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Holomorphic vector field with one zero on the Grassmannian and cohomology

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Abstract. We consider a holomorphic vector field on the complex Grassmannian constructed from a nilpotent matrix. We show that this vector field vanishes only at a single point. Using the Baum-Bott localization theorem we give a Grothendieck residue formula for the intersection numbers of the Grassmannian. Knowing that Chern classes of the tautological bundle generate the cohomology ring of the Grassmannian we can compute the ideal of relations explicitly from the residue formula. This shows that the cohomology ring of the Grassmannian is determined by holomorphic vector field around its only zero.

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1. Introduction

In this article we show that the Baum-Bott localization formula [2, Theorem 1] using holomorphic vector fields can be used to compute the cohomology ring and the intersection numbers of the complex Grassmannian. Moreover, the holomorphic vector field can be chosen such that it has only a zero point, hence the cohomology ring of the Grassmannian is determined by this vector field near its single zero. From the residue formula of Baum-Bott for Chern numbers we are able to deduce the relations between the generators of the cohomology ring of the Grassmannian, hence to compute its cohomology ring.

The structure of the article is as follows. In Section 2 we recall the definition and properties of the Grothendieck residue which we use in the sequel. In Section 3 we recall the Baum-Bott localization theorem for holomorphic vector fields (Theorem

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3.1). In Section 4 we construct the holomorphic vector field on the Grassmannian from the action of a nilpotent matrix of form (4.1). We show that this vector field has only a zero point (Proposition 4.1). The vector field of same type of matrix on $\mathbb{CP}^{n-1}$ was considered in [5] and [4, §7] to demonstrate how vector fields with isolated zeroes can be used in computing cohomology rings. In [1] this construction of holomorphic vector fields with single zero is generalized to G/P, where G is a connected linear group algebraic group defined over a an algebraically closed field of characteristic zero and P is a parabolic subgroup. We note that the Grassmannian $Gr_k(n, \mathbb{C})$ can be viewed as homogeneous space. Then we express this vector field in local coordinates around the zero point (cf. (4.8)). In Section 5 we write up a residue formula for the Chern numbers of the tangent bundle of the Grassmannian $Gr_k(n, \mathbb{C})$ by the Baum-Bott theorem and we simplify it by eliminating all but n variables (Theorem 5.6). In Section 6 we recall the properties of Chern classes and the cohomology ring of the complex Grassmannian in terms of Chern classes of the tautological and the quotient bundle. In Lemma 6.3 we show that when $n \neq 2k$ then the Chern classes of the tangent bundle also generate the cohomology ring. In Theorem 6.4 and Corollary 6.5 we reinterpret the results of Theorem 5.6 to give the final version of the residue formula in terms of Chern classes of the tautological and quotient bundle when $n \neq 2k$. Using the similarities between Local Duality property (P2) of the Grothendieck residue and Poincaré duality we can easily calculate the relations between the generators of the cohomology ring (see Subsection 6.2.1).

Finally, in Section 7 we show the connection between the Grothendieck residue formula (6.5) and the Jeffrey-Kirwan residue formula for the Grassmannian constructed as symplectic quotient (cf. (7.1) and [7, Proposition 7.2]).

2. The Grothendieck residue

In this section we recall the definition of the Grothendieck residue and its properties used in the sequel. Let $h, f_1, \dots, f_r$ be holomorphic functions in an open neighborhood $\mathcal{U} \subset \mathbb{C}^r$ of a point p. Suppose that $p \in \mathcal{U}$ is the only common zero of $f_1, \dots, f_r$ in $\mathcal{U}$.

Definition 2.1. The *Grothendieck residue* is defined as

$$\operatorname{Res}_p \left(\frac{h\, dz_1 \dots dz_r}{f_1| \dots |f_r} \right) = \frac{1}{(2\pi\sqrt{-1})^r} \int_{\Gamma(\varepsilon)} \frac{h(z) dz_1 \dots dz_r}{f_1(z) \dots f_r(z)}, \tag{2.1}$$

where $\Gamma(\varepsilon) = \{z \in \mathcal{U} : |f_i(z)| = \varepsilon_i,\ i = 1, \dots, r\}$ for a small regular value

$$\varepsilon = (\varepsilon_1, \dots, \varepsilon_r) \in \mathbb{R}^r_{>0}$$

of $(|f_1|, \dots, |f_r|) : \mathcal{U} \to \mathbb{R}^r$ and the torus $\Gamma(\varepsilon)$ is oriented according to the differential form $d(\arg f_1(z)) \dots d(\arg f_r(z))$.

We note that the Grothendieck residue does not depend on the choice of the small regular value ε. We list some properties of the Grothendieck residue, which we will use in the sequel.

(P1) The functions $f_1, \ldots, f_r$ in the denominator of the residue anticommute. That is, if σ is a permutation of indices then

$$\operatorname{Res}_p\left(\frac{h\,dz_1\ldots dz_r}{f_1|\ldots|f_r}\right) = \operatorname{sgn}(\sigma)\cdot\operatorname{Res}_p\left(\frac{h\,dz_1\ldots dz_r}{f_{\sigma(1)}|\cdots|f_{\sigma(r)}}\right),$$

where $\operatorname{sgn}(\sigma)$ is the sign of the permutation σ. This sign is the result of changing the orientation of the cycle $\Gamma(\varepsilon)$ on which we integrate.

(P2) (Local Duality) We have $\operatorname{Res}_p\left(\frac{hg\,dz_1\ldots dz_r}{f_1|\ldots|f_r}\right) = 0$ for any local holomorphic germ $g \in \mathcal{O}_p$ if and only if h belongs to the ideal $\langle f_1, \ldots, f_r\rangle_p \subset \mathcal{O}_p$ (cf. [6, p.659]). Assume that $f_1, \ldots, f_r \in \mathbb{C}[z_1, \ldots, z_r]$ are polynomials and $\{p\} = \{z \in \mathbb{C}^r \mid f_1(z) = \ldots = f_r(z) = 0\}$ is their only common zero in $\mathbb{C}^r$. We note that this is the cases for isolated zero $p = 0 \in \mathbb{C}^r$ when $f_1, \ldots, f_r$ are graded homogeneous polynomials, i.e. there are degrees $d_i, \delta_i \in \mathbb{Z}_{>0}$, for $i = 1, \ldots, r$ such that $f_i(t^{d_1}z_1, \ldots, t^{d_r}z_r) = t^{\delta_i}f_i(z_1, \ldots, z_r)$ for all $i = 1, \ldots, r$. Then we have the following version of the Local Duality. $\operatorname{Res}_p\left(\frac{hg\,dz_1\ldots dz_r}{f_1|\ldots|f_r}\right) = 0$ for any polynomial $g \in \mathbb{C}[z_1, \ldots, z_r]$ if and only if h belongs to the ideal $\langle f_1, \ldots, f_r\rangle \subset \mathbb{C}[z_1, \ldots, z_r]$ (cf. [9, p.44]).

(P3) When $f = (f_1, \ldots, f_r)$ is nondegenerate at p, i.e. the Jacobian determinant $J_f(p) = \det\left(\frac{\partial f_i}{\partial z_j}(p)\right)_{i,j=1}^r \neq 0$ then $\operatorname{Res}_p\left(\frac{h\,dz_1\ldots dz_r}{f_1|\ldots|f_r}\right) = \frac{h(p)}{J_f(p)}$, (cf. [6, p.650]).

(P4) (Transformation Law) Let $A = (A_{ij})_{i,j=1}^r$ be an r-by-r matrix with holomorphic coefficients $A_{ij} \in \mathcal{O}(\mathcal{U})$ such that p is the only locally common zero of $g_i = \sum_{j=1}^r A_{ij}f_j$, $i = 1, \ldots, r$. Then

$$\operatorname{Res}_p\left(\frac{h\,dz_1\ldots dz_r}{f_1|\ldots|f_r}\right) = \operatorname{Res}_p\left(\frac{h\det(A)\,dz_1\ldots dz_r}{g_1|\ldots|g_r}\right),$$

(cf. [6, p.657]). In the case of $p = 0 \in \mathbb{C}^r$ and $g_i = z_i^{\mu_i}$, $i = 1, \ldots, r$ with $\mu_i \geq 1$ the above residue becomes an iterated residue and can be evaluated by expanding $h\det(A)$ into power series in variables $z_1, \ldots, z_r$ around $p = 0 \in \mathbb{C}^r$ and taking the coefficient of the term $z_1^{\mu_1-1}\ldots z_r^{\mu_r-1}$.

