\star = \varphi_1(0) &= \inf_{u\in\Psi^{-1}(I_0)} -\frac{\Phi_{p,q}(u)}{\Psi(u)} \\ &\le \frac{\Phi_{p,q}(u_1)}{\mathcal{H}(u_1)} = \frac{\|\nabla u_1\|_p^p/p + \|\nabla u_1\|_q^q/q}{\|u_1\|_{q,a}^q/q + \int_\Omega F(x,u_1(x))dx} \\ &\le \frac{\|\nabla u_1\|_p^p/p + \|\nabla u_1\|_q^q/q}{1/q + 1/q' + q'\|\nabla u_1\|_p^p/p\lambda_1} \\ &< \frac{\|\nabla u_1\|_p^p/p}{q'\|\nabla u_1\|_p^p/p\lambda_1} + \frac{\|u_1\|^q}{q} = \lambda_1,\end{aligned}$$

as required. Finally, by (3.7), (3.6) and (3.2), for all $u \in W_0^{1,q}(\Omega)$, with $u \neq 0$, we have

$$\frac{\Phi_{p,q}(u)}{|\Psi(u)|} \ge \frac{\|u\|^q/q}{(1+S_f)\|u\|_{q,a}^q/q} \ge \frac{\lambda_1}{1+S_f} = \lambda_\star.$$

Hence, in particular $\lambda^\star \ge \lambda_\star$ by (3.10). □

Lemma 3.2. *If $(\mathscr{F})$–(a) holds, then the operators*

$$\mathcal{H}_1', \quad \mathcal{H}_2', \quad \Psi' : W_0^{1,q}(\Omega) \to W^{-1,q'}(\Omega)$$

are compact and $\mathcal{H}_1$, $\mathcal{H}_2$, Ψ are sequentially weakly continuous in $W_0^{1,q}(\Omega)$.

The proof is mutatis mutandis the same as the proof of the similar Lemma 3.2 of [7] and so we omit it here.

Lemma 3.3. *If $(\mathscr{F})$–(a) holds, then the functional $J_\lambda = \Phi_{p,q} + \lambda\Psi$ is coercive in $W_0^{1,q}(\Omega)$ for every $\lambda \in I$, $I = (-\infty, \lambda_1)$.*

Proof. Clearly, $(\mathscr{F})$–(a) implies that

$$|F(x,s)| \le f_0(x)|s| + f_1(x)|s|^m/m \le f_0(x) + (f_0(x) + f_1(x)/m)|s|^m \tag{3.11}$$

for a.a. $x \in \Omega$ and all $s \in \mathbb{R}$.

Fix $\lambda \in (-\infty, \lambda_1)$ and $u \in W_0^{1,q}(\Omega)$. Then, (3.2), (3.7), (3.11) and the Hölder inequality give

$$\begin{aligned}J_\lambda(u) &\ge \frac{1}{q}\|u\|^q - \frac{\lambda}{q}\|u\|_{q,a}^q - |\lambda|\int_\Omega |F(x,u)|dx \\ &\ge \frac{1}{q}\left(1 - \frac{\lambda}{\lambda_1}\right)\|u\|^q - |\lambda|\cdot\|f_0\|_1 - |\lambda|\cdot\|f_0 + f_1/m\|_\alpha \|u\|_{\alpha' m}^m \\ &\ge \frac{1}{q}\left(1 - \frac{\lambda}{\lambda_1}\right)\|u\|^q - |\lambda| C_1 - |\lambda|\, C_2\|u\|^m,\end{aligned}$$

where $C_1 = \|f_0\|_1$ and $C_2 = c_{\alpha' m}^m\|f_0 + f_1/m\|_\alpha$, where $c_{\alpha' m}$ denotes the Sobolev constant of the compact embedding $W_0^{1,q}(\Omega) \hookrightarrow\hookrightarrow L^{\alpha' m}(\Omega)$. Clearly $C_1 < \infty$, since

$f_0 \in L^\alpha(\Omega) \subset L^1(\Omega)$ by $(\mathscr{F})$–(a), being $\alpha > N/q > 1$ and Ω bounded. This shows the assertion, since $1 < m < q$ by $(\mathscr{F})$–(a). □

4. Main result for $(\mathscr{P}_\lambda)$

Following the strategies proposed in [7], here we prove the main theorem for the (p, q) problem $\mathscr{P}_\lambda$.

