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Reducing the complexity of equilibrium problems and applications to best approximation problems

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Abstract. We consider the scalar equilibrium problems governed by a bifunction in a finite-dimensional framework and we characterize the solutions by means of extreme or exposed points.

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1. Introduction

In this article, we focus on scalar equilibrium problems governed by a bifunction within a finite-dimensional framework. Through the use of classical arguments and techniques from Convex Analysis, we show that under suitable generalized convexity assumptions imposed on the bifunction, the solutions of the equilibrium problem can be characterized by means of extreme points (Corollary 4.13) or exposed points (Corollary 4.16) of the feasible domain. Our findings have significant implications for various particular instances, including variational inequalities and optimization problems, and are particularly relevant to best approximation problems, as seen in the examples of Section 4.

This paper is organized as follows. In Section 2, we introduce our general notations and we recall some useful facts from Convex Analysis, primarily focused on the best approximation problem. In Section 3, following up on the same problem, from a geometric point of view, we proved that if S is a nonempty convex subset of $\mathbb{R}^n$,

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then the elements of best approximation to an arbitrary element in $\mathbb{R}^n$ from S, can be characterized by means of the Gauss map (Remark 3.2). In fact, if S is also closed, then it is known that x^0 is the element of best approximation to an arbitrary element x^* from S, if and only if x^* is an element of the translated normal cone to S at x^0 by the vector x^0 (Proposition 3.3). This led us to Proposition 3.4, where we have proved that the set $\{x + N_S(x) \setminus \{0\} \mid x \in \operatorname{bd} S\}$ is a partition of $\mathbb{R}^n \setminus S$, where $N_S(x)$ is the normal cone to S at x.

In Section 4, we move onto equilibrium problems and reducing their complexity (Theorem 4.7), as follows. For an arbitrary nonempty set A and an arbitrary nonempty subset M of $\mathbb{R}^n$, we consider the bifunction $g : A \times \operatorname{conv} M \to \mathbb{R}$, which is assumed to be quasiconvex in the second argument. For such a bifunction, we show that the equilibrium points of g are precisely the equilibrium points of the restriction $g|_{A\times M}$. A consequence of this is Corollary 4.13, which for a nonempty convex and compact subset S of $\mathbb{R}^n$, and a bifunction $g : A \times S \to \mathbb{R}$, also assumed to be quasiconvex in the second argument, shows that the equilibrium points of g are precisely the equilibrium points of $g|_{A\times \operatorname{ext} S}$. We also point out the particular case, when $M = \operatorname{ext} S$ for some Minkowski set S, which shows that the previous hypothesis of boundedness of S is not crucial. Finally, Theorem 4.7 led us to our main result, Corollary 4.16, where we have obtained that for a nonempty convex and compact subset S of $\mathbb{R}^n$, if $g : A \times S \to \mathbb{R}$ is quasiconvex and lower semicontinous in the second argument, then the equilibrium points of g are precisely the equilibrium points of $g|_{A\times \exp S}$.

2. Notations and preliminaries

Throughout this paper $\mathbb{R}^n$ stands for the n-dimensional real Euclidean space, whose norm $\|\cdot\|$ is induced by the usual inner product $\langle\cdot,\cdot\rangle$. For any points $x, y \in \mathbb{R}^n$, we use the notations

$$\begin{aligned} [x, y] &:= \{(1-t)x + ty \mid t \in [0,1]\,\}, \\ ]x, y[&:= \{(1-t)x + ty \mid t \in]0,1[\,\}. \end{aligned}$$

Recall that a set $S \subseteq \mathbb{R}^n$ is called convex if $[x, y] \subseteq S$, for all $x, y \in S$. Of course, this is equivalent to say that $]x, y[\subseteq S$, for all $x, y \in S$.

Given a convex set $S \subseteq \mathbb{R}^n$ we denote the set of extreme points of S by

$$\operatorname{ext} S = \{x^0 \in S \mid \forall x, y \in S \,:\, x^0 = \tfrac{1}{2}(x+y) \;\Rightarrow x = y = x^0\}.$$

A point x^0 is said to be an exposed point of S if there is a supporting hyperplane H which supports S at x^0 such that $\{x^0\} = H \cap S$. We denote the set of exposed points of S by

$$\exp S = \{x^0 \in S \mid \exists c \in \mathbb{R}^n \setminus \{0\} \text{ such that } \operatorname*{argmin}_{x\in S}\langle c, x\rangle = \{x^0\}\}.$$

It is well-known that $\exp S \subseteq \operatorname{ext} S$.