(P5) Assume that $0 \in \mathcal{U}$ is the only zero of polynomials $f_1, \ldots, f_{r-1}, f_r = z_r \in \mathbb{C}[z_1, \ldots, z_r]$ in the open set $\mathcal{U} \subseteq \mathbb{C}^r$ and consider the inclusion $\iota_r : \mathbb{C}^{r-1} \to \mathbb{C}^r$, $\iota_r(z_1, \ldots z_{r-1}) = (z_1, \ldots, z_{r-1}, 0)$. Then by division with remainder with respect to z_r we have $f_j = \varphi_j z_r + \rho_j$, where $\rho_j = \iota_r^* f_j = f_j \circ \iota_r \in \mathbb{C}[z_1, \ldots z_{r-1}]$ non-zero for all $j = 1, \ldots, r-1$. If we set $\rho_r = z_r$, then $\rho_i = \sum_{j=1}^r A_{ij}f_j$ with $A_{ii} = 1$, $1 \leq i \leq r$, $A_{ir} = -\varphi_i$, $1 \leq i \leq r-1$ and $A_{ij} = 0$ otherwise. Thus, $\det(A) = 1$ and by (P4), (P2) we have

$$\operatorname{Res}_0\left(\frac{h\,dz_1\ldots dz_r}{f_1|\ldots|f_{r-1}|z_r}\right) = \operatorname{Res}_0\left(\frac{h\,dz_1\ldots dz_r}{\iota_r^*f_1|\ldots|\iota_r^*f_{r-1}|z_r}\right)$$

$$= \operatorname{Res}_0\left(\frac{\iota_r^*h\,dz_1\ldots dz_r}{\iota_r^*f_1|\ldots|\iota_r^*f_{r-1}|z_r}\right) = \operatorname{Res}_0\left(\frac{\iota_r^*h\,dz_1\ldots dz_{r-1}}{\iota_r^*f_1|\ldots|\iota_r^*f_{r-1}}\right).$$

(P6) (Pull-back) If $\varphi = (\varphi_1, \dots, \varphi_r) : \mathbb{C}^r \to \mathbb{C}^r$ is a finite map, generically m-fold cover then

$$\operatorname{Res}_p \left(\frac{h(z)\, dz_1 \dots dz_r}{f_1(z)| \dots |f_r(z)} \right) = \frac{1}{m} \sum_{\zeta \in \varphi^{-1}(p)} \operatorname{Res}_\zeta \left(\frac{h(\varphi(w)) J_\varphi(w)\, dw_1 \dots dw_r}{f_1(\varphi(w))| \dots |f_r(\varphi(w))} \right).$$

(P7) Assume that z_i has degree δ_i and $f_i \in \mathbb{C}[z_1, \dots, z_r]$ are graded homogeneous polynomials of degree d_i, for $i = 1, \dots, r$. Let h be also homogeneous polynomial of degree d. Rescaling $z \mapsto \lambda z = (\lambda^{\delta_1} z_1, \dots, \lambda^{\delta_r} z_r)$ yields

$$\begin{aligned}\operatorname{Res}_0 \left(\frac{h(z)\, dz_1 \dots dz_r}{f_1(z)| \dots |f_r(z)} \right) &= \operatorname{Res}_0 \left(\frac{h(\lambda z) d(\lambda^{\delta_1} z_1) \dots d(\lambda^{\delta_r} z_r)}{f_1(\lambda z)| \dots |f_r(\lambda z)} \right) \\ &= \lambda^D \operatorname{Res}_0 \left(\frac{h(z)\, dz_1 \dots dz_r}{f_1(z)| \dots |f_r(z)} \right),\end{aligned}$$

where $D = d + \delta_1 + \ldots + \delta_r - d_1 - \ldots - d_r$, hence the residue vanishes if $D \neq 0$, i.e. when $d \neq d_1 + \ldots + d_r - \delta_1 - \ldots - \delta_r$.

3. The Baum-Bott localization theorem

We recall the Baum-Bott theorem from [2], which states that Chern numbers of the tangent bundle can be computed by Grothendieck residues at zeroes of holomorphic vector fields. Let M be a compact complex analytic manifold of dimension $\dim_{\mathbb{C}} M = r$ and let ϑ be a holomorphic vector field on M. We assume that ϑ vanishes only at isolated points. We consider holomorphic local coordinates $z_1, \dots, z_r$ on M centered at a zero p, i.e. $z_1(p) = \ldots = z_r(p) = 0$. In these local coordinates we can write

$$\vartheta = \sum_{i=1}^{r} \vartheta_i \frac{\partial}{\partial z_i},$$

where $\vartheta_1, \dots, \vartheta_r$ are holomorphic functions in a neighborhood $\mathcal{U}_p$ of p and this point p is their only common zero in $\mathcal{U}_p$.

Let $V_p = \left(\frac{\partial \vartheta_i}{\partial z_j} \right)_{i,j=1}^{r}$ be the Jacobian. The *Chern classes* $c_i(V_p)$ of the matrix V_p are defined by the formula

$$c(V_p; t) = \sum_{i=0}^{r} c_i(V_p) t^i = \det(I + tV_p). \tag{3.1}$$

Moreover, for any multidegree $\alpha = (\alpha_1, \dots, \alpha_r) \in \mathbb{Z}^r_{\geq 0}$ such that $|\alpha| = \alpha_1 + 2\alpha_2 + \ldots + r\alpha_r = r$ the *Chern numbers* of V_p are given by $c^\alpha(V_p) = c_1(V_p)^{\alpha_1} \cdots c_r(V_p)^{\alpha_r}$.

Denote $c_i(TM) \in H^{2i}(M, \mathbb{C})$, $i = 1, \dots, r$ the Chern classes of the holomorphic tangent bundle TM. The theorem of Baum and Bott [2, Theorem 1] states that for any multidegree $\alpha = (\alpha_1, \dots, \alpha_r)$ the *Chern numbers*

$$c^\alpha(TM) = \int_M c_1(TM)^{\alpha_1} \cdots c_r(TM)^{\alpha_r} \tag{3.2}$$

of the holomorphic tangent bundle TM can be computed by Grothendieck residues at the zeroes of the holomorphic vector field ϑ as follows.

Theorem 3.1 ([2, Theorem 1]). *Let ϑ be a holomorphic vector field on a compact complex analytic manifold M of dimension $\dim_{\mathbb{C}} M = r$. Assume that ϑ vanishes at only isolated points. Then the Chern numbers of the tangent bundle of M can be computed as*

$$c^{\alpha}(TM) = \sum_{p \in \text{zeroes of } \vartheta} \operatorname{Res}_p \left(\frac{c^{\alpha}(V_p)\, dz_1 \dots dz_r}{\vartheta_1 | \dots | \vartheta_r} \right). \tag{3.3}$$

4. Nilpotent vector field on the Grassmannian

From the action of a nilpotent matrix of form (4.1) we construct a holomorphic vector field ϑ on the Grassmannian $Gr_k(n, \mathbb{C})$, which vanishes at a single point (Proposition 4.1). Moreover, in (4.8) we express this vector field in local coordinates around its zero and we use it to compute the Chern numbers of the Grassmannian (cf. (5.1)). The result of the computation is given in Theorem 5.6.

4.1. Definition of the vector field ϑ

On the complex vector space $\mathcal{M}_{k,n}(\mathbb{C})$ of k-by-n complex matrices we consider the natural left action of the group $GL_k(\mathbb{C})$ by matrix multiplication. We assume that $k < n$. On the subset of rank k matrices of $\mathcal{M}_{k,n}(\mathbb{C})$ the group $GL_k(\mathbb{C})$ acts properly and freely. The quotient $Gr_k(n, \mathbb{C})$, called the *Grassmannian*, is the set of k-dimensional complex linear subspaces of $\mathbb{C}^n$ and it has the structure of a compact complex analytic manifold. In particular, a full rank matrix $P \in \mathcal{M}_{k,n}(\mathbb{C})$ represents a point in $Gr_k(n, \mathbb{C})$, namely the k-dimensional complex linear subspace spanned by the rows of P.

On $\mathcal{M}_{k,n}(\mathbb{C})$ the multiplication by n-by-n matrices $GL_n(\mathbb{C})$ on the right commutes with the action of $GL_k(n)$ on the left, hence this right action descends to the Grassmannian $Gr_k(n, \mathbb{C})$.

For any $\ell \geq 2$ we consider nilpotent matrices of the following form

$$N_\ell = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & & & & \\ 0 & 0 & 0 & \dots & 1 \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix} \in \mathcal{M}_\ell(\mathbb{C}). \tag{4.1}$$

We consider the vector field ϑ on the Grassmannian induced by the nilpotent matrix $N_n \in \mathcal{M}_n(\mathbb{C})$ as follows. Let ϑ be the vector field associated with the holomorphic flow $Fl(t, P) = P \cdot \exp(tN_n)$, $t \in \mathbb{C}$ on the Grassmannian $Gr_k(n, \mathbb{C})$, where $P \in \mathcal{M}_{k,n}(\mathbb{C})$ is a rank k matrix representing a point on the Grassmannian.

4.2. Zeroes of the vector field ϑ

The following proposition gives a description of matrices representing the zeroes of the vector field ϑ and shows that they represent the same point on the Grassmannian.