Theorem 4.1. *Let $\mathscr{A}_{p,q}$ be as in* (2.1) *and let $\lambda_\star$ and $\lambda^\star$ be as defined in* (3.5) *and in* (3.10), *respectively. Assume that $(\mathscr{F})$–(a) and (b) hold.*

(i) If $\lambda \in [0, \lambda_\star]$, then $(\mathscr{P}_\lambda)$ has only the trivial solution.

(ii) If also $(\mathscr{F})$–(c) holds, then problem $(\mathscr{P}_\lambda)$ admits at least two nontrivial solutions for every $\lambda \in (\lambda^\star, \lambda_1)$, where $\lambda^\star = \varphi_1(0) < \lambda_1$ by Lemma 3.1.

Proof. (i) Let $u \in W_0^{1,q}(\Omega)$ be a nontrivial solution of $(\mathscr{P}_\lambda)$ for some $\lambda \geq 0$. Then,

$$\int_\Omega \mathbf{A}_{p,q}(\nabla u) \cdot \nabla \varphi \, dx = \lambda \int_\Omega \{a(x)|u|^{q-2}u + f(x, u)\}\varphi \, dx$$

for all $\varphi \in W_0^{1,q}(\Omega)$. Take $\varphi = u$ and by (2.1), (3.2), (3.4) and (3.7)

$$\begin{aligned} \lambda_1 \|u\|^q &< \lambda_1 \int_\Omega \mathbf{A}_{p,q}(\nabla u)\nabla u \, dx = \lambda_1 \lambda \int_\Omega \{a(x)|u|^q + f(x, u)u\}dx \\ &= \lambda_1 \lambda \left(\|u\|_{q,a}^q + \int_\Omega \frac{f(x, u)}{a(x)|u|^{q-1}} a(x)|u|^q dx \right) \\ &\leq \lambda_1 \lambda \left(1 + S_f\right) \|u\|_{q,a}^q \leq \lambda \left(1 + S_f\right) \|u\|^q. \end{aligned}$$

Therefore $\lambda > \lambda_\star$ by (3.5), as required.

(ii) By (2.1) the functional $\Phi_{p,q}$ is convex. Moreover, $\Phi_{p,q}$ is weakly lower semicontinuous and $\Phi'_{p,q}$ verifies condition $(\mathscr{S}_+)$ in $W_0^{1,q}(\Omega)$, as already proved in Lemma 2.2. Furthermore, $\Psi' : W_0^{1,q}(\Omega) \to W^{-1,q'}(\Omega)$ is compact and Ψ is sequentially weakly continuous in $W_0^{1,q}(\Omega)$ by Lemma 3.2. Moreover, the functional J_λ is coercive for every $\lambda \in I$, where $I = (-\infty, \lambda_1)$, thanks to Lemma 3.3.

We claim that $\Psi(W_0^{1,q}(\Omega)) \supset \mathbb{R}_0^- = (-\infty, 0]$. Indeed, $\Psi(0) = 0$ and $(\mathscr{F})$–(a) and (3.11) imply that

$$|F(x, s)| \leq f_0(x) + (1 + 1/m)C_f a(x)|s|^m$$

for a.a. $x \in \Omega$ and all $s \in \mathbb{R}$. Hence, the Hölder inequality gives

$$\begin{aligned} \Psi(u) &\leq -\frac{1}{q}\|u\|_{q,a}^q + \int_\Omega |F(x, u)|dx \\ &\leq -\frac{1}{q}\|u\|_{q,a}^q + \|f_0\|_1 + 2C_f \int_\Omega a(x)|u|^m dx \\ &\leq -\frac{1}{q}\|u\|_{q,a}^q + \|f_0\|_1 + 2C_f \|a\|_1^{(q-m)/q} \|u\|_{q,a}^m, \end{aligned}$$