The convex hull of a set $M \subseteq \mathbb{R}^n$, i.e., the smallest convex set in $\mathbb{R}^n$ containing M is denoted by $\operatorname{conv} M$.

Next, we recall the following well-known theorems (see for example [5] and [7]):

Theorem 2.1 (Minkowski (Krein-Milman)). *Every compact convex set in $\mathbb{R}^n$ is the convex hull of its extreme points.*

Theorem 2.2 (Straszewicz). *Every compact convex subset M of $\mathbb{R}^n$ admits the representation*:

$$M \quad = \quad \mathrm{cl}(\mathrm{conv}(\exp M)).$$

In the book by Breckner and Popovici [1, C 5.2.7, p. 82] we have the following remark:

Remark 2.3 (Minkowski). Let $S \subseteq \mathbb{R}^n$ be a compact convex set. Then, for each subset M of S, the following equivalence holds:

$$S = \mathrm{conv}\, M \quad \Longleftrightarrow \quad \mathrm{ext}\, S \subseteq M.$$

Definition 2.4. Let S be a nonempty subset of $\mathbb{R}^n$ and let $x^* \in \mathbb{R}^n$. A point $x^0 \in S$ is said to be *an element of best approximation to x^* from S* (or *a nearest point to x^* from S*) if

$$\|x^0 - x^*\| \leq \|x - x^*\|, \text{ for all } x \in S.$$

The problem of best approximation of x^* by elements of S consists in finding all elements of best approximation to x^* from S. The solution set

$$P_S(x^*) := \{x^0 \in S \mid \ \|x^0 - x^*\| \leq \|x - x^*\|, \text{ for all } x \in S\}$$

is called *the metric projection of x^* on S*.

Remark 2.5. The problem of best approximation is an optimization problem,

$$\begin{cases} f(x) \longrightarrow \min \\ x \in S, \end{cases}$$

whose objective function $f : \mathbb{R}^n \to \mathbb{R}$ is defined for all $x \in \mathbb{R}^n$ by

$$f(x) := \|x - x^*\|.$$

Actually, we have

$$P_S(x^*) = \underset{x \in S}{\mathrm{argmin}}\, f(x).$$

Definition 2.6. Let S be a nonempty subset of $\mathbb{R}^n$ and let $x^* \in \mathbb{R}^n$, we say that $x^0 \in S$ is a farthest point from S to x^* if

$$\|x^0 - x^*\| \geq \|x - x^*\|, \text{ for all } x \in S,$$

i.e.,

$$x^0 \in \underset{x \in S}{\mathrm{argmax}}\, \|x - x^*\|.$$

In this paper we will use the following well known results from Convex Analysis (see for example [1]).

Proposition 2.7. *Any farthest point from a nonempty set $S \subseteq \mathbb{R}^n$ to a point $x^* \in \mathbb{R}^n$ is an exposed point of S, i.e.,*

$$\underset{x \in S}{\mathrm{argmax}}\, \|x - x^*\| \subseteq \exp S.$$

Theorem 2.8 (existence of elements of best approximation). *If S is a nonempty closed subset of $\mathbb{R}^n$, then for every $x^* \in \mathbb{R}^n$ there is an element of best approximation to x^* from S. In other words, we have*

$$P_S(x^*) \neq \emptyset, \text{ i.e., } \operatorname{card}(P_S(x^*)) \geq 1.$$

Theorem 2.9 (unicity of the element of best approximation). *If $S \subseteq \mathbb{R}^n$ is a nonempty convex set and $x^* \in \mathbb{R}^n$, then there exists at most one element of best approximation to x^* from S. In other words, we have*

$$\operatorname{card}(P_S(x^*)) \leq 1.$$

Theorem 2.10 (characterization of elements of best approximation). *Let $S \subseteq \mathbb{R}^n$, let $x^0 \in S$, and let $x^* \in \mathbb{R}^n$. Then the following hold:*

(a) *If $\langle x - x^0, x^* - x^0 \rangle \leq 0$ for all $x \in S$, then x^0 is an element of best approximation to x^* from S.*
(b) *If S is convex and x^0 is an element of best approximation to x^* from S, then we have that $\langle x - x^0, x^* - x^0 \rangle \leq 0$ for all $x \in S$.*

Corollary 2.11. *Let $S \subseteq \mathbb{R}^n$ be a nonempty convex set and let $x^* \in \mathbb{R}^n$. Then*

$$P_S(x^*) = \{x^0 \in S \mid \langle x - x^0, x^* - x^0 \rangle \leq 0, \text{ for all } x \in S\}.$$

3. The inverse images of the metric projection

Our main results from Section 4 generate some interesting examples with regard to the best approximation problem. A further analysis of the characterization of the elements of best approximation (Theorem 2.10) led us to a geometric approach involving the inverse images of the metric projection, which will be presented in this section.