Proposition 4.1. *A matrix $P \in \mathcal{M}_{k,n}(\mathbb{C})$ of rank k represents a zero of the vector field ϑ on $Gr_k(n,\mathbb{C})$ if and only if there exists matrix $S \in \mathcal{M}_k(\mathbb{C})$ such that $SP = PN_n$. Moreover, any solution $(S,P) \in \mathcal{M}_k(\mathbb{C}) \times \mathcal{M}_{k,n}(\mathbb{C})$ of $SP = PN_n$ is of form $S = UN_kU^{-1}$ and $P = \begin{pmatrix} O_{k,n-k} & U \end{pmatrix}$, where $U \in GL_k(\mathbb{C})$ and $O_{k,n-k} \in \mathcal{M}_{k,n-k}(\mathbb{C})$ is the zero matrix. The point in $Gr_k(n,\mathbb{C})$ represented by $P = \begin{pmatrix} O_{k,n-k} & I_k \end{pmatrix}$ is the only zero of the vector field ϑ.*

Proof. The zeroes of the vector field ϑ correspond to the fixed points of the flow Fl, hence P represents a zero of ϑ if and only if there exists $A(t) \in GL_k(\mathbb{C})$ for all $t \in \mathbb{C}$ such that $A(t)P = P\exp(tN_n)$. Moreover, $A(t)$ is differentiable and denote $S = A'(0)$. Therefore, from the previous relation we get $SP = PN_n$ by taking the differential at $t = 0$. On the other hand, if there is a pair (S,P) with P of rank k such that $SP = PN_n$, then $S^jP = PN_n^j$ for all $j \in \mathbb{Z}_{\geq 0}$, hence $\exp(St)P = P\exp(tN_n)$. This implies that P represents a zero of ϑ.

Consider a pair $(S,P) \in \mathcal{M}_k(\mathbb{C}) \times \mathcal{M}_{k,n}(\mathbb{C})$ satisfying $SP = PN_n$. In particular, $S^nP = PN_n^n = O_{k,n-k}$ and P has rank k, thus $S^n = O_k$, i.e. S is nilpotent. The Jordan form of $S \in \mathcal{M}_k(\mathbb{C})$ implies that already $S^k = O_k$. Since $O_k = S^kP = PN_n^k$, thus we must have $P = \begin{pmatrix} O_{k,n-k} & U \end{pmatrix}$ for some matrix $U \in GL_k(\mathbb{C})$ and $S = UN_kU^{-1}$.

The rank k matrices $\begin{pmatrix} O_{k,n-k} & U \end{pmatrix}$ and $\begin{pmatrix} O_{k,n-k} & I_k \end{pmatrix}$ represent the same point in $Gr_k(n,\mathbb{C})$. □

4.3. Expression of the vector field ϑ in local coordinates at the zero point

The local parametrization on the Grassmannian around the point represented by P of Proposition 4.1 is given by the matrix $Z = \begin{pmatrix} z & I_k \end{pmatrix}$ with block $z = (z_{i,j})_{i,j}$ of $k \times (n-k)$ complex coordinates $z_{i,j}$, $i = 1,\dots,k$, $j = 1,\dots,n-k$. The ordering of coordinates will be lexicographical.

First, we compute the flow Fl of ϑ as follows. We set matrices $B(t)$ and $D(t)$ by the relation $\begin{pmatrix} B(t) & D(t) \end{pmatrix} = \begin{pmatrix} z & I_k \end{pmatrix}\exp(tN_n)$, hence $z(t) = D(t)^{-1}B(t)$ gives the local coordinates of the flow at the zero point. We note that $B(0) = z$, $D(0) = I_k$ and $D(t)$ is a matrix of polynomials in t of degree at most n and has of form $D(t) = I_k + D_1t + \ldots + D_nt^n$ with

$$D_1 = \begin{pmatrix} z_{1,n-k} & 1 & 0 & \dots & 0 \\ z_{2,n-k} & 0 & 1 & \dots & 0 \\ \vdots & & & & \\ z_{k-1,n-k} & 0 & 0 & \dots & 1 \\ z_{k,n-k} & 0 & 0 & \dots & 0 \end{pmatrix}. \tag{4.2}$$

Finally, in local coordinates z the vector field ϑ can be computed as

$$\vartheta = \sum_{i=1}^{k}\sum_{j=1}^{n-k} \frac{d}{dt}\left(D^{-1}(t)B(t)\right)_{i,j}(0)\frac{\partial}{\partial z_{i,j}} = \sum_{i=1}^{k}\sum_{j=1}^{n-k}(-D_1z + zN_{n-k})_{i,j}\frac{\partial}{\partial z_{i,j}}.$$

Note that zN_{n-k} is the matrix got from z by shifting its columns to the right and inserting zeroes in the first column. In more details, if $\vartheta = \sum_{i=1}^{k}\sum_{j=1}^{n-k}\vartheta_{i,j}\frac{\partial}{\partial z_{i,j}}$ then

$$\vartheta_{i,1} = -z_{i,n-k}z_{1,1} - z_{i+1,1}, \qquad (i = 1,\dots,k-1), \tag{4.3}$$

$$\vartheta_{k,1} = -z_{k,n-k}z_{1,1}, \tag{4.4}$$

$$\vartheta_{i,j} = -z_{i,n-k}z_{1,j} - z_{i+1,j} + z_{i,j-1}, \qquad (i = 1,\dots,k-1,\ j = 2,\dots,n-k), \tag{4.5}$$

$$\vartheta_{k,j} = -z_{k,n-k}z_{1,j} + z_{k,j-1}, \qquad (j = 2,\dots,n-k). \tag{4.6}$$

If we introduce notations $z_{k+1,j} = 0$ for $j = 1,\dots,n-k$ and $z_{i,0} = 0$ for $i = 1,\dots,k$ then we can give a uniform description for these functions. For any $i = 1,\dots,k$ and $j = 1,\dots,n-k$ we have

$$\vartheta_{i,j} = -z_{i,n-k}z_{1,j} - z_{i+1,j} + z_{i,j-1}. \tag{4.7}$$

With these notations the nilpotent vector field around the zero equals to

$$\vartheta = \sum_{i=1}^{k}\sum_{j=1}^{n-k}\left(-z_{i,n-k}z_{1,j} - z_{i+1,j} + z_{i,j-1}\right)\frac{\partial}{\partial z_{i,j}}. \tag{4.8}$$

5. Residue formula for the Grassmannian

We start by writing up the Baum-Bott residue formula (3.3) for the vector field (4.8) on the Grassmannian $Gr_k(n,\mathbb{C})$ and we prove a series of lemmas about the building blocks of this residue formula to reduce the number of variables from $k(n-k)$ to n. At the end of this section in Theorem 5.6 we get a simplified residue formula for Chern numbers of the Grassmannian. In Section 6 this formula will be reinterpreted in terms of Chern classes of the tautological and quotient bundle on the Grassmannian (Theorem 6.4 and Corollary 6.5) to get an even simpler formula.

By Theorem 3.1 we get the following formula for Chern numbers (3.2) of the tangent bundle of the Grassmannian:

$$c^{\alpha}(TGr_k(n,\mathbb{C})) = \mathrm{Res}_0\left(\frac{\prod_{i=1}^{k}\prod_{j=1}^{n-k}dz_{i,j}}{\prod_{i=1}^{k}\prod_{j=1}^{n-k}\vartheta_{i,j}}c^{\alpha}(V)\right), \tag{5.1}$$

where $V = \left(\dfrac{\partial\vartheta_{i,j}}{\partial z_{h,l}}\right)_{(i,j),(h,l)}$ with lexicographical ordering on pairs (i,j) and (h,l). This ordering is compatible with the ordering of functions $\vartheta_{i,j}$ in the denominator of (5.1). We will drop the vertical line notation from the denominator in the Grothendieck residue, but we will keep track of the order of functions.

We make the following change of variables. We express the variables $z_{a,b}$, $a = 1,\dots,k$, $b = 1,\dots,n-k$ in terms of $z_{1,j}$, $j = 1,\dots,n-k-1$, $z_{i,n-k}$, $i = 1,\dots,k$, and $\vartheta_{i,j}$, $i = 1,\dots,k-1$, $j = 1,\dots,n-k-1$. We keep the common variables, while the others can expressed in new variables as follows.

Lemma 5.1. *For any $i = 1, \dots, k-1$ and $j = 1, \dots, n-k-1$ we have*

$$z_{i+1,j} = -\sum_{\ell=0}^{\min\{i,\, j-1\}} z_{i-\ell,n-k}\, z_{1,j-\ell} - \sum_{\ell=0}^{\min\{i-1,\, j-1\}} \vartheta_{i-\ell,j-\ell} \tag{5.2}$$

with notation $z_{0,n-k} = -1$.