since $a \in L^1(\Omega)$, being $\alpha > N/q > 1$ and Ω bounded. Therefore,

$$\lim_{\substack{\|u\|_{q,a}\to\infty \\ u\in W_0^{1,q}(\Omega)}} \Psi(u) = -\infty,$$

thanks to the restriction $1 < m < q$ in assumption $(\mathscr{F})$–(a). Hence, the claim follows by the continuity of Ψ in $W_0^{1,q}(\Omega)$ and by (3.2).

Thus, $(\inf_{W_0^{1,q}(\Omega)} \Psi, \sup_{W_0^{1,q}(\Omega)} \Psi) \supset \mathbb{R}_0^-$. By (3.8) for every $u \in \Psi^{-1}(I_0)$ we have

$$\varphi_1(r) \le \frac{\Phi_{p,q}(u)}{r - \Psi(u)} \quad \text{for all } r \in (\Psi(u), 0),$$

so that

$$\limsup_{r\to 0^-} \varphi_1(r) \le -\frac{\Phi_{p,q}(u)}{\Psi(u)} \quad \text{for all } u \in \Psi^{-1}(I_0).$$

In other words, by (3.10)

$$\limsup_{r\to 0^-} \varphi_1(r) \le \varphi_1(0) = \lambda^\star. \tag{4.1}$$

From $(\mathscr{F})$–(a) and (b), that is (3.3) and (3.4), it follows the existence of a positive real number $\kappa > 0$ such that

$$|F(x,s)| \le \kappa\, a(x)|s|^\gamma \quad \text{for a.a. } x \in \Omega \text{ and all } s \in \mathbb{R}. \tag{4.2}$$

To this aim, denoting by ℓ_0 the limit number in (3.3), there exists $\delta > 0$ such that $|F(x,s)| \le (\ell_0 + 1)a(x)|s|^\gamma$ for a.a. $x \in \Omega$ and all s, with $|s| < \delta$. Fix s, with $|s| \ge \delta$, then by (3.6) for a.a. $x \in \Omega$

$$|F(x,s)| \le \frac{S_f}{q}|s|^{q-\gamma}a(x)|s|^\gamma \le \frac{S_f\delta^{q-\gamma}}{q}a(x)|s|^\gamma,$$

being $\gamma > q$ by $(\mathscr{F})$–(b). Hence, $\kappa = \max\{\ell_0 + 1, S_f\delta^{q-\gamma}/q\}$ and (4.2) holds.
We note in passing that the embedding $W_0^{1,q}(\Omega) \hookrightarrow L^\gamma(\Omega; a)$ is continuous. Indeed, by the Hölder inequality, with $1/\wp + 1/\alpha + \gamma/q^\star = 1$, where $\wp$ is the crucial exponent

$$\wp = \frac{\alpha' q^\star}{q^\star - \gamma\alpha'} > 1,$$

being $\gamma \in (q, q^\star/\alpha')$, as assumed in $(\mathscr{F})$–(b), we have

$$\int_\Omega a(x)|u|^\gamma dx \le |\Omega|^{1/\wp}\|a\|_\alpha \|u\|_{q^\star}^\gamma \le \tilde{C}\|u\|^\gamma, \tag{4.3}$$

where $\tilde{C} = c_{q^\star}^\gamma |\Omega|^{1/\wp}\|a\|_\alpha$ and $c_{q^\star}$ is the Sobolev constant for the continuous embedding $W_0^{1,q}(\Omega) \hookrightarrow L^{q^\star}(\Omega)$.

