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Theorems regarding starlikeness and convexity

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Abstract. Giving the sharp version of an univalence condition means to give the final response to an open question. We prove in this paper the sharp version of a starlikeness condition. The basic tool in our study is the convolution theory. We present a short history of the problem before the proof of the main result.

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1. Introduction

The class $\mathcal{A}$ is the set of analytic functions f defined in the open unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ by the power series

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n.$$

The subset of $\mathcal{A}$ which contains univalent functions is denoted by S.
The class K consists of all functions $f \in S$ for which the domain $f(\mathbb{D})$ is convex in $\mathbb{C}$. We have

$$K = \{f \in \mathcal{A} : \mathrm{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > 0,\ z \in \mathbb{D}\}.$$

The class of starlike functions is defined as follows

$$S^* = \{f \in S : f(\mathbb{D})\text{is starlike with respect to the origin}\}.$$

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We have

$$S^* = \{f \in \mathcal{A} : \mathrm{Re}\left(\frac{zf'(z)}{f(z)}\right) > 0,\ z \in \mathbb{D}\}.$$

The class of strongly starlike functions of order α, $\alpha \in (0,1]$ is defined by

$$SS^*(\alpha) = \{f \in \mathcal{A} : \left|\arg\left(\frac{zf'(z)}{f(z)}\right)\right| < \frac{\alpha\pi}{2},\ z \in \mathbb{D}\}. \text{ We have } SS^*(\alpha) \subset S^*.$$

In [8] it is proved that

$$f \in \mathcal{A}, \quad \mathrm{Re}[f'(z) + zf''(z)] > 0,\ z \in \mathbb{D} \Rightarrow f \in S^* \tag{1.1}$$

and in [6] this result is improved as follows

$$f \in \mathcal{A}, \quad \mathrm{Re}[f'(z) + zf''(z)] > -\frac{\pi^2 - 6}{24 - \pi^2} \approx -0.27,\ z \in \mathbb{D} \Rightarrow f \in S^*.$$

These two results lead to a big number of papers regarding these questions and analogous ones. We present a few improvements in the followings.

In [6] the conjecture is stated that the biggest c for which

$$f \in \mathcal{A}, \quad \mathrm{Re}\left[f'(z) + zf''(z)\right] > -c, \forall z \in \mathbb{D} \Rightarrow f \in S^*$$

is $c = \frac{\ln 4 - 1}{2 - \ln 4} \approx 0.629$.
This conjecture was proved in [10].
Professor P. Mocanu improved the result (1.1) in an other direction. He proved in [4] the implication

$$f \in \mathcal{A}, \quad \mathrm{Re}\left[f'(z) + \frac{z}{2}f''(z)\right] > 0,\ z \in \mathbb{D} \Rightarrow f \in S^*. \tag{1.2}$$

In [3] Corollary 5.5j.1 it is given the following improvement of (1.2).

Theorem 1.1. *If $f \in \mathcal{A}$ and $-1 < \gamma < \gamma_0 = 1.869\ldots$, then*

$$\mathrm{Re}\left[f'(z) + \frac{1}{1+\gamma}zf''(z)\right] > 0,\ z \in \mathbb{D} \Rightarrow f \in S^*, \tag{1.3}$$

($\frac{1}{1+\gamma_0} \approx 0.348$).

Two improvements of (1.2) and (1.3) are given in [9] and [12].

Theorem 1.2. [9] *The biggest value of c for which the implication holds*

$$f \in \mathcal{A}, \quad \mathrm{Re}\left[f'(z) + \frac{z}{2}f''(z)\right] > -c,\ z \in \mathbb{D} \Rightarrow f \in S^*, \tag{1.4}$$

it is $c = \frac{3 - \ln 16}{\ln 16 - 2} \approx 0.294$.

Theorem 1.3. [12] *The implication holds*

$$f \in \mathcal{A}, \quad \mathrm{Re}\left[f'(z) + \frac{z}{7}f''(z)\right] > 0,\ z \in \mathbb{D} \Rightarrow f \in S^*. \tag{1.5}$$

In [12] it is also proved that if λ_0 is the smallest positive value for which the implication holds

$$f \in \mathcal{A}, \quad \Big[f'(z) + \lambda_0 z f''(z)\Big] > 0,\ z \in \mathbb{D} \Rightarrow f \in S^*,$$

then we have $\lambda_0 \in (\frac{1}{8}, \frac{1}{7})$.

The following improvements of (1.1) are proved in [5].