Proof. We fix a pair (i,j) with $1 \le i \le k-1$ and $1 \le j \le n-k-1$. From relations (4.7) we construct the following recursion between variables on the same diagonal as $z_{i+1,j}$

$$z_{i+1-\ell,j-\ell} = -z_{i-\ell,n-k} z_{1,j-\ell} + z_{i-\ell,j-1-\ell} - \vartheta_{i-\ell,j-\ell}, \tag{5.3}$$

for $\ell = 0, \dots, \min\{i-1, j-1\}$. We specify the following edge cases.
For $\ell = i-1 = \min\{i-1, j-1\}$ we have

$$\begin{aligned} z_{2,j-i+1} &= -z_{1,n-k} z_{1,j-i+1} + z_{1,j-i} - \vartheta_{1,j-i+1} \\ &= -z_{1,n-k} z_{1,j-i+1} - z_{0,n-k} z_{1,j-i} - \vartheta_{1,j-i+1}, \end{aligned}$$

with notation $z_{0,n-k} = -1$. Moreover, for $\ell = j-1 = \min\{i-1, j-1\}$ we have

$$\begin{aligned} z_{i-j+2,1} &= -z_{i-j+1,n-k} z_{1,1} + z_{i-j+1,0} - \vartheta_{i-j+1,1} \\ &= -z_{i-j+1,n-k} z_{1,1} - \vartheta_{i-j+1,1} \end{aligned}$$

by earlier introduced notation $z_{i-j+1,0} = 0$.

Finally, substituting the relations (5.3) into each other yields the statement of the lemma. □

The differential form in the numerator of (5.1) can be expressed in terms of new variables as follows.

Lemma 5.2.

$$\prod_{i=1}^{k} \prod_{j=1}^{n-k} dz_{i,j} = \prod_{i=1}^{k-1} \Big(\prod_{j=1}^{n-k-1} d\vartheta_{i,j} \Big) dz_{i,n-k} \Big(\prod_{j=1}^{n-k-1} dz_{1,j} \Big) dz_{k,n-k}.$$

Proof. Taking the differential of (5.2) for every $i = 1, \dots, k-1$ and $j = 1, \dots, n-k-1$, then substituting into the product of differentials basically replaces $dz_{i+1,j}$ with $-d\vartheta_{i,j}$, thus we get

$$\begin{aligned} \prod_{i=1}^{k} \prod_{j=1}^{n-k} dz_{i,j} &= (dz_{1,1} \dots dz_{1,n-k-1} dz_{1,n-k})\, ((-d\vartheta_{1,1}) \dots (-d\vartheta_{1,n-k-1}) dz_{2,n-k}) \\ &\quad \dots ((-d\vartheta_{k-1,1}) \dots (-d\vartheta_{k-1,n-k-1}) dz_{k,n-k}) \\ &= (d\vartheta_{1,1} \dots d\vartheta_{1,n-k-1} dz_{1,n-k}) \dots \\ &\quad (d\vartheta_{k-1,1} \dots d\vartheta_{k-1,n-k-1} dz_{k-1,n-k})\, (dz_{1,1} \dots dz_{1,n-k-1} dz_{k,n-k}) \\ &= \Big(\prod_{i=1}^{k-1} \Big(\prod_{j=1}^{n-k-1} d\vartheta_{i,j} \Big) dz_{i,n-k} \Big) \Big(\prod_{j=1}^{n-k-1} dz_{1,j} \Big) dz_{k,n-k}. \end{aligned}$$

□

The functions $\vartheta_{k,j}$, $j = 1, \ldots, n-k$ and $\vartheta_{i,n-k}$, $i = 1, \ldots, k-1$ in the denominator of the residue (5.1) can be expressed in terms of new variables as follows.

Lemma 5.3. *For $i = k$ or $j = n - k$ we have*

$$\vartheta_{i,j} = -\sum_{h=-1}^{\min\{i,j-1\}} z_{i-h,n-k}\, z_{1,j-h} - \sum_{h=1}^{\min\{i-1,j-1\}} \vartheta_{i-h,j-h} \tag{5.4}$$

with notations $z_{1,n-k+1} = 1$, $z_{0,n-k} = -1$ and $z_{k+1,j} = 0$.

Proof. In the first case when $i = k$ we substitute (5.2) into (4.6) and we get

$$\begin{aligned}
\vartheta_{k,j} &= -z_{k,n-k}\, z_{1,j} - \sum_{\ell=0}^{\min\{k-1,j-2\}} z_{k-1-\ell,n-k}\, z_{1,j-1-\ell} - \sum_{\ell=0}^{\min\{k-2,j-2\}} \vartheta_{k-1-\ell,j-1-\ell} \\
&= -z_{k,n-k}\, z_{1,j} - \sum_{h=1}^{\min\{k,j-1\}} z_{k-h,n-k}\, z_{1,j-h} - \sum_{h=1}^{\min\{k-1,j-1\}} \vartheta_{k-h,j-h} \\
&= -\sum_{h=-1}^{\min\{k,j-1\}} z_{k-h,n-k}\, z_{1,j-h} - \sum_{h=1}^{\min\{k-1,j-1\}} \vartheta_{k-h,j-h}
\end{aligned}$$

by earlier introduced notation $z_{k+1,j} = 0$.

In the second case when $j = n - k$, we substitute (5.2) into (4.5) in place of $z_{i,n-k-1}$ and we get

$$\begin{aligned}
\vartheta_{i,n-k} = &- z_{i,n-k}\, z_{1,n-k} - z_{i+1,n-k} - \sum_{\ell=0}^{\min\{i-1,n-k-2\}} z_{i-1-\ell,n-k}\, z_{1,n-k-1-\ell} - \\
&- \sum_{\ell=0}^{\min\{i-2,n-k-2\}} \vartheta_{i-1-\ell,n-k-1-\ell} \\
= &- z_{i,n-k}\, z_{1,n-k} - z_{i+1,n-k} z_{1,n-k+1} - \sum_{h=1}^{\min\{i,n-k-1\}} z_{i-h,n-k}\, z_{1,n-k-h} \\
&- \sum_{h=1}^{\min\{i-1,n-k-1\}} \vartheta_{i-h,n-k-h} \\
= &- \sum_{h=-1}^{\min\{i,n-k-1\}} z_{i-h,n-k}\, z_{1,n-k-h} - \sum_{h=1}^{\min\{i-1,n-k-1\}} \vartheta_{i-h,n-k-h}
\end{aligned}$$

by setting $z_{1,n-k+1} = 1$. □

We introduce notation for the first sum of (5.4)

$$\zeta_{i,j} = -\sum_{h=-1}^{\min\{i,j-1\}} z_{i-h,n-k} z_{1,j-h} \quad \text{for} \quad i = k \text{ or } j = n - k, \tag{5.5}$$

and $\zeta_{i,j} = \vartheta_{i,j}$ otherwise. Hence by (5.4) we have $\zeta_{i,j} = \vartheta_{i,j} + \sum_{h=1}^{\min\{i-1,j-1\}} \vartheta_{i-h,j-h}$ if $i = k$ or $j = n-k$ and $\vartheta_{i,j} = \zeta_{i,j}$ otherwise.

Remark 5.4. The transformation matrix between $(\vartheta_{i,j})_{i=\overline{1,k},\, j=\overline{1,n-k}}$ and $(\zeta_{i,j})_{i=\overline{1,k},\, j=\overline{1,n-k}}$ (lexicographically ordered) is lower triangular with 1 on the diagonal.

Next, we compute the final component of the residue formula (5.1), namely $c(V;t)$ (cf. (3.1)). We introduce the following notations:

$$\widetilde{u}_i = -z_{i,n-k},\ i = 0,\dots,k+1 \quad \text{and} \quad w_j = z_{1,n-k+1-j} = z_{1,j^*},\ j = 0,\dots,n-k, \tag{5.6}$$

where $j^* = n-k+1-j$. By earlier notations $\widetilde{u}_0 = -z_{0,n-k} = 1$, $w_0 = z_{1,n-k+1} = 1$, $\widetilde{u}_{k+1} = -z_{k+1,n-k} = 0$ and $\widetilde{u}_1 = -w_1 = -z_{1,n-k}$.

We associate with $w = (w_1,\dots,w_{n-k})$ the following matrix

$$\Lambda_w = \begin{pmatrix} w_1 & -1 & 0 & \dots & 0 \\ w_2 & 0 & -1 & \dots & 0 \\ \vdots & & & & \\ w_{n-k-1} & 0 & 0 & \dots & -1 \\ w_{n-k} & 0 & 0 & \dots & 0 \end{pmatrix}. \tag{5.7}$$

Lemma 5.5. *Using notations of (5.6) we have*

$$c(V;t) = \det\Big(\sum_{i=0}^{k} \widetilde{u}_i t^i (I - t\Lambda_w)^{k-i}\Big).$$

Proof. From (4.7) we can compute

$$\frac{\partial \vartheta_{1,n-k}}{\partial z_{1,n-k}} = -2z_{1,n-k},\ \frac{\partial \vartheta_{i,j}}{\partial z_{i,n-k}} = -z_{1,j}$$

and $\dfrac{\partial \vartheta_{i,j}}{\partial z_{1,j}} = -z_{i,n-k}$ if $i \neq 1$ or $j \neq n-k$, $\dfrac{\partial \vartheta_{i,j}}{\partial z_{i+1,j}} = -1$, $\dfrac{\partial \vartheta_{i,j}}{\partial z_{i,j-1}} = 1$ and $\dfrac{\partial \vartheta_{i,j}}{\partial z_{h,\ell}} = 0$ otherwise.

