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Multiple solution for a fourth-order nonlinear eigenvalue problem with singular and sublinear potential

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Dedicated to the memory of Professor Csaba Varga

Abstract. Let (M, g) be a Cartan-Hadamard manifold. For certain positive numbers μ and λ, we establish the multiplicity of solutions to the problem

$$\Delta_g^2 u - \Delta_g u + u = \mu \frac{u}{d_g(x_0, x)^4} + \lambda \alpha(x) f(u), \quad \text{in } M,$$

where $x_0 \in M$, while $f : \mathbb{R} \to \mathbb{R}$ is continuous function, superlinear at zero and sublinear at infinity.

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1. Introduction

The biharmonic non-linear Schrödinger equation

$$i\partial_t \psi + a\Delta^2 \psi + b\Delta\psi + c|\psi|^{2w}\psi = 0 \quad \text{in } \mathbb{R} \times \mathbb{R}^d,$$

where $a, w > 0$ and $b, c \in \mathbb{R}$, $c \neq 0$ has been introduced by Karpman and Shagalov [13]. The problem, because of its physical applications, has received much attention in recent years. After a Lyapunov-Schmidt type reduction, i.e., a separation of variables the previous problem reduces to a fourth-order elliptic equation. With the aid of variational methods, the existence and multiplicity of nontrivial solutions for such problems have been extensively studied in the literature over the last decades, see for instance [4, 5, 9, 16] and reference therein.

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Similarly, in recent years singular fourth order equations have been widely studied because of their wide application to physical models such as non-Newtonian fluids, see for instance [1, 3, 12, 11, 6, 17, 18].

Most of the aforementioned papers provide existence and multiplicity results by employing different techniques as variational methods, genus theory, the Nehari manifold etc.

As far as we know, no result is available in the literature concerning singular foruth order Schrödinger systems on non-compact Riemannian manifolds. Motivated by this fact, the purpose of the present paper is to provide multiplicity results in the case of the singular foruth order Schrödinger system in such a non-compact setting. Since this problem is very general, we shall restrict our study to Hadamard manifolds (simply connected, complete Riemannian manifolds with non-positive sectional curvature).

To be more precise, let (M, g) be a d-dimensional Hadamard manifold, with $d \geq 5$ and we shall consider the following problem

$$\begin{cases} \Delta_g^2 u - \Delta_g u + u = \mu \frac{u}{d_g(x_0,x)^4} + \lambda \alpha(x) f(u), & \text{in } M \\ u \in W_g^{2,2}(M) \end{cases} \tag{$\mathscr{P}_{\lambda,\mu}$}$$

where f is a given function, while λ and μ are positive constants, and $\alpha \in L^1(M) \cap L^\infty(M)$. On the nonlinearity $f : \mathbb{R} \to \mathbb{R}$ we assume that

(f_1) is superlinear at zero, i.e. $\lim\limits_{s\to 0} \dfrac{f(s)}{s} = 0$,

(f_2) is sublinear at infinity, i.e., $\lim\limits_{s\to\infty} \dfrac{f(s)}{s} = 0$,

(f_3) denoting by $F(s) = \displaystyle\int_0^s f(t)\,\mathrm{d}t$, finally we assume that $\sup\limits_{s\in\mathbb{R}} F(s) > 0$.

Our main result reads as follows:

Theorem 1.1. *Let (M, g) be a d-dimensional Hadamard manifold, with $d \geq 5$ and $f : \mathbb{R} \to \mathbb{R}$ be continuous function which satisfies (f_1), (f_2) and (f_3) and $\alpha \in L^1(M) \cap L^\infty(M)$ be a non-zero, non-negative function which depends on $d_g(x_0, \cdot)$ and satisfies $\sup\limits_{R>0} \operatorname*{essinf}\limits_{d_g(x_0,x)\leq R} \alpha(x) > 0$. Then for every $\mu \in \left[0, \frac{d^2(d-4)^2}{16}\right)$ there exist an open interval $I_\mu \subset (0, +\infty)$ and a real number $\sigma_\mu > 0$ such that for every $\lambda \in I_\mu$ the problem $(\mathscr{P}_{\lambda,\mu})$ has at least two distinct nontrivial weak solutions in $W_g^{2,2}(M)$ whose $W_g^{2,2}$-norms are less than σ_μ.*

The proof of Theorem 1.1 is based on a three critical point result of Bonanno [2] (which is actually a refinement of a general principle of Ricceri [20, 19]), combined with a compact embedding result(see Farkas, Kristály and Mester [8]) combined with variational arguments.