Theorem 1.4. *If $f \in \mathcal{A}$ then the implications hold:*

$$\mathrm{Re}\Big[f'(z) + zf''(z)\Big] > 0,\ z \in \mathbb{D} \Rightarrow \Big|\arg\frac{zf'(z)}{f(z)}\Big| < \frac{\pi}{3}; \tag{1.6}$$

$$\Big|\arg[f'(z) + zf''(z)]\Big| < \frac{2\pi}{3},\ z \in \mathbb{D} \Rightarrow \mathrm{Re}\,\frac{zf'(z)}{f(z)} > 0,\ z \in \mathbb{D}; \tag{1.7}$$

$$\mathrm{Re}\Big[f'(z) + \frac{z}{2}f''(z)\Big] > 0,\ z \in \mathbb{D} \Rightarrow \Big|\arg\frac{zf'(z)}{f(z)}\Big| < \frac{4\pi}{9},\ z \in \mathbb{D}; \tag{1.8}$$

$$\Big|\arg[f'(z) + zf''(z)]\Big| < \frac{5\pi}{9},\ z \in \mathbb{D} \Rightarrow \mathrm{Re}\,\frac{zf'(z)}{f(z)} > 0,\ z \in \mathbb{D}. \tag{1.9}$$

In [3] it is proved the following result. (Corollary5.2d.1)

Theorem 1.5. [3] *If $f \in \mathcal{A}$ and $\mathrm{Re}[zf''(z)] > -\frac{3}{7}$, $z \in \mathbb{D}$, then the function F defined by $F(z) = \frac{2}{z}\int_0^z f(t)dt$ belongs to the class K.*

The sharp version of this theorem is the next result and it is given in [11].

Theorem 1.6. [11] *If $f \in \mathcal{A}$ and*

$$\mathrm{Re}[zf''(z)] > -1,\ z \in \mathbb{D},$$

then the function F defined by $F(z) = \frac{2}{z}\int_0^z f(t)dt$ belongs to the class S^*. If $\mathrm{Re}[zf''(z)] > -c$, $z \in \mathbb{D}$ and $c > 1$, then there are functions $f \in \mathcal{A}$ which verify this inequality and F does not belong to S.

This theorem can be given in the following equivalent form.

Theorem 1.7. *If $F \in \mathcal{A}$ is a function with the property*

$$\mathrm{Re}\Big(zF''(z) + \frac{z^2}{3}F'''(z)\Big) > -\frac{2}{3},\ z \in \mathbb{D}, \tag{1.10}$$

then F belongs to the class K.

2. Preliminaries

Our main result it is analogous to Theorem 1.7 and Theorem 1.6. In order to prove it, we need the following lemmas.

Lemma 2.1. [7] *If $f \in \mathcal{A}$, then f is starlike if and only if*

$$\frac{f(z)}{z} * \frac{h_T(z)}{z} \neq 0 \text{ for every } z \in \mathbb{D} \text{ and } T \in \mathbb{R},$$

where

$$h_T(z) = \frac{iT\frac{z}{1-z} + \frac{z}{(1-z)^2}}{1+iT} = 1 + \sum_{n=1}^{\infty} \frac{n+1+iT}{1+iT} z^n, \; T \in \mathbb{R}, z \in \mathbb{D}.$$

Lemma 2.2. [7] *Let $\mathcal{H}(\mathbb{D})$ be the set of analytic functions in $\mathbb{D}$. The class of functions with positive real part is denoted by $\mathcal{P}$ and it is defined by the equality*

$$\mathcal{P} = \{f \in \mathcal{H}(\mathbb{D}) : f(0) = 1, \; \mathrm{Re}\, f(z) > 0, \; z \in \mathbb{D}\}.$$

Provided that $g(0) = 1$ we have $f(z) * g(z) \neq 0$, $(\forall) z \in \mathbb{D}$, $(\forall)$ $f \in \mathcal{P}$ if

$$\mathrm{Re}\, g(z) > \frac{1}{2}, \; z \in \mathbb{D}.$$

3. Main result

Theorem 3.1. *Let $f \in \mathcal{A}$. If*

$$\mathrm{Re}[zf''(z) + \frac{z^2}{2} f'''(z)] > -c, \; z \in \mathbb{D}, \tag{3.1}$$

where $c = \frac{1}{4(2\ln 2 - 1)}$, then we have $f \in S^*$. The result is sharp.