Hence, by (3.2), (3.7), (4.2) and (4.3) for every $u \in W_0^{1,q}(\Omega)$, we get

$$|\Psi(u)| \le \frac{1}{q\lambda_1}\|u\|^q + C_\gamma\|u\|^\gamma, \tag{4.4}$$

where $C_\gamma = \tilde{C}\,\kappa$. Therefore, given $r < 0$ and $v \in \Psi^{-1}(r)$ we have by (2.1)

$$r = \Psi(v) \ge -\frac{1}{q\lambda_1}\|v\|^q - C_\gamma\|v\|^\gamma \ge -\frac{1}{\lambda_1}\Phi_{p,q}(v) - \ell\,\Phi_{p,q}(v)^{\gamma/q}, \tag{4.5}$$

where $\ell = C_\gamma q^{\gamma/q}$.

Since the functional $\Phi_{p,q}$ is bounded below, coercive and lower semicontinuous on the reflexive Banach space $W_0^{1,q}(\Omega)$, it is easy to see that $\Phi_{p,q}$ is also coercive on the sequentially weakly closed non–empty set $\Psi^{-1}(r)$ thanks to Lemma 3.2. Therefore, by Theorem 6.1.1 of [3], there exists an element $u_r \in \Psi^{-1}(r)$ such that

$$\Phi_{p,q}(u_r) = \inf_{v\in\Psi^{-1}(r)} \Phi_{p,q}(v).$$

By (3.9), we get

$$\varphi_2(r) \geq -\frac{1}{r} \inf_{v\in\Psi^{-1}(r)} \Phi_{p,q}(v) = \frac{\Phi_{p,q}(u_r)}{|r|},$$

being $u \equiv 0 \in \Psi^{-1}(I^r)$. From (4.5) we obtain

$$\begin{aligned} 1 &\leq \frac{1}{\lambda_1} \cdot \frac{\Phi_{p,q}(u_r)}{|r|} + \ell|r|^{\gamma/p-1}\left(\frac{\Phi_{p,q}(u_r)}{|r|}\right)^{\gamma/q} \\ &\leq \frac{\varphi_2(r)}{\lambda_1} + \ell|r|^{\gamma/q-1}\varphi_2(r)^{\gamma/q}. \end{aligned} \tag{4.6}$$

There are now two possibilities to be considered. Either φ_2 is locally bounded at 0^-, so that the above inequality shows at once that

$$\liminf_{r\to 0^-} \varphi_2(r) \geq \lambda_1, \tag{4.7}$$

being $\gamma > q$ by $(\mathscr{F})$–(b), or $\limsup_{r\to 0^-} \varphi_2(r) = \infty$. In both cases, (4.1) and Lemma 3.1 yield that

$$\limsup_{r\to 0^-} \varphi_1(r) \leq \lambda^\star < \lambda_1 \leq \limsup_{r\to 0^-} \varphi_2(r).$$

Hence, for all integers $n \geq n^\star = 1 + [2/(\lambda_1 - \lambda^\star)]$ there exists a number $r_n < 0$ so close to 0^- that $\varphi_1(r_n) < \lambda^\star + 1/n < \lambda_1 - 1/n < \varphi_2(r_n)$. In particular,

$$[\lambda^\star + 1/n, \lambda_1 - 1/n] \subset (\varphi_1(r_n), \varphi_2(r_n)) = (\varphi_1(r_n), \varphi_2(r_n)) \cap I \tag{4.8}$$

for all $n \geq n^\star$, where $I = (-\infty, \lambda_1)$ is the interval of λ's on which J_λ is coercive in $W_0^{1,q}(\Omega)$ by Lemma 3.3. Therefore, since all the assumptions of Theorem 2.1, Part (a) of (ii) of [7] are satisfied and $u \equiv 0$ is a critical point of J_λ, problem $(\mathscr{P}_\lambda)$ admits at least two nontrivial solutions for all $\lambda \in (\varphi_1(r_n), \varphi_2(r_n))$ and all $n \geq n^\star$. In conclusion, problem $(\mathscr{P}_\lambda)$ admits at least two nontrivial solutions for all $\lambda \in (\lambda^\star, \lambda_1)$, since

$$(\lambda^\star, \lambda_1) = \bigcup_{n=n^\star}^{\infty} [\lambda^\star + 1/n, \lambda_1 - 1/n] \subset \bigcup_{n=n^\star}^{\infty} (\varphi_1(r_n), \varphi_2(r_n))$$

by (4.8). $\square$

5. The nonlinear eigenvalue problem $(\mathcal{P}_\lambda)$

In this last section we treat the different somehow simpler nonlinear eigenvalue problem $(\mathcal{P}_\lambda)$, for which the involved assumption $(\mathscr{F})$–(c) is replaced by a more direct transparent request, which is much easier to verify.