2. Preliminaries

Let (M, g) be a complete non-compact Riemannian manifold with $\dim M = d$. Let T_xM be the tangent space at $x \in M$, $TM = \bigcup_{x\in M} T_xM$ be the tangent bundle, and $d_g : M \times M \to [0, +\infty)$ be the distance function associated to the Riemannian metric g. Let $B_g(x, \rho) = \{y \in M : d_g(x, y) < \rho\}$ be the open metric ball with center x and radius $\rho > 0$; if dv_g is the canonical volume element on (M, g), the volume of a bounded open set $\Omega \subset M$ is $\mathrm{Vol}_g(\Omega) = \int_\Omega \mathrm{d}v_g = \mathcal{H}^d(\Omega)$. If $\mathrm{d}\sigma_g$ denotes the $(d-1)$-dimensional Riemannian measure induced on $\partial\Omega$ by g, then

$$\mathrm{Area}_g(\partial\Omega) = \int_{\partial\Omega} \mathrm{d}\sigma_g = \mathcal{H}^{d-1}(\partial\Omega)$$

stands for the area of $\partial\Omega$ with respect to the metric g. Hereafter, $\mathcal{H}^l$ denotes the l-dimensional Hausdorff measure.

Let $p > 1$. The norm of $L^p(M)$ is given by

$$\|u\|_p = \left(\int_M |u|^p \mathrm{d}v_g\right)^{1/p}.$$

Let $u : M \to \mathbb{R}$ be a function of class C^1. If (x^i) denotes the local coordinate system on a coordinate neighbourhood of $x \in M$, and the local components of the differential of u are denoted by $u_i = \frac{\partial u}{\partial x_i}$, then the local components of the gradient $\nabla_g u$ are $u^i = g^{ij}u_j$. Here, g^{ij} are the local components of $g^{-1} = (g_{ij})^{-1}$. In particular, for every $x_0 \in M$ one has the eikonal equation

$$|\nabla_g d_g(x_0, \cdot)| = 1 \text{ a.e. on } M. \tag{2.1}$$

When no confusion arises, if $X, Y \in T_xM$, we simply write $|X|$ and $\langle X, Y\rangle$ instead of the norm $|X|_x$ and inner product $g_x(X, Y) = \langle X, Y\rangle_x$, respectively.

The $L^p(M)$ norm of $\nabla_g u : M \to TM$ is given by

$$\|\nabla_g u\|_p = \left(\int_M |\nabla_g u|^p \mathrm{d}v_g\right)^{\frac{1}{p}}.$$

The space $W_g^{2,2}(M)$ is the completion of $C_0^\infty(M)$ with respect to the norm

$$\|u\|^2_{W_g^{1,2}(M)} = \|u\|_2^p + \|\nabla_g u\|_2^2 + \|\Delta_g u\|_2^2.$$

Let G be a compact connected subgroup of $\mathrm{Isom}_g(M)$, and let $\mathcal{O}_G^x = \{\xi x : \xi \in G\}$ be the orbit of the element $x \in M$. The action of G on $W_g^{2,2}(M)$ is defined by

$$(\xi u)(x) = u(\xi^{-1}x) \text{ for all } x \in M,\ \xi \in G,\ u \in W_g^{1,p}(M), \tag{2.2}$$

where $\xi^{-1} : M \to M$ is the inverse of the isometry ξ. We say that a continuous action of a group G on a complete Riemannian manifold M is *coercive* (see Tintarev [22, Definition 7.10.8] or Skrzypczak and Tintarev [21, Definition 1.2]) if for every $t > 0$, the set

$$\mathscr{O}_t = \{x \in M : \mathrm{diam}\mathcal{O}_G^x \le t\}$$

is bounded.

Let $m(y,\rho)$ be the maximal number of mutually disjoint geodesic balls with radius ρ on $\mathcal{O}_G^y$

$$m(y,\rho) = \sup\{n \in \mathbb{N} : \exists \xi_1, \dots, \xi_n \in G : B_g(\xi_i y, \rho) \cap B_g(\xi_j y, \rho) = \emptyset, \forall i \neq j\}$$

We also define

$$W_{g,G}^{2,2}(M) = \{u \in W_g^{2,2}(M) : \xi u = u \text{ for all } \xi \in G\}$$

be the subspace of G-invariant functions of $W_g^{2,2}(M)$.