Remark 3.1. From a geometric point of view, the property $\langle x - x^0, x^* - x^0 \rangle \leq 0$ for all $x \in S$ in assertion (b) of Theorem 2.10 shows that $x^* - x^0$ belongs to the so-called normal cone to S at x^0, i.e.,

$$N_S(x^0) = \{d \in \mathbb{R}^n \mid \langle x - x^0, d \rangle \leq 0, \text{ for all } x \in S\}.$$

For $S \subseteq \mathbb{R}^n$ a nonempty convex set, we recall the *Gauss map* of S, introduced in [4], which is a set-valued map, defined as follows:

$$G_S : \mathbb{R}^n \rightrightarrows S^{n-1}, \qquad G_S(x) := N_S(x) \cap S^{n-1},$$

where S^{n-1} is the unit sphere in $\mathbb{R}^n$.

Remark 3.2. Let $S \subseteq \mathbb{R}^n$ be a nonempty convex set and let $x^* \in \mathbb{R}^n \setminus S$. Then x^0 is an element of best approximation to x^* from S if and only if

$$\frac{x^* - x^0}{\|x^* - x^0\|} \in G_S(x^0).$$

Indeed, x^0 is an element of best approximation to x^* from S if and only if for all $x \in S$

$$\begin{aligned}\langle x - x^0, x^* - x^0\rangle \leq 0 \quad &\Leftrightarrow \quad \left\langle x - x^0, \frac{x^* - x^0}{\|x^* - x^0\|}\right\rangle \leq 0 \\ &\Leftrightarrow \quad \frac{x^* - x^0}{\|x^* - x^0\|} \in N_S(x^0) \\ &\Leftrightarrow \quad \frac{x^* - x^0}{\|x^* - x^0\|} \in G_S(x^0).\end{aligned}$$

Let $S \subseteq \mathbb{R}^n$ be a nonempty closed convex set. By Theorems 2.8 and 2.9 it follows that, for all $x^* \in \mathbb{R}^n$, $P_S(x^*)$ is a singleton. So, in this case, P_S can be considered as a single valued mapping.

The following result is well-known, yet we include it because it is one of the main ingredient of our main result of this section, Proposition 3.4.

Proposition 3.3. *Let $S \subseteq \mathbb{R}^n$ be a nonempty closed convex set. Then, for all $x^* \in \mathbb{R}^n$, we have that $P_S(x^*) = x^0$ if and only if $x^* \in x^0 + N_S(x^0)$.*

Proof. Let $x^* \in \mathbb{R}^n$. Since S is closed and convex, there exists $x^0 \in S$ such that $P_S(x^*) = x^0$. It follows by Corollary 2.11 that

$$\langle x - x^0, x^* - x^0\rangle \leq 0, \text{ for all } x \in S$$

which, by Remark 3.1, is equivalent to

$$x^* - x^0 \in N_S(x^0) \iff x^* \in x^0 + N_S(x^0),$$

and the statement is completely proved. □

Proposition 3.4. *Let $S \subseteq \mathbb{R}^n$ be a nonempty closed convex set. Then, the family*

$$\{x + N_S(x) \setminus \{0\} \mid x \in \operatorname{bd} S\}$$

is a partition of $\mathbb{R}^n \setminus S$.

Proof. We need to show that,

$$\mathbb{R}^n \setminus S = \bigcup_{x \in \operatorname{bd} S} (x + N_S(x) \setminus \{0\})$$

and $(x + N_S(x) \setminus \{0\}) \cap (y + N_S(y) \setminus \{0\}) \neq \emptyset$ implies $x = y$.

Let $x^* \in \mathbb{R}^n \setminus S$ and $x^0 \in S$ such that $P_S(x^*) = x^0$. By Proposition 3.3, we obtain that $x^* \in x^0 + N_S(x^0) \setminus \{0\}$, yet

$$x^0 + N_S(x^0) \setminus \{0\} \subseteq \bigcup_{x \in S} (x + N_S(x) \setminus \{0\}).$$

Subsequently, we get that

$$\mathbb{R}^n \setminus S \subseteq \bigcup_{x \in S} (x + N_S(x) \setminus \{0\}).$$