To this aim, let us denote by

$$B_0 = \{x \in \mathbb{R}^N \,:\, |x - x_0| \le r_0\}$$

the closed ball of $\mathbb{R}^N$ centered at a point $x_0 \in \mathbb{R}^N$ and of radius $r_0 > 0$. As in the previous paper [7], for the somehow simpler problem $(\mathcal{P}_\lambda)$, the *ad hoc* hypothesis $(\mathscr{F})$–(c) is replaced by the less stringent condition

$(\mathscr{F})$–(c') *Assume that there exist* $x_0 \in \Omega$, $s_0 \in \mathbb{R}$ *and* $r_0 > 0$ *so small that* $B_0 \subset \Omega$ *and*

$$\operatorname*{ess\,inf}_{B_0} F(x, |s_0|) = \mu_0 > 0, \qquad \operatorname*{ess\,sup}_{B_0} \max_{|t| \le |s_0|} |F(x,t)| = M_0 < \infty.$$

Clearly, when f does not depend on x, condition $(\mathscr{F})$–(c') simply reduces to the request that $F(s_0) > 0$ at a point $s_0 \in \mathbb{R}$, as first assumed in [11] by Kristály, Lisei and Varga. In this new setting, we derive the next result which improves the main theorem of [11] and extends Corollary 3.6 of [7] to the (p,q)–Laplacian case.

Theorem 5.1. *Let* $\mathscr{A}_{p,q}$ *be as in* (2.1) *and let* $f : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ *satisfy conditions* $(\mathscr{F})$–(a) *and* (b).

(*i*) *If* $\lambda \in [0, \ell_\star]$, *where* $\ell_\star = \lambda_1/S_f$, *then problem* $(\mathcal{P}_\lambda)$ *has only the trivial solution.*
(*ii*) *If furthermore* f *verifies* $(\mathscr{F})$–(c'), *then there exists* $\ell^\star \ge \ell_\star$ *such that* $(\mathcal{P}_\lambda)$ *admits at least two nontrivial solutions for all* $\lambda \in (\ell^\star, \infty)$.

Proof. Using the notation of (2.1) and Lemma 2.2, the energy functional $\mathcal{J}_\lambda$, associated to problem $(\mathcal{P}_\lambda)$, is given by $\mathcal{J}_\lambda = \Phi_{p,q} + \lambda \Psi_2$, where $\Phi_{p,q}$ is defined in Lemma 2.2 and

$$\Psi_2(u) = -\int_\Omega F(x, u(x))dx \quad \text{for all } u \in W_0^{1,q}(\Omega).$$

First, note that $\mathcal{J}_\lambda$ is coercive in $W_0^{1,q}(\Omega)$ for every $\lambda \in \mathbb{R}$. Indeed, as shown in the proof of Lemma 3.3, by (2.1) for all $u \in W_0^{1,q}(\Omega)$

$$\mathcal{J}_\lambda(u) \ge \frac{1}{q}\|u\|^q - |\lambda| \int_\Omega |F(x,u)|dx \ge \frac{1}{q}\|u\|^q - |\lambda| C_1 - |\lambda|\, C_2 \|u\|^m,$$

where $C_1 = \|f_0\|_1$, $C_2 = c_{\alpha' m}^m \|f_0 + f_1/m\|_\alpha$ and $c_{\alpha' m}$ denotes as before the Sobolev constant of the compact embedding $W_0^{1,q}(\Omega) \hookrightarrow\hookrightarrow L^{\alpha' m}(\Omega)$. This shows the claim, since $1 < m < q$ by $(\mathscr{F})$–(a). Hence, here $I = \mathbb{R}$.