Theorem 2.1 ([8], Theorem 1.1). *Let (M,g) be a d-dimensional Hadamard manifold, and let G be a compact connected subgroup of $\mathrm{Isom}_g(M)$ such that $\mathrm{Fix_M}(G) \neq \emptyset$. Then the following statements are equivalent:*

(i) *G is coercive;*
(ii) $\mathrm{Fix}_M(G)$ *is a singleton*;
(iii) *$m(y,\rho) \to \infty$ as $d_g(x_0,y) \to \infty$.*

Moreover, from any of the above statements it follows that the embedding $W_{g,G}^{2,2}(M) \subset W_{g,G}^{1,2}(M) \hookrightarrow L^q(M)$ is compact for every $2 \le q < 2^\# = \frac{2d}{d-4}$ if $1 < p < d$.

In order to prove Theorem 1.1, we recall an abstract tool, which is the following critical point result of Bonanno [2] (which is actually a refinement of a general principle of Ricceri [20, 19]):

Theorem 2.2 ([2], Theorem 2.1). *Let X be a separable and reflexive real Banach space, and let $\Phi, J : X \to \mathbb{R}$ be two continuously Gâteaux differentiable functionals, such that $\Phi(u) \ge 0$ for every $u \in X$. Assume that there exist $u_0, u_1 \in X$ and $\rho > 0$ such that*

(1) $\Phi(u_0) = J(u_0) = 0$,
(2) $\rho < \Phi(u_1)$,
(3) $\displaystyle\sup_{\Phi(u)<\rho} J(u) < \rho \frac{J(u_1)}{\Phi(u_1)}$.

Further, put

$$\overline{a} = \zeta\rho\left(\rho\frac{J(u_1)}{\Phi(u_1)} - \sup_{\Phi(u)<\rho} J(u)\right)^{-1}, \quad \text{where } \zeta > 1,$$

and assume that the functional $\Phi - \lambda J$ is sequentially weakly lower semicontinuous, satisfies the Palais-Smale condition and

(4) $\displaystyle\lim_{\|u\|\to\infty} (\Phi(u) - \lambda J(u)) = +\infty$, *for all $\lambda \in [0, \overline{a}]$.*

Then there exists an open interval $\Lambda \subset [0, \overline{a}]$ and a number $\mu > 0$ such that for each $\lambda \in \Lambda$, the equation $\Phi'(u) - \lambda J'(u) = 0$ admits at least three solutions in X having norm less than μ.

We conclude this section by stating the Rellich inequality: if (M,g) is a Hadamard manifold with $\dim M = d \ge 5$, then we have the following inequality (see for instance [15])

$$\int_M (\Delta_g u)^2 \, \mathrm{d}v_g \ge \frac{d^2(d-4)^2}{16} \int_M \frac{u^2}{d_g^4(x_0,x)} \, \mathrm{d}v_g, \ \forall u \in W_g^{2,2}(M). \tag{2.3}$$

where the constant is $\frac{d^2(d-4)^2}{16}$ sharp, but are never achieved

3. Proof of the main result

As in usual case we associate the energy functional with the problem $(\mathscr{P}_{\lambda,\mu})$, $E_{\lambda,\mu} : M \to \mathbb{R}$,

$$E_{\lambda,\mu}(u) = \int_M (\Delta_g u)^2 + |\nabla_g u|^2 + u^2 \, \mathrm{d}v_g \\ - \mu \int_M \frac{u^2}{d_g(x_0, x)^4} \, \mathrm{d}v_g - \lambda \int_M \alpha(x) F(u) \, \mathrm{d}v_g.$$