Proof. The condition (3.1) is equivalent to

$$\frac{c + zf''(z) + \frac{z^2}{2} f'''(z)}{c} \in \mathcal{P},$$

and the Herglotz representation formula gives

$$\frac{c + zf''(z) + \frac{z^2}{2} f'''(z)}{c} = \int_0^{2\pi} \frac{1 + e^{-it}z}{1 - e^{-it}z} d\mu(t), z \in \mathbb{D}. \tag{3.2}$$

If

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

then (3.2) leads to

$$1 + \frac{1}{c} \sum_{n=2}^{\infty} a_n \frac{n^2(n-1)}{2} z^{n-1} = 1 + 2 \sum_{n=1}^{\infty} z^n \int_0^{2\pi} e^{-int} d\mu(t), z \in \mathbb{D}.$$

Thus we get

$$a_n = \frac{4c}{n^2(n-1)} \int_0^{2\pi} e^{-i(n-1)t} d\mu(t), \; n \in N, \; n \geq 2$$

and

$$f(z) = z + 4c \sum_{n=2}^{\infty} \frac{z^n}{n^2(n-1)} \int_0^{2\pi} e^{-i(n-1)t} d\mu(t). \tag{3.3}$$

According to Lemma 2.1 the condition of starlikeness can be rewritten as follows

$$\frac{f(z)}{z} * \frac{h_T(z)}{z} = \left(1 + 4c \sum_{n=1}^{\infty} \frac{z^n}{n(n+1)^2} \int_0^{2\pi} e^{-int} d\mu(t)\right) * \left(1 + \sum_{n=1}^{\infty} \frac{n+1+iT}{1+iT} z^n\right)$$

$$= \left(1 + 2\sum_{n=1}^{\infty} z^n \int_0^{2\pi} e^{-int} d\mu(t)\right) * \left(1 + 2c\sum_{n=1}^{\infty} \frac{n+1+iT}{(1+iT)(n+1)^2 n} z^n\right) \neq 0. \tag{3.4}$$

Lemma 2.2 implies that the condition of starlikeness (3.4) holds if

$$\operatorname{Re}\left(1 + 2c\sum_{n=1}^{\infty} \frac{n+1+iT}{(1+iT)(n+1)^2 n} z^n\right) > \frac{1}{2}, \quad (\forall)\ z \in \mathbb{D}, T \in \mathbb{R} \tag{3.5}$$

or equivalently

$$\operatorname{Re}\left(\frac{1}{4c} + \sum_{n=1}^{\infty} \frac{n(1-iT)+1+T^2}{(1+T^2)(n+1)^2 n} e^{in\theta}\right) \geq 0, \qquad (\forall)\ z \in \mathbb{D}, T \in \mathbb{R}. \tag{3.6}$$

Our aim is to prove the equality:

$$\min_{\substack{\theta \in [0,2\pi] \\ T \in \mathbb{R}}} M(\theta, T) = \frac{1}{4c} - (2\ln 2 - 1), \tag{3.7}$$

where

$$M(\theta, T) = \operatorname{Re}\left(\frac{1}{4c} + \sum_{n=1}^{\infty} \frac{n(1-iT)+1+T^2}{(1+T^2)(n+1)^2 n} e^{in\theta}\right)$$

$$= \frac{1}{4c} + \frac{1}{1+T^2} \operatorname{Re}\Big[\sum_{n=1}^{\infty} \frac{e^{in\theta}}{n(n+1)} - iT\sum_{n=1}^{\infty} \frac{e^{in\theta}}{(n+1)^2} + T^2 \sum_{n=1}^{\infty} \frac{e^{in\theta}}{n(n+1)^2}\Big].$$

We use the integral representations

$$\sum_{n=1}^{\infty} \frac{e^{in\theta}}{(n+1)^2} = \int_0^1\int_0^1 tu \frac{e^{i\theta} - tu}{1 + t^2u^2 - 2tu\cos\theta} dtdu, \theta \in (0, 2\pi)$$

$$\sum_{n=1}^{\infty} \frac{e^{in\theta}}{n(n+1)} = \int_0^1\int_0^1 u \frac{e^{i\theta} - tu}{1 + t^2u^2 - 2tu\cos\theta} dtdu, \ \ \theta \in (0, 2\pi)$$

and we obtain

$$\begin{aligned} M(\theta, T) \quad = \quad & \frac{1}{4c} + \frac{1}{1+T^2}\bigg(\int_0^1\int_0^1 u \frac{\cos\theta - tu}{1 + t^2u^2 - 2tu\cos\theta} dtdu + \\ & + T\int_0^1\int_0^1 \frac{tu\sin\theta}{1 + t^2u^2 - 2tu\cos\theta} dtdu + \\ & + T^2 \int_0^1\int_0^1 u(1-t)\frac{\cos\theta - tu}{1 + t^2u^2 - 2tu\cos\theta} dtdu\bigg). \end{aligned}$$