(*i*) This part of the statement is proved using the same argument produced for the proof of Theorem 4.1–(*i*). Let $u \in W_0^{1,q}(\Omega)$ be a nontrivial solution of $(\mathcal{P}_\lambda)$ for some $\lambda \ge 0$. Then, by (2.1) and (3.4)

$$\begin{aligned} \lambda_1 \|u\|^q &< \lambda_1 \int_\Omega \mathbf{A}_{p,q}(\nabla u) \cdot \nabla u dx = \lambda_1 \lambda \int_\Omega f(x,u) u dx \le \lambda_1 \lambda S_f \|u\|_{q,a}^q \\ &\le \lambda S_f \|u\|^q \end{aligned}$$

thanks to (3.2). Thus, if u is a nontrivial (weak) solution of $(\mathcal{P}_\lambda)$, then necessarily $\lambda > \ell_\star = \lambda_1/S_f$, as required.

(ii) The proof of this part is again strongly based on an application of Theorem 2.1, Part (a) of (ii) of [7] and the fact that $u \equiv 0$ is a critical point of $\mathcal{J}_\lambda$. The new key functions φ_1 and φ_2 are now given by

$$\begin{aligned} \varphi_1(r) &= \inf_{u\in\Psi_2^{-1}(I_r)} \frac{\displaystyle\inf_{v\in\Psi_2^{-1}(r)} \Phi_{p,q}(v) - \Phi_{p,q}(u)}{\Psi_2(u) - r}, \quad I_r = (-\infty, r), \\ \varphi_2(r) &= \sup_{u\in\Psi_2^{-1}(I^r)} \frac{\displaystyle\inf_{v\in\Psi_2^{-1}(r)} \Phi_{p,q}(v) - \Phi_{p,q}(u)}{\Psi_2(u) - r}, \quad I^r = (r, \infty). \end{aligned} \tag{5.1}$$

We first show that there exists $u_0 \in W_0^{1,q}(\Omega)$ such that $\Psi_2(u_0) < 0$, so that the crucial number

$$\ell^\star = \varphi_1(0) = \inf_{u\in\Psi_2^{-1}(I_0)} -\frac{\Phi_{p,q}(u)}{\Psi_2(u)}, \quad I_0 = (-\infty, 0), \tag{5.2}$$

is well defined. To this aim, take $\sigma \in (0,1)$ and put

$$B = \{x \in \mathbb{R}^N : |x - x_0| \le \sigma r_0\}, \quad B_1 = \left\{x \in \mathbb{R}^N : |x - x_0| \le r_1\right\},$$

where $r_1 = (1+\sigma)r_0/2$. Hence,

$$B \subset B_1 \subset B_0 \subset \Omega.$$

Clearly, $F(x,0) = 0$ a.e. in Ω, so that $s_0 \ne 0$ in $(\mathscr{F})$–(c'). Put $v_0 = |s_0|\chi_{B_1}$ in Ω and fix ε, with $0 < \varepsilon < (1-\sigma)r_0/2$. Denote by ρ_ε the convolution kernel of fixed radius ε and define

$$u_0 = \rho_\varepsilon * v_0 \quad \text{in } \Omega.$$

Hence, $u_0 \equiv |s_0|$ in B, $0 \le u_0 \le |s_0|$ in Ω, $u_0 \in C_c^\infty(\Omega)$ and $\operatorname{supp} u_0 \subset B_0$. Therefore, $u_0 \in W_0^{1,q}(\Omega)$. By $(\mathscr{F})$–(c'),

$$\begin{aligned} \Psi_2(u_0) &= -\int_B F(x,|s_0|)dx - \int_{B_0\setminus B} F(x,u_0(x))dx \le M_0 \int_{B_0\setminus B} dx - \mu_0 \int_B dx \\ &\le \omega_N r_0^N \left[M_0(1-\sigma^N) - \mu_0\sigma^N\right], \end{aligned}$$

where ω_N is the measure of the unit ball in $\mathbb{R}^N$. Then, taking $\sigma \in (0,1)$ so close to 1^- that $\sigma^N > M_0/(\mu_0 + M_0)$, we get that $\Psi_2(u_0) < 0$, as claimed.