Based on the assumption of the continuous function f, a standard argument shows that $E_{\lambda,\mu} : W_g^{2,2}(M) \to \mathbb{R}$ is of class C^1 and its critical points are exactly the weak solutions of the studied problem. Therefore, it is enough to show the existence of multiple critical points of $E_{\lambda,\mu}$. For further use, let us denote by

$$\Phi_{\mu,0}(u) = \int_M (\Delta_g u)^2 + |\nabla_g u|^2 + u^2 \, \mathrm{d}v_g - \mu \int_M \frac{u^2}{d_g(x_0, x)^4} \, \mathrm{d}v_g$$

and

$$J_0(u) = \int_M \alpha(x) F(u) \, \mathrm{d}v_g.$$

Having in our mind the compactness result, see Theorem 2.1, we restrict the energy functional to the space $W_{g,G}^{2,2}(M)$. For simplicity, in the following we denote

$$\mathcal{E}_{\lambda,\mu} = E_{\lambda,\mu}|_{W_{g,G}^{2,2}(M)}, \ \Phi_\mu = \Phi_{\mu,0}|_{W_{g,G}^{2,2}(M)}, \ \text{ and } J = J_0|_{W_{g,G}^{2,2}(M)}.$$

Lemma 3.1. *Let G be a compact connected subgroup of* $\mathrm{Isom}_g(M)$ *with* $\mathrm{Fix}_M(G) = \{x_0\}$. *Then $E_{\lambda,\mu}$ is G-invariant.*

Proof of Lemma 3.1. Let $u \in W_g^{2,2}(M)$ and $\sigma \in G$ be arbitrarily fixed. Since $\sigma : M \to M$ is an isometry on M, by (2.2), for every $x \in M$ we have

$$\nabla_g(\sigma u)(x) = D\sigma_{\sigma^{-1}(x)} \nabla_g u(\sigma^{-1}(x)),$$

where $D\sigma_{\sigma^{-1}(x)} : T_{\sigma^{-1}(x)} M \to T_x M$ denotes the differential of σ at the point $\sigma^{-1}(x)$. Note that the (signed) Jacobian determinant of σ is 1 and $D\sigma_{\sigma^{-1}(x)}$ preserves inner products. Therefore, by using the latter facts, relation (2.2) and a change of variables $y = \sigma^{-1}(x)$, it turns out that

$$\begin{aligned} &\int_M \left(|\nabla_g(\sigma u)(x)|_x^2 + |(\sigma u)(x)|^2\right) \mathrm{d}v_g(x) \\ &= \int_M \left(|\nabla_g u(\sigma^{-1}(x))|_{\sigma^{-1}(x)}^2 + |u(\sigma^{-1}(x))|^2\right) \mathrm{d}v_g(x) \\ &= \int_M \left(|\nabla_g u(y)|_y^2 + |u(y)|^2\right) \mathrm{d}v_g(y), \end{aligned}$$

We claim that

$$\Delta_g \left((\sigma \circ u)(x)\right) = \Delta_g u(\sigma^{-1}(x)).$$

To prove this claim, we choose an arbitrary test function φ, then we consider the following integral

$$\begin{aligned}
&\int_M \Delta_g\left((\sigma\circ u)(x)\right)\varphi(\sigma^{-1}(x))\mathrm{d}v_g(x)\\
&= -\int_M \langle D\sigma_{\sigma^{-1}(x)}\nabla_g u(\sigma^{-1}(x)), D\sigma_{\sigma^{-1}(x)}\varphi(\sigma^{-1}(x))\rangle \mathrm{d}v_g(x)\\
&= -\int_M \langle \nabla_g u(\sigma^{-1}(x)), \varphi(\sigma^{-1}(x))\rangle \mathrm{d}v_g(x)\\
&= -\int_M \langle \nabla_g u(y), \varphi(y)\rangle \mathrm{d}v_g(y)\\
&= \int_M \Delta_g u(y)\varphi(y)\mathrm{d}v_g(y)\\
&= \int_M \Delta_g u(\sigma^{-1}(x))\varphi(\sigma^{-1}(x))\mathrm{d}v_g(x),
\end{aligned}$$

the arbitrariness of the function φ proves the claim. Finally, since $\sigma \in G$ and $\alpha \in L^1(M)\cap L^\infty(M)$ is a non-zero, non-negative function which depends on $d_g(x_0,\cdot)$ and $\mathrm{Fix}_M(G)=\{x_0\}$, it turns out that for every $u \in W^{2,2}_{g,G}(M)$, we have $J_0(\sigma u) = J_0(u)$, which concludes the proof. □

The principle of symmetric criticality of Palais (see Kristály, Rădulescu and Varga [14, Theorem 1.50]) and the previous Lemma imply that the critical points of $\mathcal{E}_{\lambda,\mu} = E_{\lambda,\mu}|_{W^{2,2}_{g,G}(M)}$ are also critical points of the original functional $E_{\lambda,\mu}$. Therefore, it is enough to find critical points of $\mathcal{E}_{\lambda,\mu}$.