Let $W = [W_{i,j}]_{i,j=1}^{n-k}$ be the matrix with $W_{i+1,i} = 1$, $W_{i,n-k} = -z_{1,i} = -w_{n-k+1-i}$ and $W_{i,j} = 0$ otherwise (centrally symmetric image of $-\Lambda_w$). Denote $I = I_{n-k}$ and let $U_i = -z_{i,n-k}I = \widetilde{u}_i I$ for $i = 0,\dots,k$. Then $V = \left(\dfrac{\partial \vartheta_{i,j}}{\partial z_{h,\ell}}\right)_{(i,j),(h,\ell)}$ is the $k(n-k)$-by-$k(n-k)$ matrix composed of $(n-k)$-by-$(n-k)$ blocks $V^{i,j}$, $i,j = 1,\dots,k$ as follows. Let $V^{1,1} = U_1 + W$ and for $i > 1$ let $V^{i,1} = U_i$, $V^{i,i} = W$ and $V^{i,i+1} = -I$, while other blocks are zeroes. Then we compute

$$\begin{aligned} c(V;t) &= \det(I_{k(n-k)} + tV) \\ &= \begin{vmatrix} tU_1 + I + tW & -tI & 0 & \dots & 0 & 0 \\ tU_2 & I + tW & -tI & \dots & 0 & 0 \\ \vdots & & & & & \\ tU_{k-1} & 0 & 0 & \dots & I + tW & -tI \\ tU_k & 0 & 0 & \dots & 0 & I + tW \end{vmatrix} \end{aligned}$$

as follows. To get rid of $I+tW$ from the first $(k-1)$ diagonal blocks, we add successively to the $(k-1)^{\mathrm{th}},\dots,1^{\mathrm{st}}$ columns of blocks the subsequent column of blocks multiplied by $t^{-1}(I+tW)$. After this step there will be blocks

$$tU_k+t^{1-k}(I+tW)^k,\ t^{2-k}(I+tW)^{k-1},\dots,t^{-1}(I+tW)^2,\ I+tW$$

in the last row. Next, we multiply the first $k-1$ rows of blocks by

$$t^{1-k}(I+tW)^{k-1},\dots,t^{-2}(I+tW)^2,\ t^{-1}(I+tW)$$

respectively and we add them to the last row to get blocks of zeroes in the last row except the first column, where there will be $\sum_{i=0}^{k}(I+tW)^{k-i}t^{1-k+i}U_i$.

Summing up, we get a determinant with blocks $-tI$ above the diagonal, $tU_1,\dots,tU_{k-1}$ in the first $(k-1)$ rows of the first column, respectively, and $\sum_{i=0}^{k}(I+tW)^{k-i}t^{1-k+i}U_i$ in the last row of the first columns, while other blocks are zeroes. Finally, we move the first column of blocks to the end to get an block-upper triangular determinant in exchange of a $(-1)^{(k-1)(n-k)^2}$-sign. Hence,

$$\begin{aligned}\det(I_{k(n-k)}+tV)&=(-1)^{(k-1)(n-k)^2+(k-1)(n-k)}\cdot\det\Big(\sum_{i=0}^{k}(I+tW)^{k-i}t^iU_i\Big)\\&=\det\Big(\sum_{i=0}^{k}\widetilde{u}_it^i(I+tW)^{k-i}\Big)=\det\Big(\sum_{i=0}^{k}\widetilde{u}_it^i(I-t\Lambda_w)^{k-i}\Big),\end{aligned}$$

where the last equality is by reflecting the determinant first with respect to the diagonal and then to the anti-diagonal, which sends W to $-\Lambda_w$, and it is compatible with matrix addition and multiplication. □

Consider the polynomial ring $\mathbb{C}[u,w]=\mathbb{C}[u_1,\dots,u_k,w_1,\dots,w_{n-k}]$ and polynomials

$$P_\ell(u,v)=\sum_{s=0}^{\ell}u_sw_{\ell-s},\qquad \forall\ell=1,\dots,n, \tag{5.8}$$

with notations $u_0=w_0=1$. We state the reshaped Baum-Bott residue formula for the Grassmannian $Gr_k(n,\mathbb{C})$ with only n variables.

Theorem 5.6. *For any multidegree $\alpha=(\alpha_1,\dots,\alpha_{k(n-k)})\in\mathbb{Z}_{\geq0}^{k(n-k)}$ with*

$$|\alpha|=\alpha_1+\dots+\alpha_{k(n-k)}=k(n-k)$$

the Chern numbers can be computed as

$$\begin{aligned}c^\alpha(TGr_k(n,\mathbb{C}))&=\int_{Gr_k(n,\mathbb{C})}\prod_{s=1}^{k(n-k)}c_s(TGr_k(n,\mathbb{C}))^{\alpha_s}\\&=\operatorname{Res}_0\left(\frac{\prod_{i=1}^{k}du_i\prod_{j=1}^{n-k}dw_j}{\prod_{\ell=1}^{n}P_\ell(u,w)}\Delta^\alpha(u,w)\right),\end{aligned}$$

where

$$\Delta(u,w;t)=\sum_{\ell=0}^{k(n-k)}\Delta_\ell t^\ell=\det\Big(\sum_{i=0}^{k}u_it^i(I-t\Lambda_w)^{k-i}\Big)$$

and

$$\Delta^{\alpha}(u,w) = \Delta^{\alpha_1} \cdot \ldots \cdot \Delta^{\alpha_{k(n-k)}}.$$

Proof. By the Baum-Bott Theorem we have the residue formula (5.1). In this formula we replace the differential form by the one in Lemma 5.2. We also reorder the functions in the denominator and by (P1) the functions in the denominator are anti-commuting just like the differential forms in the numerator. Thus, the residue (5.1) becomes

$$\mathrm{Res}_0\left(\frac{\prod_{i=1}^{k-1}\left(\prod_{j=1}^{n-k-1} d\vartheta_{i,j}\right)dz_{i,n-k}\left(\prod_{j=1}^{n-k-1} dz_{1,j}\right)dz_{k,n-k}}{\prod_{i=1}^{k}\prod_{j=1}^{n-k}\vartheta_{i,j}}c^{\alpha}(V)\right) = \tag{5.9}$$

$$= \mathrm{Res}_0\left(\frac{\prod_{i=1}^{k-1}\prod_{j=1}^{n-k-1} d\vartheta_{i,j}\prod_{i=1}^{k} dz_{i,n-k}\prod_{j=1}^{n-k-1} dz_{1,j}}{\prod_{i=1}^{k-1}\prod_{j=1}^{n-k-1}\vartheta_{i,j}\prod_{i=1}^{k}\vartheta_{i,n-k}\prod_{j=1}^{n-k-1}\vartheta_{k,j}}c^{\alpha}(V)\right). \tag{5.10}$$

Recall that we have relations $\vartheta_{i,j} = \zeta_{i,j} - \sum_{h=1}^{\min\{i-1,j-1\}}\vartheta_{i-h,j-h}$ for $i = k$ or $j = n-k$ by (5.5) and Lemma 5.3. Thus, when $i = k$ or $j = n-k$ we can replace $\vartheta_{i,j}$ with $\zeta_{i,j}$ in (5.10) by Transformation Law (P4) and Remark 5.4 to get

$$\mathrm{Res}_0\left(\frac{\prod_{i=1}^{k-1}\prod_{j=1}^{n-k-1} d\vartheta_{i,j}\prod_{i=1}^{k} dz_{i,n-k}\prod_{j=1}^{n-k-1} dz_{1,j}}{\prod_{i=1}^{k-1}\prod_{j=1}^{n-k-1}\vartheta_{i,j}\prod_{i=1}^{k}\zeta_{i,n-k}\prod_{j=1}^{n-k-1}\zeta_{k,j}}c^{\alpha}(V)\right). \tag{5.11}$$

We note that $c^{\alpha}(V)$ depends only on $z_{1,1},\ldots,z_{1,n-k},\ldots,z_{k,n-k}$ (see Lemma 5.5). Thus, applying property (P5) to (5.11) yields

$$\mathrm{Res}_0\left(\frac{\prod_{i=1}^{k} dz_{i,n-k}\prod_{j=1}^{n-k-1} dz_{1,j}}{\prod_{i=1}^{k}\zeta_{i,n-k}\prod_{j=1}^{n-k-1}\zeta_{k,j}}c^{\alpha}(V)\right) =$$

$$= \mathrm{Res}_0\left(\frac{\prod_{i=1}^{k} dz_{i,n-k}\prod_{j=2}^{n-k} dz_{1,j^*}}{\prod_{i=1}^{k}\zeta_{i,n-k}\prod_{j=2}^{n-k}\zeta_{k,j^*}}c^{\alpha}(V)\right), \tag{5.12}$$

where $j^* = n-k+1-j$.