Furthermore, by (3.6) and (3.2), for all $u \in W_0^{1,q}(\Omega)$, with $u \not\equiv 0$, we easily obtain that

$$\frac{\Phi_{p,q}(u)}{|\Psi_2(u)|} \ge \frac{\|u\|^q/q}{S_f\|u\|_{q,a}^q/q} \ge \frac{\lambda_1}{S_f} = \ell_\star.$$

Hence, $\ell^\star \ge \ell_\star$ by (5.2).

In particular, for all $u \in \Psi_2^{-1}(I_0)$, we have by (5.1)

$$\varphi_1(r) \le \frac{\Phi_{p,q}(u)}{r - \Psi_2(u)} \quad \text{for all } r \in (\Psi_2(u), 0).$$

Hence, (4.1) holds in the form $\limsup_{r\to 0^-} \varphi_1(r) \le \varphi_1(0) = \ell^\star$, where now $\varphi_1(0)$ is given by (5.1) and (5.2). Also in this new setting (4.2) and (4.3) are still valid and (4.4) simply reduces to

$$|\Psi_2(u)| \le C_\gamma \|u\|^\gamma \quad \text{for all } u \in W_0^{1,q}(\Omega),$$

with the same constant $C_\gamma > 0$. Taking $r < 0$ and $v \in \Psi_2^{-1}(r)$, we get

$$r = \Psi_2(v) \ge -C_\gamma \|v\|^\gamma \ge -C_\gamma (q\Phi_{p,q}(v))^{\gamma/q}.$$

Therefore, by (5.1), since $u \equiv 0 \in \Psi_2^{-1}(I^r)$,

$$\varphi_2(r) \ge \frac{1}{|r|} \inf_{v\in\Psi_2^{-1}(r)} \Phi_{p,q}(v) \ge \kappa |r|^{q/\gamma - 1},$$

where $\kappa = C_\gamma^{-q/\gamma}/q$. This implies that $\lim_{r\to 0^-} \varphi_2(r) = \infty$, being $\gamma > q$ by $(\mathscr{F})$–(b). In conclusion, we have proved that

$$\limsup_{r\to 0^-} \varphi_1(r) \le \varphi_1(0) = \ell^\star < \lim_{r\to 0^-} \varphi_2(r) = \infty. \tag{5.3}$$

This shows that for all integers $n \ge n^\star = 2 + [\ell^\star]$ there exists $r_n < 0$ so close to 0^- that $\varphi_1(r_n) < \ell^\star + 1/n < n < \varphi_2(r_n)$. Hence, since all the assumptions of Theorem 2.1, Part (a) of (ii) of [7] are satisfied and $u \equiv 0$ a critical point of $\mathcal{J}_\lambda$, problem $(\mathcal{P}_\lambda)$ admits at least two nontrivial solutions for all

$$\lambda \in \bigcup_{n=n^\star}^{\infty} (\varphi_1(r_n), \varphi_2(r_n)) \supset \bigcup_{n=n^\star}^{\infty} [\ell^\star + 1/n, n] = (\ell^\star, \infty),$$

since here $I = \mathbb{R}$ is the interval of λ's in which the main functional $\mathcal{J}_\lambda$ is coercive in $W_0^{1,q}(\Omega)$. □

It is apparent from the main definitions (3.5), (3.10), Theorem 5.1 and (5.2) that $0 < \lambda_\star < \ell_\star \le \ell^\star \le \lambda^\star$. Hence, Theorem 5.1 provides also the useful information that $0 < \lambda_\star < \lambda^\star$.

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