Lemma 3.2. *For every $\mu \in \left[0, \frac{d^2(d-4)^2}{16}\right)$ and $\lambda \in \mathbb{R}_+$, the functional $\mathcal{E}_{\lambda,\mu}$ is sequentially weakly lower semicontinuous on $W^{2,2}_{g,G}(M)$.*

Proof. First we prove that the functional Φ_μ is sequentially weakly lower semicontinuous on $W^{2,2}_g(M)$. To this end, we consider $u, v \in W^{2,2}_g(M)$ and $t \in [0,1]$, and thus

$$\begin{aligned}
\Phi_\mu(tu+(1-t)v) &= \int_M \left(\Delta_g(tu+(1-t)v)\right)^2 \mathrm{d}v_g + \int_M |\nabla_g(tu+(1-t)v)|^2 \mathrm{d}v_g\\
&\quad + \int_M (tu+(1-t)v)^2 \mathrm{d}v_g - \mu\int_M \frac{(tu+(1-t)v)^2}{d_g^4(x_0,x)}\mathrm{d}v_g\\
&\le \int_M \left(\Delta_g(tu+(1-t)v)\right)^2 \mathrm{d}v_g + \int_M t|\nabla_g u|^2 + (1-t)|\nabla_g v|^2 \mathrm{d}v_g\\
&\quad + \int_M tu^2 + (1-t)v^2 \mathrm{d}v_g - \mu\int_M \frac{(tu+(1-t)v)^2}{d_g^4(x_0,x)}\mathrm{d}v_g.
\end{aligned}$$

Now, using the following identity

$$(ta+(1-t)b)^2 = ta^2 + (1-t)b^2 - t(1-t)(a-b)^2,$$

we get that

$$\Phi_\mu(tu+(1-t)v) \le t\Phi_\mu(u) + (1-t)\Phi_\mu(v) - t(1-t)\left(\int_M (\Delta_g(u-v))^2 \,\mathrm{d}v_g - \mu\int_M \frac{(u-v)^2}{d_g^4(x_0,x)}\,\mathrm{d}v_g\right).$$

Using the Rellich inequality (2.3) (see also Kristály and Repovs [15]), one has that

$$\int_M (\Delta_g(u-v))^2 \,\mathrm{d}v_g - \mu\int_M \frac{(u-v)^2}{d_g^4(x_0,x)}\,\mathrm{d}v_g \ge 0,$$

for every $u, v \in W_g^{2,2}(M)$, thus

$$\Phi_\mu(tu+(1-t)v) \le t\Phi_\mu(u) + (1-t)\Phi_\mu(v).$$

Thus Φ_μ is positive and convex, therefore is sequentially weakly lower semicontinous.

It remains to prove that J is sequentially weakly continuous. To this end, consider a sequence $\{u_k\}_k$ in $W_{g,G}^{2,2}(M)$ which converges weakly to $u \in W_{g,G}^{2,2}(M)$, and suppose that

$$J(u_k) \not\to J(u_k) \quad \text{as } k \to \infty.$$

Thus, there exist $\varepsilon > 0$ and a subsequence of $\{u_n\}_n$, denoted again by $\{u_n\}_n$, such that $u_n \to u$ in $L^\infty(M)$ and

$$0 < \varepsilon \le |J(u_k) - J(u)|, \quad \text{for every } k \in \mathbb{N}.$$

Thus, by the mean value theorem, there exists $\theta_k \in (0,1)$ such that

$$\begin{aligned} 0 < \varepsilon &\le |\langle J'(u+\theta_k(u_k-u)), u_k - u\rangle| \\ &\le \int_M \alpha(x)|f(u+\theta_k(u_k-u))| \cdot |u_k - u| \,\mathrm{d}v_g. \end{aligned}$$

Using the assumptions (f_1), (f_2) and the Hölder inequality the last term tends to 0, which provides a contradiction. ☐