$M(\theta,T)$ can be written in the following equivalent form

$$\begin{aligned}
M(\theta,T) &= \frac{1}{4c}-\frac{1}{1+T^2}\int_0^1\!\!\int_0^1\frac{u}{1+tu}dtdu-\frac{T^2}{1+T^2}\int_0^1\!\!\int_0^1\frac{u(1-t)}{1+tu}dtdu\\
&+\frac{1}{1+T^2}\Big[\int_0^1\!\!\int_0^1 u\frac{\cos\theta-tu}{1+t^2u^2-2tu\cos\theta}dtdu+\int_0^1\!\!\int_0^1\frac{u}{1+tu}dtdu\\
&+T\int_0^1\!\!\int_0^1\frac{tu\sin\theta}{1+t^2u^2-2tu\cos\theta}dtdu\\
&+T^2\Big(\int_0^1\!\!\int_0^1 u(1-t)\frac{\cos\theta-tu}{1+t^2u^2-2tu\cos\theta}dtdu+\int_0^1\!\!\int_0^1\frac{u(1-t)}{1+tu}dtdu\Big)\Big].
\end{aligned}$$

$$\begin{aligned}
M(\theta,T) &= \frac{1}{4c}-\int_0^1\!\!\int_0^1\frac{u}{1+tu}dtdu+\frac{T^2}{1+T^2}\int_0^1\!\!\int_0^1\frac{tu}{1+tu}dtdu\\
&+\frac{1}{1+T^2}\Big((1+\cos\theta)\int_0^1\!\!\int_0^1\frac{u(1-tu)}{(1+tu)(1+t^2u^2-2tu\cos\theta)}dt\\
&+T\sin\theta\int_0^1\!\!\int_0^1\frac{tu}{1+t^2u^2-2tu\cos\theta}dtdu\\
&+T^2(1+\cos\theta)\int_0^1\!\!\int_0^1\frac{u(1-t)(1-tu)}{(1+tu)(1+t^2u^2-2tu\cos\theta)}dtdu\Big).
\end{aligned}$$

$$\begin{aligned}
M(\theta,T) &= \frac{1}{4c}-\int_0^1\!\!\int_0^1\frac{u}{1+tu}dtdu\\
&+\frac{T^2}{1+T^2}(1+\cos\theta)\int_0^1\!\!\int_0^1\frac{u(1-t)(1-tu)}{(1+tu)(1+t^2u^2-2tu\cos\theta)}dtdu\\
&+\frac{1}{1+T^2}\Big((1+\cos\theta)\int_0^1\!\!\int_0^1\frac{u(1-tu)}{(1+tu)(1+t^2u^2-2tu\cos\theta)}dtdu\\
&+T\sin\theta\int_0^1\!\!\int_0^1\frac{tu}{1+t^2U^2-2tu\cos\theta}dtdu+T^2\int_0^1\!\!\int_0^1\frac{tu}{1+tu}dtdu\Big).
\end{aligned}$$

A simple calculation leads to

$$M(\theta,T)=\frac{1}{4c}-\int_0^1\!\!\int_0^1\frac{u}{1+tu}dtdu+L_1(\theta,T)+\frac{1}{1+T^2}L_2(\theta,T). \tag{3.8}$$

where

$$L_1(\theta,T)=\frac{T^2}{1+T^2}(1+\cos\theta)\int_0^1\!\!\int_0^1\frac{u(1-t)(1-tu)}{(1+tu)(1+t^2u^2-2tu\cos\theta)}dtdu$$

and

$$\begin{aligned}
L_2(\theta,T) &= (1+\cos\theta)\int_0^1\!\!\int_0^1\frac{u(1-tu)}{(1+tu)(1+t^2u^2-2tu\cos\theta)}dtdu+\\
&+T\sin\theta\int_0^1\!\!\int_0^1\frac{tu}{1+t^2u^2-2tu\cos\theta}dtdu+T^2\int_0^1\!\!\int_0^1\frac{tu}{1+tu}dtdu.
\end{aligned}$$