In order to prove the opposite inclusion, let us consider $x \in S$ and $u \in N_S(x) \setminus \{0\}$. If we assume that $x + u \in S$ then, by the definition of $N_S(x) \setminus \{0\}$,

$$\langle x + u - x, u\rangle = \langle u, u\rangle \leq 0 \implies u = 0$$

which contradicts the fact that $u \in N_S(x) \setminus \{0\}$. Thus $x + N_S(x) \setminus \{0\} \subseteq \mathbb{R}^n \setminus S$, for all $x \in S$, i.e.,

$$\bigcup_{x \in S} (x + N_S(x) \setminus \{0\}) \subseteq \mathbb{R}^n \setminus S.$$

Therefore, we have proved that

$$\mathbb{R}^n \setminus S = \bigcup_{x \in S} (x + N_S(x) \setminus \{0\}).$$

However, since $x + N_S(x) \setminus \{0\}$ is nonempty if and only if x is an boundary point of S, we obtain that

$$\mathbb{R}^n \setminus S = \bigcup_{x \in \operatorname{bd} S} (x + N_S(x) \setminus \{0\}).$$

If $(x + N_S(x) \setminus \{0\}) \cap (y + N_S(y) \setminus \{0\}) \neq \emptyset$, then there is an $u \in N_S(x) \setminus \{0\}$ and $v \in N_S(y) \setminus \{0\})$ such that $x + u = y + v$. Furthermore, since $u \in N_S(x) \setminus \{0\}$, we obtain $\langle y - x, u \rangle = \langle u - v, u \rangle \leq 0$, therefore $\|u\|^2 \leq \langle u, v \rangle$. By similar reasoning, since $v \in N_S(y) \setminus \{0\}$, we obtain $\|v\|^2 \leq \langle u, v \rangle$. Thus,

$$0 \leq \|x - y\|^2 = \|u - v\|^2 = \|u\|^2 - 2\langle u, v \rangle + \|v\|^2 \leq 0$$

Henceforth, $\|x - y\| = 0$, which implies $x = y$.

□

Remark 3.5. Alternatively, one may argue as follows. By Proposition 3.3, we have that for all $x \in S$, the set $x + N_S(x)$ is the inverse image $P_S^{-1}(x)$, of x through P_S. If we consider the restriction $P_S|_{\mathbb{R}^n \setminus S}$ of the mapping P_S, then for all $x \in S$, we have that the set $x + N_S(x) \setminus \{0\}$ is the inverse image of x through the restriction $P_S|_{\mathbb{R}^n \setminus S}$. By the equivalence relation induced by $\ker P_S|_{\mathbb{R}^n \setminus S}$, we obtain that the family

$$\{x + N_S(x) \setminus \{0\} \mid x \in \operatorname{bd} S\}$$

is a partition of $\mathbb{R}^n \setminus S$.

Example 3.6. For $n = 2$, consider the set

$$M = \{x_1 = (1, 1), x_2 = (-1, 1), x_3 = (-1, -1), x_4 = (1, -1)\} \subseteq \mathbb{R}^2$$

and

$$S = \operatorname{conv} M = [-1, 1] \times [-1, 1],$$

as in Figure 1. Obviously, S is a nonempty closed convex subset of $\mathbb{R}^2$ and

$$\operatorname{bd} S =]x_1, x_2[\cup]x_2, x_3[\cup]x_3, x_4[\cup]x_1, x_4[\cup \{x_1, x_2, x_3, x_4\}.$$

On the four vertices of the square, we have

$$\begin{aligned}
N_S(x_1) \setminus \{0\} &= \{(v_1, v_2) \mid v_1 \geq 0, v_2 \geq 0\} \setminus \{(0, 0)\} \\
N_S(x_2) \setminus \{0\} &= \{(v_1, v_2) \mid v_1 \leq 0, v_2 \geq 0\} \setminus \{(0, 0)\} \\
N_S(x_3) \setminus \{0\} &= \{(v_1, v_2) \mid v_1 \leq 0, v_2 \leq 0\} \setminus \{(0, 0)\} \\
N_S(x_4) \setminus \{0\} &= \{(v_1, v_2) \mid v_1 \geq 0, v_2 \leq 0\} \setminus \{(0, 0)\}.
\end{aligned}$$

On the other points of the boundary, we have

$$N_S(x) \setminus \{0\} = \begin{cases} \{(0,v) \mid v > 0\}, \text{ for all } x \in]x_1, x_2[\\ \{(v,0) \mid v < 0\}, \text{ for all } x \in]x_2, x_3[\\ \{(0,v) \mid v < 0\}, \text{ for all } x \in]x_3, x_4[\\ \{(v,0) \mid v > 0\}, \text{ for all } x \in]x_1, x_4[. \end{cases}$$

It is easy to see that $\mathbb{R}^n \setminus S = \bigcup\limits_{x \in \operatorname{bd} S} (x + N_S(x) \setminus \{0\})$.