Lemma 3.3. *For every $\mu \in \left[0, \frac{d^2(d-4)^2}{16}\right)$ and $\lambda \in \mathbb{R}_+$, the functional $\mathcal{E}_{\lambda,\mu}$ is coercive and satisfies the Palais-Smale condition.*

Proof. First we prove that the functional $\mathcal{E}_{\lambda,\mu}$ is coercive. Let us fix $\mu \in \left[0, \frac{n^2(n-4)^2}{16}\right)$ and $\lambda \in \mathbb{R}_+$. We denote $\overline{\mu} = \frac{n^2(n-4)^2}{16}$. By the (f_1) and (f_2) for every $\varepsilon > 0$, there exists $\delta_\varepsilon \in (0,1)$ such that

$$|f(s)| \le \varepsilon|s| \ \text{ for all} |s| \le \delta_\varepsilon \ \text{ and } |s| \ge \delta_\varepsilon^{-1}.$$

Since $f \in C(\mathbb{R},\mathbb{R})$, there also exists a number $M_\varepsilon > 0$ such that

$$\frac{|f(s)|}{|s|^q} \le M_\varepsilon \ \text{ for all } |s| \in [\delta_\varepsilon, \delta_\varepsilon^{-1}],$$

where $q \in (0,1)$. Therefore

$$|f(s)| \le \varepsilon|s| + M_\varepsilon|s|^q, \quad \text{for all } s \in \mathbb{R}. \tag{3.1}$$

Thus, for every $u \in W^{2,2}_{g,G}(M)$ we have

$$\begin{aligned}\mathcal{E}_{\lambda,\mu} &\geq \frac{1}{2}\left(1-\frac{\mu}{\overline{\mu}}\right)\|u\|^2 - \lambda\int_M \alpha(x)|F(u)|\,\mathrm{d}v_g \\ &\geq \frac{1}{2}\left(1-\frac{\mu}{\overline{\mu}}\right)\|u\|^2 - \frac{1}{2}\lambda\|\alpha\|_\infty\varepsilon\|u\|^2 - \frac{\lambda M_\varepsilon C}{q+1}\|u\|^{q+1}.\end{aligned}$$

If $\|u\| \to \infty$ we conclude that $\mathcal{E}_{\lambda,\mu}(u) \to \infty$ as well, i.e. $\mathcal{E}_{\lambda,\mu}$ is coercive. Now, let $\{u_k\}_k$ be a sequence in $W^{2,2}_{g,G}(M)$ such that $\{\mathcal{E}_{\lambda,\mu}(u_k)\}_k$ is bounded and $\|\mathcal{E}'_{\lambda,\mu}(u_k)\|_* \to 0$. Since $\mathcal{E}_{\lambda,\mu}$ is coercive, the sequence $\{u_k\}_k$ is bounded in $W^{2,2}_{g,G}(M)$. Therefore, up to a subsequence, $u_k \rightharpoonup u$ weakly in $W^{2,2}_{g,G}(M)$ for some $u \in W^{2,2}_{g,G}(M)$.
Hence, due to Theorem Theorem 2.1, it follows that $u_k \to u$ strongly in $L^p(M)$.
In particular, we have that

$$\mathcal{E}'_{\lambda,\mu}(u)(u-u_k) \to 0 \quad \text{and} \quad \mathcal{E}'_{\lambda,\mu}(u_k)(u-u_k) \to 0 \quad \text{as} \quad k \to \infty. \tag{3.2}$$

On the one hand, it is easy to verify that

$$\begin{aligned}\left(1-\frac{\mu}{\overline{\mu}}\right)\|u_k-u\|^2 \leq & \|u_k-u\|^2 - \mu\int_M \frac{(u_k-u)^2}{d_g^4(x_0,x)}\,\mathrm{d}v_g \\ =&\mathcal{E}'_{\lambda,\mu}(u)(u-u_k) + \mathcal{E}'_{\lambda,\mu}(u_k)(u-u_k) \\ &+ \lambda\int_M \alpha(x)[f(u_k)-f(u)](u_k(x)-u(x))\,\mathrm{d}v_g.\end{aligned}$$

On the other hand, by means of (f_1) and (f_2) one has that

$$\int_M \alpha(x)[f(u_k)-f(u)](u_k(x)-u(x))\,\mathrm{d}v_g \to 0,$$

as $k \to \infty$. Thus we proved that $\|u_k - u\| \to 0$, which proves the claim. □