By Lemma 5.5 we have $c(V;t) = \Delta(\widetilde{u},w;t)$, hence $c^{\alpha}(V) = \Delta^{\alpha}(\widetilde{u},w)$. Moreover, from (5.5) and (5.6) for $i = k$ and $j = 1,\ldots,n-k-1$ we have

$$\zeta_{k,j} = \sum_{h=-1}^{\min\{k,j-1\}}\widetilde{u}_{k-h}w_{n-k+1-j+h} = P_{n+1-j}(\widetilde{u},w),$$

hence $\zeta_{k,j^*} = P_{k+j}(\widetilde{u},w)$. Similarly, from (5.5) and (5.6) for $j = n-k$ and $i = 1,\ldots,k$ we have

$$\zeta_{i,n-k} = \sum_{h=-1}^{\min\{i,n-k-1\}}\widetilde{u}_{i-h}w_{h+1} = P_{i+1}(\widetilde{u},w).$$

We recall that $\widetilde{u}_1 = -w_1$. Thus, (5.12) becomes

$$\mathrm{Res}_0\left(\frac{\prod_{i=1}^{k} dz_{i,n-k}\prod_{j=2}^{n-k} dz_{1,j^*}}{\prod_{i=1}^{k}\zeta_{i,n-k}\prod_{j=2}^{n-k}\zeta_{k,j^*}}c^{\alpha}(V)\right) = \mathrm{Res}_0\left(\frac{\prod_{i=2}^{k} d\widetilde{u}_i\prod_{j=1}^{n-k} dw_j}{\prod_{i=2}^{n} P_i(\widetilde{u},w)}\Delta^{\alpha}(\widetilde{u},w)\right). \tag{5.13}$$

Finally, to get a more symmetric formula we separate $\widetilde{u}_1$ from w_1. Therefore, let $P_1 = P_1(u,w) = u_1 + w_1$ and $u_i = \widetilde{u}_i$ for $i = 2, \ldots, k$, hence $u_1 = P_1 - w_1 = P_1 + \widetilde{u}_1$. Thus,

$$\operatorname{Res}_0\left(\frac{\prod_{i=1}^{k} du_i \prod_{j=1}^{n-k} dw_j}{\prod_{i=1}^{n} P_i(u,w)} \Delta^\alpha(u,w)\right) \stackrel{(\dagger)}{=}$$

$$\stackrel{(\dagger)}{=} \operatorname{Res}_0\left(\frac{dP_1 \prod_{i=2}^{k} du_i \prod_{j=1}^{n-k} dw_j}{P_1 \prod_{i=2}^{n} P_i(u,w)} \Delta^\alpha(u,w)\right) \stackrel{(\ddagger)}{=}$$

$$\stackrel{(\ddagger)}{=} \operatorname{Res}_0\left(\frac{dP_1 \prod_{i=2}^{k} d\widetilde{u}_i \prod_{j=1}^{n-k} dw_j}{P_1 \prod_{i=2}^{n} P_i(\widetilde{u},w)} \Delta^\alpha(\widetilde{u},w)\right) \stackrel{(\dagger\dagger)}{=}$$

$$\stackrel{(\dagger\dagger)}{=} \operatorname{Res}_0\left(\frac{\prod_{i=2}^{k} du_i \prod_{j=1}^{n-k} d\widetilde{w}_j}{\prod_{i=2}^{n} P_i(\widetilde{u},w)} \Delta^\alpha(\widetilde{u},w)\right).$$

In (†) we replaced du_1 with dP_1. In (‡) we made substitution $u_1 = P_1 + \widetilde{u}_1$ and, moreover, we applied Local Duality (P2) to get rid of P_1 from $\Delta^\alpha(u,w)$, appeared after the substitution and to get $\Delta^\alpha(\widetilde{u},w)$. Furthermore, we used Transformation Law (P4) to remove P_1 from $P_2(u,w), \ldots, P_n(u,w)$ after the aforementioned substitution and to get $P_2(\widetilde{u},w), \ldots, P_n(\widetilde{u},w)$ in the denominator. Last, in (††) we used property (P5) to eliminate P_1 from the residue and we got back the right hand side of (5.13). □

6. The residue formula and cohomological relations

We will give an interpretation of variables u_i's and w_j's of Theorem 5.6 in terms of Chern classes of the tautological and quotient bundle on the Grassmannian. Thus, in Theorem 6.4 we can give an even simpler version of Theorem 5.6.

6.1. Cohomology ring of the complex Grassmannian

First, we recall the properties of Chern classes from [3, Ch. IV], then we recall the generators and relations of the cohomology ring of the complex Grassmannian in Theorem 6.1 (cf. [3, Proposition 23.2]).

6.1.1. Chern classes. To a complex vector bundle $\mathcal{E}$ (of rank p) over a manifold M one can associate cohomological classes $c_i(\mathcal{E}) \in H^{2i}(M,\mathbb{C})$, $i = 1, \ldots, p$, called the i^{th} *Chern class* ($c_0(\mathcal{E}) = 1$ and $c_i(\mathcal{E}) = 0$ when $i > p$). One can arrange them into a sequence $c(\mathcal{E};t) = 1 + c_1(\mathcal{E})t + \ldots + c_p(\mathcal{E})t^p$ and $c(\mathcal{E}) = c(\mathcal{E};1)$ is called the *total Chern class.*

Usually, one uses Chern roots to calculate with Chern classes. This is based on the *Splitting Principle* ([3, Ch. IV, §21]): one can pretend that the bundle $\mathcal{E}$ is a direct sum of complex line bundles. The Chern classes $\eta_1, \ldots, \eta_p \in H^2(M,\mathbb{C})$ of these hypothetical line bundles are the *Chern roots* of $\mathcal{E}$, hence $c_i(\mathcal{E}) = e_i(\eta) = \sum_{1 \le \ell_1 < \cdots < \ell_i \le p} \eta_{\ell_1} \cdots \eta_{\ell_i}$ is the i^{th} elementary symmetric polynomial of the Chern roots ($e_0(\eta) = 1$) and $c(\mathcal{E};t) = \prod_{i=1}^{p}(1 + t\eta_i)$. For example, the dual bundle $\mathcal{E}^*$ has Chern roots $-\eta_1, \ldots, -\eta_p$, hence $c(\mathcal{E}^*;t) = c(\mathcal{E};-t)$. Or, if $\mathcal{F}$ is another bundle over M

with Chern roots $\phi_1, \dots, \phi_q$ then $\mathcal{E} \oplus \mathcal{F}$ has Chern roots $\eta_1, \dots, \eta_p, \phi_1, \dots, \phi_q$, hence $c(\mathcal{E} \oplus \mathcal{F}; t) = c(\mathcal{E}; t)c(\mathcal{F}; t)$ (*Whitney product formula*, cf. [3, (20.10.3)]). Similarly, $\mathcal{E} \otimes \mathcal{F}$ has Chern roots $\eta_i + \phi_j$, $i = 1, \dots, p$, $j = 1, \dots, q$, hence $c(\mathcal{E} \otimes \mathcal{F}; t) = \prod_{i=1}^{p} \prod_{j=1}^{q} (1 + t\eta_i + t\phi_j)$, which can be expressed in terms of Chern classes of $c_i(\mathcal{E})$ and $c_j(\mathcal{F})$ (see Lemma 6.2). In particular, when $\mathcal{E}$ and $\mathcal{F}$ are line bundles then $c_1(\mathcal{E} \otimes \mathcal{F}) = c_1(\mathcal{E}) + c_1(\mathcal{F})$ (cf. [3, (21.9)]). The Chern roots of the trivial bundle are zero, thus $c(M \times \mathbb{C}^s; t) = 1$.

6.1.2. Generators and relations of the cohomology ring of the complex Grassmannians. There is a tautological exact sequence of complex vector bundles over the complex Grassmannian $Gr_k(n, \mathbb{C})$

$$0 \to \mathcal{L} \to Gr_k(n, \mathbb{C}) \times \mathbb{C}^n \to \mathcal{Q} \to 0, \tag{6.1}$$

where $\mathcal{L} = \{(U, u) \in Gr_k(n, \mathbb{C}) \times \mathbb{C}^n \mid u \in U\}$ is the tautological (rank k) complex vector bundle and $\mathcal{Q}$ is the quotient vector bundle (of rank $n-k$). The tautological exact sequence (6.1) induces the relation of total Chern classes $c(\mathcal{L})c(\mathcal{Q}) = c(\mathcal{L} \oplus \mathcal{Q}) = c(Gr_k(n, \mathbb{C}) \times \mathbb{C}^n) = 1$, hence we get the following relations between the Chern classes of $\mathcal{L}$ and $\mathcal{Q}$:

$$\sum_{i+j=\ell} c_i(\mathcal{L})c_j(\mathcal{Q}) = 0, \qquad \forall \ell = 1, \dots, n, \tag{6.2}$$

($0 \le i \le k$, $0 \le j \le n-k$). From the first $n-k$ relation one can recursively express each Chern class of the quotient bundle $\mathcal{Q}$ in terms of Chern classes of the tautological bundle $\mathcal{L}$. Substituting them into the remaining k relations we get relations between Chern classes of $\mathcal{L}$.