It is easily seen that $L_1(\theta, T) \geq 0,\ (\forall)\ \theta \in (0, 2\pi)$ and $T \in \mathbb{R}$. We will prove that $L_2(\theta, T) \geq 0,\ \ (\forall)\ \ \theta \in (0, 2\pi)$ and $T \in \mathbb{R}$.
$L_2(\theta, T)$ is a polynomial of degree two with respect to the variable T.
We have

$$\begin{aligned}\Delta_2(\theta) &= \sin^2\theta \left(\int_0^1\!\!\int_0^1 \frac{tu}{1+t^2u^2-2tu\cos\theta} dtdu \right)^2 \\ &\quad - 4(1+\cos\theta) \int_0^1\!\!\int_0^1 \frac{u(1-tu)}{(1+tu)(1+t^2u^2-2tu\cos\theta)} dtdu \int_0^1\!\!\int_0^1 \frac{tu}{1+tu} dtdu.\end{aligned}$$

We calculate the integrals and we get

$$\int_0^1\!\!\int_0^1 \frac{tu}{1+tu} dtdu = 1 - \frac{\pi^2}{12} > \frac{1-\log 2}{2} = \frac{1}{2} \int_0^1\!\!\int_0^1 \frac{t^2u}{1-t^2u^2} dtdu.$$

This inequality implies

$$\begin{aligned}\Delta_2(\theta) &\leq \sin^2\theta \left(\int_0^1 \int_0^1 \frac{tu}{1+t^2u^2-2tu\cos\theta} dtdu \right)^2 \\ &\quad - 4(1+\cos\theta) \int_0^1 \int_0^1 \frac{u(1-tu)}{(1+tu)(1+t^2u^2-2tu\cos\theta)} dtdu \int_0^1 \int_0^1 \frac{t^2u}{1-t^2u^2} dtdu \\ &= 4L_3(\theta)\cos^2\frac{\theta}{2}.\end{aligned}$$

We have

$$\begin{aligned}L_3(\theta) &= \left(\int_0^1\!\!\int_0^1 \frac{tu\sin\frac{\theta}{2}}{1+t^2u^2-2tu\cos\theta} dtdu \right)^2 \\ &\quad - 2\int_0^1\!\!\int_0^1 \frac{u(1-tu)}{(1+tu)(1+t^2u^2-2tu\cos\theta)} dtdu \int_0^1\!\!\int_0^1 \frac{t^2u}{1-t^2u^2} dtdu. \qquad (3.9)\end{aligned}$$

The inequality Cauchy - Schwarz implies

$$\begin{aligned}&\int_0^1\!\!\int_0^1 \frac{u(1-tu)}{(1+tu)(1+t^2u^2-2tu\cos\theta)} dtdu \int_0^1\!\!\int_0^1 \frac{t^2u}{1-t^2u^2} dtdu \\ &\geq \left(\int_0^1\!\!\int_0^1 \frac{tu}{(1+tu)\sqrt{1+t^2u^2-2tu\cos\theta}} dtdu \right)^2. \qquad (3.10)\end{aligned}$$

Thus in order to prove $\Delta_2(\theta) \leq 0,\ \theta \in (0, 2\pi)$ it is enough to show that

$$\int_0^1\!\!\int_0^1 \frac{tu\sin\frac{\theta}{2}}{1+t^2u^2-2tu\cos\theta} dtdu \leq \int_0^1\!\!\int_0^1 \frac{tu}{(1+tu)\sqrt{1+t^2u^2-2tu\cos\theta}} dtdu. \quad (3.11)$$

The inequality (3.11) holds if we show that

$$\frac{tu}{(1+tu)\sqrt{1+t^2u^2-2tu\cos\theta}} \geq \frac{tu\sin\frac{\theta}{2}}{1+t^2u^2-2tu\cos\theta} \qquad (3.12)$$

in case of $t, u \in [0, 1]$ and $\theta \in (0, 2\pi)$. A short calculation shows that the inequality (3.12) is equivalent to

$$(1+\cos\theta)(1-t)^2 \geq 0, t, u \in [0, 1], \theta \in (0, 2\pi)$$

and we have proved $L_3(\theta,T)\le 0$ for $\theta\in(0,2\pi)$, $T\in\mathbb{R}$ and this is equivalent to $\Delta_2(\theta)\le 0,\ \theta\in(0,2\pi)$.
Thus the inequalities hold $L_1(\theta,T)\ge 0$ $L_2(\theta,T)\ge 0$ for $\theta\in(0,2\pi)$, $T\in\mathbb{R}$ and $L_1(\pi,0)=L_2(\pi,0)=0$. This implies that

$$\inf_{\substack{\theta\in(0,2\pi)\\ T\in\mathbb{R}}} M(\theta,T)=\frac{1}{4c}-\int_0^1\int_0^1\frac{u}{1+tu}dtdu=\frac{1}{4c}-(2\ln 2-1). \tag{3.13}$$

Finally if $\frac{1}{4c}-(2\ln 2-1)=0\Leftrightarrow c=\frac{1}{4(2\ln 2-1)}$, then

$$M(\theta,T)\ge 0 \text{ for every } \theta\in(0,2\pi),\ T\in\mathbb{R}$$

and this implies the starlikeness of the function f. □

The integral version of the proved theorem is given in the next corollary.