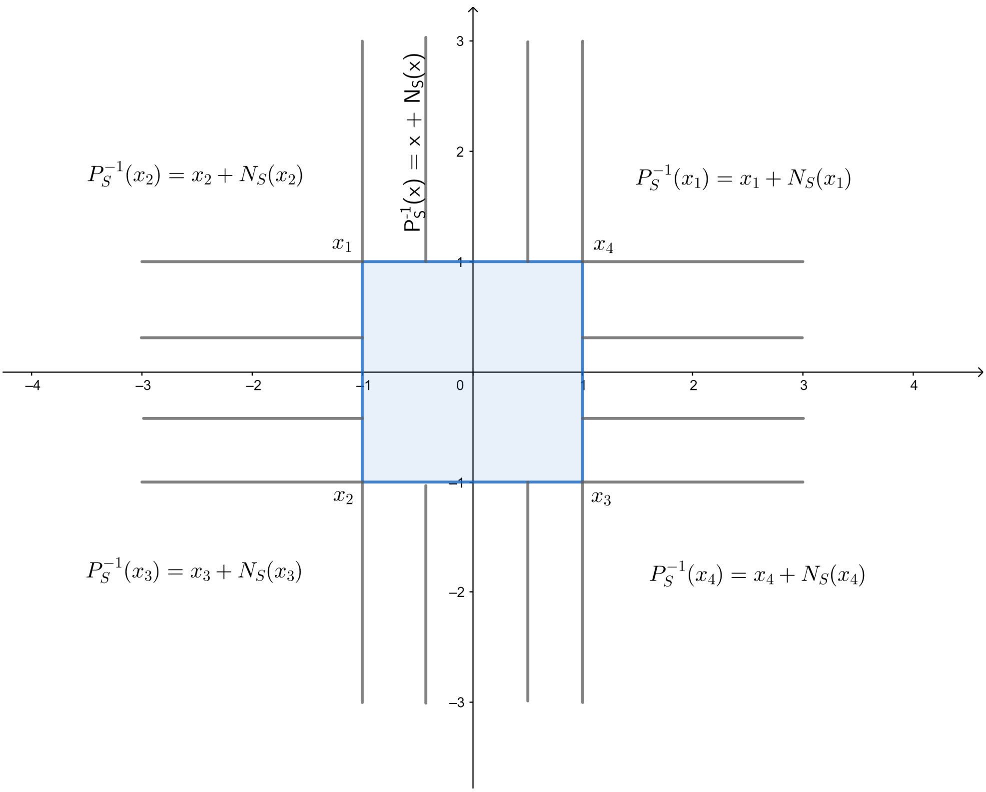

FIGURE 1. Proposition 3.4 applied for the particular case of a square in $\mathbb{R}^2$

4. Equilibrium problems

The equilibrium problem, introduced in [6], has been formulated in a more general way in [2, p. 18]. We propose a slightly modified definition. Let $g : A \times B \to \mathbb{R}$ be a "bifunction", where A and B are nonempty sets.

Definition 4.1. The *equilibrium problem* with respect to $g : A \times B \to \mathbb{R}$ and a couple of subsets $A' \subseteq A$ and $B' \subseteq B$, consists in finding the elements $x^0 \in A'$ satisfying

$$g(x^0, x) \leq 0 \text{ for all } x \in B'.$$

The set of all solutions of the equilibrium problem will be denoted by

$$\text{eq}(g \mid A', B') := \{x^0 \in A' \mid g(x^0, x) \leq 0, \ \forall\, x \in B'\}.$$

Remark 4.2. It is easy to see that

$$\text{eq}(g \mid A', \emptyset) = A'$$

and that

$$\text{eq}(g \mid A', B') \subseteq \text{eq}(g \mid A', B''), \ \forall B'' \subseteq B'$$

Example 4.3 (optimization problems). Consider a minimization problem

$$\begin{cases} f(x) \longrightarrow \min \\ x \in S, \end{cases}$$

where $f : \mathbb{R}^n \to \mathbb{R}$ is a function and $S \subseteq \mathbb{R}^n$ is a nonempty set. By defining the bifunction $g : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ as

$$g(u, v) := f(u) - f(v), \ \forall\, (u, v) \in \mathbb{R}^n \times \mathbb{R}^n,$$

we obtain

$$\text{eq}(g \mid S, S) = \operatorname*{argmin}_{x \in S} f(x).$$

Example 4.4 (variational inequalities). Let $T : S \to \mathbb{R}^n$ be a function defined on a nonempty set $S \subseteq \mathbb{R}^n$. The problem of finding $x^0 \in S$ such that

$$\langle T(x^0), x - x^0 \rangle \geq 0, \forall\, x \in S$$

is called a variational inequality. Denote by sol(VI) the set of its solutions. By defining the bifunction $g : S \times S \to \mathbb{R}$ as

$$g(u, v) := \langle T(u), u - v \rangle, \ \forall\, (u, v) \in S \times S,$$

we obtain

$$\text{eq}(g \mid S, S) = \text{sol(VI)}$$

Example 4.5 (the best approximation problem).