The Chern classes $c_1(\mathcal{L}), \dots, c_k(\mathcal{L})$ of $\mathcal{L}$ generate the cohomology ring $H^*(Gr_k(n, \mathbb{C}), \mathbb{C})$ with real coefficients, i.e. the ring homomorphism $\mathbb{C}[x_1, \dots, x_k] \to H^*(Gr_k(n, \mathbb{C}), \mathbb{C})$, $x_i \mapsto c_i(\mathcal{L})$ is surjective. Moreover, the above mentioned relation are the only relations between them in the cohomological ring. Nevertheless, one includes the Chern classes of the quotient bundle $\mathcal{Q}$ among the generators for easier description of relations. In this latter case we have the following description of the cohomology ring of the complex Grassmannian $Gr_k(n, \mathbb{C})$.

Theorem 6.1 (cf. [3, Proposition 23.2]**).** *The graded ring morphism induced by $x_i \mapsto c_i(\mathcal{L})$ and $y_j \mapsto c_j(\mathcal{Q})$ induces an isomorphism of graded rings*

$$H(Gr_k(n, \mathbb{C}), \mathbb{C}) \simeq \mathbb{C}[x_1, \dots, x_k, y_1, \dots, y_{n-k}] / \langle P_1(x, y), \dots, P_n(x, y) \rangle,$$

where $P_\ell(x, y) = \sum_{i+j=\ell} x_i y_j$, $\ell = 1, \dots, n$ with convention $x_0 = y_0 = 1$ and $\deg x_i = 2i$ for $i = 1, \dots, k$, $\deg y_j = 2j$ for $j = 1, \dots, n-k$.

6.2. Reinterpretation of the residue formula in terms of tautological and quotient bundle

Since Theorem 5.6 is in terms of the Chern classes of the tangent bundle we have to show that they also generate the cohomology ring.

The tangent bundle of $Gr_k(n, \mathbb{C})$ can be given as $TGr_k(n, \mathbb{C}) \simeq Hom(\mathcal{L}, \mathcal{Q}) = \mathcal{L}^* \otimes \mathcal{Q}$. Thus, if $\sigma_1, \dots, \sigma_k$ and $\tau_1, \dots, \tau_{n-k}$ are Chern roots of $\mathcal{L}$ and $\mathcal{Q}$, respectively, then

$$c(TGr_k(n,\mathbb{C});t) = c(\mathcal{L}^* \otimes \mathcal{Q};t) = \prod_{i=1}^{k}\prod_{j=1}^{n-k}(1 - t\sigma_i + t\tau_j). \tag{6.3}$$

Similarly to [8, Lemma 1] we have the following formula.

Lemma 6.2. *If $e_i(\sigma) = e_i(\sigma_1,\dots,\sigma_k)$, $i = 1,\dots,k$ and $e_j(\tau) = e_j(\tau_1,\dots,\tau_{n-k})$, $j = 1,\dots,n-k$ are elementary symmetric polynomials in formal variables σ_ℓ and τ_ℓ, respectively, then*

$$\prod_{i=1}^{k}\prod_{j=1}^{n-k}(1 + t\sigma_i + t\tau_j) = \det\Big(\sum_{i=0}^{k} e_i(\sigma)t^i(I + t\Lambda_{e(\tau)})^{k-i}\Big),$$

where $\Lambda_{e(\tau)}$ is defined in (5.7). Thus,

$$c(\mathcal{L}^* \otimes \mathcal{Q};t) = \det\Big(\sum_{i=0}^{k} c_i(\mathcal{L}^*)t^i(I + t\Lambda_{c(\mathcal{Q})})^{k-i}\Big).$$

Proof. In the proof of [8, Lemma 1] it was shown that the matrix $\Lambda_{e(\tau)}$ is diagonalizable, $\Lambda_{e(\tau)} = E\operatorname{diag}(\tau_1,\dots\tau_{n-k})E^{-1}$, hence

$$\det\Big(\sum_{i=0}^{k} e_i(\sigma)t^i(I + t\Lambda_{e(\tau)})^{k-i}\Big) =$$

$$= \det\Big(\sum_{i=0}^{k} e_i(\sigma)t^i E\operatorname{diag}(1 + t\tau_1,\dots,1 + t\tau_{n-k})^{k-i}E^{-1}\Big) =$$

$$= \det\Big(E\operatorname{diag}\Big(\sum_{i=0}^{k} e_i(\sigma)t^i(1 + t\tau_1),\dots,\sum_{i=0}^{k} e_i(\sigma)t^i(1 + t\tau_{n-k})\Big)^{k-i}E^{-1}\Big) =$$

$$= \prod_{j=1}^{n-k}\sum_{i=0}^{k} e_i(\sigma)t^i(1 + t\tau_j)^{k-i} = \prod_{j=1}^{n-k}\prod_{i=1}^{k}(1 + t\tau_j + t\sigma_i).$$

□

Lemma 6.3. *The Chern classes of the tangent bundle of the Grassmannian $Gr_k(n,\mathbb{C})$ also generate the cohomology ring when $n \neq 2k$.*

Proof. The relation (6.2) reads as $c_\ell(\mathcal{Q})+c_{\ell-1}(\mathcal{Q})c_1(\mathcal{L})+c_{\ell-2}(\mathcal{Q})c_2(\mathcal{L})+\cdots+c_\ell(\mathcal{L}) = 0$ for $\ell \le n-k$, hence the Chern classes $c_j(Q)$ of the quotient bundle can be expressed recursively in terms of Chern classes of the tautological bundle $\mathcal{L}$,

$$c_j(\mathcal{Q}) = -c_j(\mathcal{L}) + \textit{polynomial of lower order classes of } \mathcal{L}. \tag{6.4}$$

Then by (6.3) and Lemma 6.2 we have

$$c(TGr_k(n,\mathbb{C});t) = \det\Big(\sum_{i=0}^{k} c_i(\mathcal{L}^*)t^i\big(I + t\Lambda_{c(\mathcal{Q})}\big)^{k-i}\Big)$$
$$= \det\Big(\sum_{j=0}^{n-k} c_j(\mathcal{Q})t^j\big(I + t\Lambda_{c(\mathcal{L}^*)}\big)^{n-k-j}\Big),$$

thus $c_\ell(TGr_k(n,\mathbb{C})) = (n-k)c_\ell(\mathcal{L}^*) + kc_\ell(\mathcal{Q})+$ *polynomial of lower order classes*. Hence, by (6.4) we get $c_\ell(TGr_k(n,\mathbb{C})) = [(-1)^\ell(n-k) - k]c_\ell(\mathcal{L}) +$ *polynomial of lower order classes of* $\mathcal{L}$. If $n \neq 2k$ then the coefficient of $c_\ell(\mathcal{L})$ does not vanish, hence it can be expressed recursively in terms of Chern classes of the tangent bundle. □

Theorem 5.6 can be reformulated using Chern classes of the tautological and the quotient bundle.

Theorem 6.4. *Assume that $n \neq 2k$. For any polynomial*

$$R(x,y) \in \mathbb{C}[x_1,\dots,x_k,y_1,\dots,y_{n-k}]$$

we have

$$\int_{Gr_k(n,\mathbb{C})} R(c(\mathcal{L}^*),c(\mathcal{Q}^*)) = \operatorname{Res}_0\left(\frac{\prod_{i=1}^k dx_i \prod_{j=1}^{n-k} dy_j}{\prod_{\ell=1}^n P_\ell(x,y)} R(x,y)\right), \qquad (6.5)$$

where $P_\ell(x,y) = \sum_{i+j=\ell} x_i y_j$, with convention $x_0 = y_0 = 1$.