Corollary 3.2. *If $f\in\mathcal{A}$ and $\mathrm{Re}(zf''(z))>-c, z\in\mathbb{D}$, where $c=\frac{1}{2(2\ln 2-1)}$, then the image function F defined by the Alexander operator*

$$F(z)=A(f)(z)=\int_0^z\frac{f(t)}{t}dt,\ z\in\mathbb{D}$$

belongs to the class S^*.

Proof. Differentiating the equality $F(z)=\int_0^z\frac{f(t)}{t}dt$ three times we get

$$zF''(z)+\frac{z^2}{2}F'''(z)=\frac{1}{2}zf''(z), z\in\mathbb{D}. \tag{3.14}$$

The conditions of the corollary and the equality (3.14) imply

$$zF''(z)+\frac{z^2}{2}F'''(z)>\frac{-1}{4(2\ln 2-1)}, z\in\mathbb{D}. \tag{3.15}$$

The inequality (3.15) and Theorem 3.1 show that $F\in S^*$. □

The next theorem shows that $c=\frac{1}{4(2\ln 2-1)}$ can not be replaced by a bigger number in case of Theorem 3.1.

Theorem 3.3. *If $c_1>\frac{1}{4(2\ln 2-1)}$, then there are functions $f\in\mathcal{A}$ which verify the condition*

$$zf''(z)+\frac{z^2}{2}f'''(z)>-c_1,\ z\in\mathbb{D} \tag{3.16}$$

and are not univalent.

Proof. The condition (3.16) implies

$$f(z)=z+4c_1\sum_{n=2}^{\infty}\frac{z^n}{n^2(n-1)}d\mu(t),\ z\in\mathbb{D}$$

and it follows that

$$f'(z)=1+4c_1\sum_{n=2}^{\infty}\frac{z^n}{n(n-1)}d\mu(t),\ z\in\mathbb{D}.$$

The condition $f'(z) \neq 0,\ z \in \mathbb{D}$ is a necessary condition of the univalence. On the other hand we have

$$f'(z) = 1 + 4c_1 \sum_{n=1}^{\infty} \frac{z^n}{n(n-1)} \int_0^{2\pi} e^{-int} d\mu(t) =$$

$$\left(1 + 2\sum_{n=1}^{\infty} z^n \int_0^{2\pi} e^{-int} d\mu(t)\right) * \left(1 + 2c_1 \sum_{n=1}^{\infty} \frac{z^n}{n(n+1)}\right).$$

Lemma 2.2 implies that $f'(z) \neq 0, z \in \mathbb{D}$ if and only if

$$\operatorname{Re}\left[1 + 2c_1 \sum_{n=1}^{\infty} \frac{z^n}{n(n+1)}\right] > \frac{1}{2}, z \in \mathbb{D},$$

and this is equivalent to

$$\operatorname{Re}\left[1 + 4c_1 \sum_{n=1}^{\infty} \frac{z^n}{n(n+1)}\right] > 0, z \in \mathbb{D}.$$

In particular from radial continuity we get

$$1 + 4c_1 \sum_{n=1}^{\infty} \frac{(-1)^n}{n(n+1)} \geq 0,$$

or equivalently

$$\frac{1}{4(2\ln 2 - 1)} = \frac{1}{4\left(\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n(n+1)}\right)} \geq c_1$$

and this contradicts the condition of the theorem $c_1 > \frac{1}{4(2\ln 2 - 1)}$. This contradiction shows that there are points $z^* \in \mathbb{D}$ such that $f'(z^*) = 0$ and consequently the function f is not univalent. $\square$

4. Concluding remarks

The method of convolution leads to sharp results in case of linear starlikeness and convexity conditions. In the presented theorems all the sharp ones are proved by convolution. Theorem 1.4 is an example, which was proved by differential subordination and we think that this method it is much useful in this case. It would be interesting to give the sharp version of an implication from Theorem 1.4.

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