1. The problem of best approximation of x^* by elements of S fits the model described in Example 4.3, where

   $$f(x) = \|x - x^*\| \text{ and } g(u, v) := f(u) - f(v)$$

   hence

   $$\text{eq}(g \mid S, S) = \operatorname*{argmin}_{x \in S} \|x - x^*\|.$$

2. Another way of seeing the best approximation problem as an equilibrium problem is to consider the bifunction $g : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ defined by
$$g(u, v) := \langle v - u, x^* - u \rangle, \ \forall\, (u, v) \in \mathbb{R}^n \times \mathbb{R}^n.$$
According to Theorem 2.10,
$$\text{eq}(g \mid S, S) = \{x^0 \in S \mid \langle x - x^0, x^* - x^0 \rangle \le 0, \ \forall\, x \in S\} \subseteq P_S(x^*),$$
the equality being true whenever S is convex, i.e.,
$$\text{eq}(g \mid S, S) = P_S(x^*).$$
Actually, by considering the function $T : S \to \mathbb{R}^n$ defined by $T(x) = x - x^*$ for all $x \in S$, we recover
$$g(u, v) = \langle T(u), u - v \rangle, \forall\, (u, v) \in S \times S,$$
hence, under the convexity assumption on S we can reduce the best approximation problem to a variational inequality:
$$P_S(x^*) = \text{sol(VI)}.$$

Example 4.6 (the farthest point problem). Let $S \subseteq \mathbb{R}^n$ be a nonempty set and let $x^* \in \mathbb{R}^n$. The problem of finding the farthest points from S to x^* fits the model described in Example 4.3, where
$$f(x) = -\|x - x^*\| \,\text{and}\, g(u, v) := f(u) - f(v),$$
hence
$$\text{eq}(g \mid S, S) = \operatorname*{argmax}_{x \in S} \|x - x^*\|$$

Theorem 4.7. *Let A be a nonempty set, let $S = \text{conv}M$ for some nonempty set $M \subseteq \mathbb{R}^n$ and let $g : A \times S \to \mathbb{R}$ be a bifunction. If for every $u \in A$, the function $h = g(u, \cdot) : S \to \mathbb{R}$ is quasiconvex, i.e.,*
$$h((1 - t)v' + tv'') \le \max\{h(v'), h(v'')\}$$
for all $v, v' \in S$ and $t \in [0, 1]$, then
$$\text{eq}(g \mid A, S) = \{x^0 \in A \mid g(x^0, x) \le 0, \ \forall\, x \in M\},$$
i.e.,
$$\text{eq}(g \mid A, S) = \text{eq}(g \mid A, M).$$

Proof. We denote by
$$E := \{x^0 \in A \mid g(x^0, x) \le 0, \ \forall\, x \in M\}.$$
It is obvious that $\text{eq}(g \mid A, S) \subseteq E$. In order to prove the converse, let $x^0 \in E$ and $x \in S$, arbitrary chosen. Since $x \in S = \text{conv}\, M$, this implies that there exists $k \in \mathbb{N}$, $x^1, x^2, ..., x^k \in M$ and $t_1, t_2, ..., t_k \ge 0$ such that $\sum_{i=1}^k t_i = 1$ and that $x = \sum_{i=1}^k t_i x^i$. Therefore, given that $g(x^0, \cdot)$ is quasiconvex, we have
$$g(x^0, x) = g(x^0, \sum_{i=1}^k t_i x^i) \le \max\{g(x^0, x^i) \mid i = 1, \dots, k\} \le 0,$$
since $x^1, x^2, ..., x^k \in M$. $\square$

Remark 4.8. Consider the minimization problem described in Example 4.3, where $g(u,v) = f(u) - f(v)$ for all $u, v \in S$. Since for every $u \in S$ we have $g(u,\cdot) = f(u) - f$, the quasiconvexity of $g(u,\cdot)$ for some $u \in S$ reduces to the the quasiconcavity of f. Thus we deduce from the Theorem 4.7 the following result.