Proof. On the polynomial ring $\mathbb{C}[x_1,\dots,x_k,y_1,\dots,y_{n-k}]$ we consider the grading induced by $\deg x_i = 2i$, $i = 1,\dots,k$ and $\deg y_j = 2j$, $j = 1,\dots,n-k$. First, assume that $R(x,y)$ is graded homogeneous polynomial of degree $2k(n-k) = \dim_{\mathbb{R}} Gr_k(n,\mathbb{C})$. By (6.3) and Lemma 6.2 we have

$$c(TGr_k(n,\mathbb{C});t) = c(\mathcal{L}^*\otimes\mathcal{Q};t) = \prod_{i=1}^{k}\prod_{j=1}^{n-k}(1 - t\sigma_i + t\tau_j) =$$
$$= \prod_{i=1}^{k}\prod_{j=1}^{n-k}(1 - t\sigma_i - t(-\tau_j)) = \det\Big(\sum_{i=0}^{k} e_i(\sigma)(-t)^i(I - t\Lambda_{e(-\tau)})^{k-i}\Big) =$$
$$= \det\Big(\sum_{i=0}^{k} e_i(-\sigma)t^i(I - t\Lambda_{e(-\tau)})^{k-i}\Big) =$$
$$= \det\Big(\sum_{i=0}^{k} c_i(\mathcal{L}^*)t^i(I - t\Lambda_{c(\mathcal{Q}^*)})^{k-i}\Big) = \Delta(c(\mathcal{L}^*),c(\mathcal{Q}^*);t),$$

hence $c_\ell(TGr_k(n,\mathbb{C})) = \Delta_\ell(c(\mathcal{L}^*),c(\mathcal{Q}^*))$ (cf. Theorem 5.6). Thus, we can write Theorem 5.6 in the form

$$\int_{Gr_k(n,\mathbb{C})} \Delta^\alpha(c(\mathcal{L}^*),c(\mathcal{Q}^*)) = \operatorname{Res}_0\left(\frac{\prod_{i=1}^k dx_i \prod_{j=1}^{n-k} dy_j}{\prod_{\ell=1}^n P_\ell(x,y)}\Delta^\alpha(x,y)\right),$$

for any multidegree $\alpha = (\alpha_1, \dots, \alpha_{k(n-k)})$ with $|\alpha| = k(n-k)$. Since the Chern classes of the tangent bundle generate the cohomology ring, any polynomial $R(c(\mathcal{L}^*), c(\mathcal{Q}^*))$ can be expressed as a linear combination of $\Delta^\alpha(c(\mathcal{L}^*), c(\mathcal{Q}^*))$'s, thus (6.5) follows.

Finally, when R is homogeneous of degree $\deg R \neq 2k(n-k)$ then the left hand side of (6.5) vanishes by definition. Moreover, the right hand side also vanishes by (P7), since $\deg R \neq \sum_{\ell=1}^n \deg P_\ell - \sum_{i=1}^k 2i - \sum_{j=1}^{n-k} 2j$. □

Corollary 6.5. *Under the assumptions of Theorem 6.4 we have*

$$\int_{Gr_k(n,\mathbb{C})} R(c(\mathcal{L}), c(\mathcal{Q})) = (-1)^{k(n-k)} \operatorname{Res}_0 \left(\frac{\prod_{i=1}^k dx_i \prod_{j=1}^{n-k} dy_j}{\prod_{\ell=1}^n P_\ell(x,y)} R(x,y) \right). \tag{6.6}$$

6.2.1. Cohomology relations from the residue formula. One benefit of the formula (6.5) or (6.6) is that we can easily deduce the relations of the cohomology ring using Poincaré duality.

Proof of Theorem 6.1. We prove the theorem for $n \neq 2k$. By Poincaré duality a (homogeneous) cohomology class $\alpha = A(c(\mathcal{L}), c(\mathcal{Q}))$ vanishes exactly when

$$\int_{Gr_k(n,\mathbb{C})} \alpha\beta = 0$$

for every class $\beta = B(c(\mathcal{L}), c(\mathcal{Q}))$. By (6.6) and Local Duality (P2) follows that $\alpha = A(c(\mathcal{L}), c(\mathcal{Q}))$ vanishes exactly when $A(x,y) \in \langle P_1(x,y), \dots, P_n(x,y)\rangle$, hence this latter being the ideal of relations. □

7. Iterated residues

Since the Chern classes $c_i(\mathcal{L}^*)$, $i = 1, \dots, k$ generates the cohomology ring, hence we will give an iterated residue formula for $\int_{Gr_k(n,\mathbb{C})} \Phi(c(\mathcal{L}^*))$, where Φ is a polynomial in k variables.

We introduce shorter notations $t' = (t_1, \dots, t_k)$ and $t'' = (t_{k+1}, \dots, t_n)$ and we consider the finite map $F : \mathbb{C}^n \to \mathbb{C}^n$ defined by

$$F(t_1, \dots, t_n) = (e_1(t'), \dots, e_k(t'), e_1(t''), \dots, e_{n-k}(t'')),$$

where e_i denotes the i^{th} elementary symmetric polynomial.

Below, we use property (P6) to pull back the residue (6.5) along F. We note that F generically is a $k!(n-k)!$-fold cover and the pull-back of polynomials

$$F^* P_\ell(x,y) = \sum_{i=0}^{\ell} e_i(t') e_{\ell-i}(t'') = e_\ell(t_1, \dots, t_n) = e_\ell(t).$$

Moreover, the Jacobian

$$\begin{aligned} J_F(t) &= \det[e_{i-1}(t_1, \dots, \widehat{t_j}, \dots, t_k)]_{i,j=1}^k \cdot \det[e_{i-1}(t_{k+1}, \dots, \widehat{t_{k+j}}, \dots, t_n)]_{i,j=1}^{n-k} \\ &= \prod_{1 \le i < j \le k} (t_i - t_j) \prod_{k+1 \le h < l \le n} (t_h - t_l), \end{aligned}$$

where $\widehat{t_j}$ means that t_j is omitted.

$$\operatorname{Res}_0\left[\frac{\prod_{i=1}^k dx_i \prod_{j=1}^{n-k} dy_j}{\prod_{i=1}^n P_i(x,y)}\Phi(x)\right] =$$

$$= \frac{1}{k!(n-k)!}\operatorname{Res}_0\left[\frac{\prod_{i=1}^k de_i(t') \prod_{j=1}^{n-k} de_j(t'')}{\prod_{\ell=1}^n e_\ell(t)}\Phi(e(t'))\right]$$

$$= \frac{1}{k!(n-k)!}\operatorname{Res}_0\left[\frac{\prod_{\ell=1}^n dt_\ell}{\prod_{\ell=1}^n e_\ell(t)}\prod_{1\le i<j\le k}(t_i-t_j)\prod_{k+1\le h<l\le n}(t_h-t_l)\Phi(e(t'))\right].$$

Next, we use the Transformation Law (P4) for the transformation $\left[t_i^n\right]_{i=1}^n = \left[(-1)^{j-1}t_i^{n-j}\right]_{i,j=1}^n\left[e_j(t)\right]_{j=1}^n$. Finally, the coefficient of $t_{k+1}^{n-1}\dots t_n^{n-1}$ in $\prod_{k+1\le h\ne l\le n}(t_h-t_l)\prod_{i=1}^k\prod_{h=k+1}^n(t_h-t_i)$ is $(n-k)!$. Thus,

$$\frac{1}{k!(n-k)!}\operatorname{Res}_0\left[\frac{\prod_{\ell=1}^n dt_\ell}{\prod_{\ell=1}^n e_\ell(t)}\prod_{1\le i<j\le k}(t_i-t_j)\prod_{k+1\le h<l\le n}(t_h-t_l)\Phi(e(t'))\right] =$$

$$= \frac{1}{k!(n-k)!}\operatorname{Res}_0\left[\frac{\prod_{\ell=1}^n dt_\ell}{\prod_{\ell=1}^n t_\ell^n}\prod_{1\le i<j\le k}(t_i-t_j)\prod_{k+1\le h<l\le n}(t_h-t_l)\right.$$
$$\left.\prod_{1\le a<b\le n}(t_b-t_a)\Phi(e(t'))\right] =$$

$$= \frac{1}{k!(n-k)!}\operatorname{Res}_0\left[\frac{\prod_{\ell=1}^n dt_\ell}{\prod_{\ell=1}^n t_\ell^n}\prod_{1\le i\ne j\le k}(t_i-t_j)\prod_{k+1\le h\ne l\le n}(t_h-t_l)\right.$$
$$\left.\prod_{i=1}^k\prod_{h=k+1}^n(t_h-t_i)\Phi(e(t'))\right] =$$

$$= \frac{1}{k!}\operatorname{Res}_0\left[\frac{\prod_{i=1}^k dt_i}{\prod_{i=1}^k t_i^n}\prod_{1\le i\ne j\le k}(t_i-t_j)\Phi(e(t'))\right] =$$

$$= \frac{1}{k!}\operatorname*{Res}_{t_1=0}\dots\operatorname*{Res}_{t_k=0}\left[\frac{\prod_{i=1}^k dt_i}{\prod_{i=1}^k t_i^n}\prod_{1\le i\ne j\le k}(t_i-t_j)\Phi(e(t'))\right],$$

hence

$$\int_{Gr_k(n,\mathbb{C})}\Phi(c(\mathcal{L}^*)) = \frac{1}{k!}\operatorname*{Res}_{t_1=0}\dots\operatorname*{Res}_{t_k=0}\left[\frac{\prod\limits_{i=1}^k dt_i}{\prod\limits_{i=1}^k t_i^n}\prod_{1\le i\ne j\le k}(t_i-t_j)\Phi(e(t_1,\dots,t_k))\right]. \tag{7.1}$$

This iterated residue formula agrees with the Jeffrey-Kirwan formula for the Grassmannian constructed as symplectic quotient (cf. [7, Proposition 7.2]).

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