Corollary 4.9. *Assume that* $S = \operatorname{conv} M$ *for some nonempty set* $M \subseteq \mathbb{R}^n$. *If* $f : S \to \mathbb{R}$ *is a quasiconcave function, then*

$$\operatorname*{argmin}_{x\in S} f(x) = \{x^0 \in S \mid f(x^0) \leq f(x), \forall\, x \in M\} \supseteq \operatorname*{argmin}_{x\in M} f(x).$$

Moreover, $\operatorname*{argmin}_{x\in S} f(x)$ *is nonempty if and only if so is* $\operatorname*{argmin}_{x\in M} f(x)$, *hence*

$$\min\, f(S) = \min\, f(M).$$

The assumptions on quasiconvexity of $g(u,\cdot)$ in Theorem 4.7 and quasiconcavity of f in Corollary 4.9 are essential, as shown by the next example (Example 4.10). Moreover, under the hypothesis of Corollary 4.9, the inclusion

$$\operatorname*{argmin}_{x\in S} f(x) \subseteq \operatorname*{argmin}_{x\in M} f(x)$$

does not hold in general, as shown by Example 4.11.

Example 4.10. Let $n = 1$, $M = \{-1, 1\}$, $S = \operatorname{conv} M = [-1, 1]$. Consider the function

$$\begin{cases} f : S \to \mathbb{R} \\ f(x) = x^2 \end{cases}$$

and the bifunction

$$\begin{cases} g : \mathbb{R}^2 \to \mathbb{R} \\ g(u,v) = f(u) - f(v),\ \forall\,(u,v) \in \mathbb{R}^2. \end{cases}$$

Clearly, the f is not quasiconcave, hence function $g(u,\cdot) : S \to \mathbb{R}$ is not quasiconvex for any $u \in S$, in view of Remark 4.8. It is easy to see that

$$\begin{aligned} \operatorname{eq}(g \mid S, S) &= \operatorname*{argmin}_{x\in S} f(x) \\ &= \{0\} \\ &\not\supseteq \operatorname{eq}(g \mid S, M) \\ &= \{x^0 \in S \mid g(x^0, x) \leq 0, \forall\, x \in M\} \\ &= S. \end{aligned}$$

Of course, this example also shows that the quasiconcavity assumption imposed on f in Corollary 4.8 is essential, because

$$\operatorname*{argmin}_{x\in S} f(x) = \{0\} \not\supseteq \operatorname*{argmin}_{x\in M} f(x) = M.$$

Example 4.11. Let $n = 1$, $M = \{-1, 1\}$, $S = \operatorname{conv} M = [-1, 1]$. Consider the function $f : S \to \mathbb{R}$ defined as $f(x) = \max\{0, x\}$, for all $x \in S$. Obviously, f is nondecreasing, hence quasiconcave. However,

$$\operatorname*{argmin}_{x\in S} f(x) = [-1, 0] \not\subseteq \operatorname*{argmin}_{x\in M} f(x) = \{-1\}.$$

Corollary 4.12. *Assume that* $S = \operatorname{conv} M$ *for some nonempty set* $M \subseteq \mathbb{R}^n$. *and let* $T : S \to \mathbb{R}^n$ *be an arbitrary function. Then the set of solutions*

$$\mathrm{sol(VI)} := \{x^0 \in S \mid \langle T(x^0), x - x^0\rangle \geq 0, \forall\, x \in S\}$$

to the variational inequality introduced in Example 4.4, admits the following representation

$$\mathrm{sol(VI)} = \{x^0 \in S \mid \langle T(x^0), x - x^0\rangle \geq 0, \forall\, x \in M\}.$$

Corollary 4.13. *If* $S \subseteq \mathbb{R}^n$ *is a nonempty convex compact set and function* $g(u, \cdot)$ *is quasiconvex on* S *for every* $u \in A$, *then*

$$\mathrm{eq}(g \mid A, S) = \{x^0 \in A \mid g(x^0, x) \leq 0,\ \forall\, x \in \mathrm{ext} S\}$$

i.e.,

$$\mathrm{eq}(g \mid A, S) = \mathrm{eq}(g \mid A, \mathrm{ext} S).$$

Proof. Follows by Theorem 4.7 and Minkowski's theorem (Theorem 2.1). □

Note that the conclusion of Corollary 4.13 still holds if S is a so-called "Minkowski set" (sets which were introduced in [3], i.e., closed, possibly unbounded sets which can be represented as the convex hull of their extreme points), yet the hypothesis of closeness is crucial as it shown in the next example.

Example 4.14. Let $n = 1, S =]-1, 1[$. Consider the function $f : S \to \mathbb{R}$ defined by $f(x) = -x^2$ and the bifunction $g : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$g(u, v) := f(u) - f(v),\ \forall\, (u, v) \in \mathbb{R}^2.$$

Clearly, $\forall\, u \in S$ the function $g(u, \cdot) : S \to \mathbb{R}$ is quasiconvex (even convex). It is easy to see that

$$\mathrm{eq}(g \mid S, S) = \operatorname*{argmin}_{x \in S} f(x) = \emptyset \neq \mathrm{eq}(g \mid S, \mathrm{ext} S) = \mathrm{eq}(g \mid S, \emptyset) = S.$$

Theorem 4.15. *Let* A *be a nonempty set, let* $S = \mathrm{cl}(\mathrm{conv}(M))$ *for some nonempty set* $M \subseteq \mathbb{R}^n$ *and let* $g : A \times S \to \mathbb{R}$ *be a bifunction. If function* $g(u, \cdot)$ *is quasiconvex and lower semicontinous on* S *for every* $u \in A$, *then*

$$\mathrm{eq}(g \mid A, S) = \{x^0 \in A \mid g(x^0, x) \leq 0,\ \forall\, x \in M\},$$

i.e.,

$$\mathrm{eq}(g \mid A, S) = \mathrm{eq}(g \mid A, M).$$

Proof. Let $x^0 \in A$ such that $g(x^0, x) \leq 0, \forall\, x \in M$. We prove that

$$g(x^0, y) \leq 0, \forall\, y \in S.$$

Let $y \in S$. By Theorem 4.7, it follows that

$$\begin{aligned} & \{x^0 \in A \mid g(x^0, x) \leq 0,\ \forall\, x \in M\} \\ = \ & \{x^0 \in A \mid g(x^0, x) \leq 0,\ \forall\, x \in \operatorname{conv} M\}. \end{aligned}$$

Hence

$$g(x^0, x) \leq 0,\ \forall\, x \in \operatorname{conv} M. \qquad (*)$$

Since $S = \mathrm{cl}(\mathrm{conv}\, M)$ and $y \in S$, it follows that there exists a sequence $(y^k)_{k\in\mathbb{N}}$ in $\mathrm{conv}\, M$ which converges to y.
According to $(*)$, we have $g(x^0, y^k) \leq 0, \forall\, k \in \mathbb{N}$, i.e.,

$$y^k \in L := \{z \in S \mid g(x^0, z) \leq 0\},\ \forall\, k \in \mathbb{N}.$$

On the other hand, the function $g(x^0, \cdot) : S \to \mathbb{R}$ is lower semicontinuous so, the level set L is closed with respect to the induced topology in S from $\mathbb{R}^n$ and, since S is closed, we deduce that L is a closed subset of $\mathbb{R}^n$, hence

$$y = \lim_{k\to\infty} y^k \in \mathrm{cl}\, L = L.$$

Thereby, $g(x^0, y) \leq 0$ and, since y was arbitrary chosen from S, we obtain that

$$\mathrm{eq}(g \mid A, S) \supseteq \{x^0 \in S \mid g(x^0, x) \leq 0,\ \forall\, x \in M\}.$$

The reverse inclusion is obvious. □

An immediate consequence of Theorem 4.15 and Straszewicz's theorem (Theorem 2.2) is the following corollary (Corollary 4.16), where we characterize solutions of an equilibrium problem by means of exposed points. Finally, another consequence of Theorem 4.15, by also using Remark 4.8 is given in Corollary 4.17.

Corollary 4.16. *Let A be a nonempty set, let $S \subseteq \mathbb{R}^n$ be a nonempty convex compact set and let $g : A \times S \to \mathbb{R}$ be a bifunction. If function $g(u, \cdot)$ is quasiconvex and lower semicontinous on S for every $u \in A$, then*

$$\mathrm{eq}(g \mid A, S) = \{x^0 \in A \mid g(x^0, x) \leq 0,\ \forall\, x \in \exp S\},$$

i.e.,

$$\mathrm{eq}(g \mid A, S) = \mathrm{eq}(g \mid A, \exp S).$$

Corollary 4.17. *Assume that $S = \mathrm{cl}(\mathrm{conv}\, M)$ for some nonempty set $M \subseteq \mathbb{R}^n$. If $f : S \to \mathbb{R}$ is a quasiconcave upper semicontinous function, then*

$$\operatorname*{argmin}_{x\in S} f(x) = \{x^0 \in S \mid f(x^0) \leq f(x), \forall\, x \in M\} \supseteq \operatorname*{argmin}_{x\in M} f(x).$$

Moreover, $\operatorname*{argmin}_{x\in S} f(x)$ is nonempty if and only if so is $\operatorname*{argmin}_{x\in M} f(x)$, hence

$$\min\, f(S) = \min\, f(M).$$

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