Lemma 3.4. *For every* $\mu \in \left[0, \frac{d^2(d-4)^2}{16}\right)$

$$\lim_{\rho\to 0^+} \frac{\sup\{J(u):\ \Phi_\mu(u) < \rho\}}{\rho} = 0.$$

Proof. Fix $\mu \in [0,\overline{\mu})$. Using again (f_1), for every $\varepsilon > 0$ there exists $\delta > 0$

$$|f(s)| < \frac{\varepsilon}{4}\left(1-\frac{\mu}{\overline{\mu}}\right)\|\alpha\|_\infty^{-1}\kappa_2^{-2}|s| \text{ for all } |s| < \delta.$$

For fixed $p > 2$, one has the following inequality

$$|F(s)| \leq \frac{\varepsilon}{4}\left(1-\frac{\mu}{\overline{\mu}}\right)\|\alpha\|_\infty^{-1}\kappa_2^{-2}|s| + c(\varepsilon)|s|^p \ \text{ for all } s \in \mathbb{R}.$$

For $\rho > 0$ define the sets

$$S^1_\rho = \{u : \Phi_\mu(u) < \rho\}; \quad S^2_\rho = \{u : (1-\mu/\overline{\mu})\|u\| < 2\rho\}.$$

Using the Rellich inequality, we have that $S^1_\rho \subseteq S^2_\rho$. Moreover, for every $u \in S^2_\rho$ we have that

$$J(u) = \int_M \alpha(x)F(u)\,\mathrm{d}v_g \leq \frac{\varepsilon}{2}\rho + c\rho^{\frac{p}{2}}.$$

Thus there exists $\rho(\varepsilon) > 0$ such that for every $0 < \rho < \rho(\varepsilon)$

$$0 \leq \frac{\sup\limits_{u \in S_\rho^1} J(u)}{\rho} \leq \frac{\sup\limits_{u \in S_\rho^2} J(u)}{\rho} \leq \frac{\varepsilon}{2} + c'\rho^{\frac{p-2}{2}} < \varepsilon,$$

which completes the proof. □

Proof of Theorem 1.1. Fix $\mu \in [0, \overline{\mu})$. We recall that $\sup\limits_{R>0} \operatorname*{essinf}\limits_{d_g(x_0,x)\leq R} \alpha(x) > 0$, thus we choose an $R_0 > 0$ such that $\alpha_{R_0} := \operatorname*{essinf}\limits_{d_g(x_0,x)\leq R_0} \alpha(x) > 0$.

From the assumption (f_3) there exists $s_0 > 0$ such that $F(s_0) > 0$. Let $u_\varepsilon \in W_{g,G}^{2,2}(M)$ such that $u_\varepsilon(x) = s_0$ for any $x \in B_g(x_0, \varepsilon R_0)$, $u_\varepsilon(x) = 0$ for any $M \setminus B_g(x_0, R_0)$, and $\|u_\varepsilon\|_\infty \leq |s_0|$. We also have

$$\begin{aligned} J(u_\varepsilon) &\geq \alpha_{R_0} F(s_0) \mathrm{Vol}_g(B_g(x_0, \varepsilon R_0)) \\ &\quad - \|\alpha\|_\infty \max_{|t|\leq|s_0|} |F(t)| \mathrm{Vol}_g\left(B_g(x_0, R_0) \setminus B_g(x_0, \varepsilon R_0)\right), \end{aligned}$$

For ε close enough to 1, the right-hand side of the last inequality becomes strictly positive; choose such a number, say ε_0. Now, taking into account Lemma 3.4, one can fix a small number $\rho = \rho(\varepsilon_0)$ such that

$$2\rho < \left(1 - \frac{\mu}{\overline{\mu}}\right) \|u\|^2,$$

$$\frac{\sup\{J(u) : \Phi_\mu(u) < \rho\}}{\rho} < \frac{2J(u_{\varepsilon_0})}{\|u_{\varepsilon_0}\|^2}.$$

In Theorem 2.2 we choose $u_1 = u_{\varepsilon_0}$ and $u_0 = 0$, and observe that the hypotheses (2) and (3) are satisfied. We define

$$\overline{a} = \frac{1+\rho}{\frac{J(u_{\varepsilon_0})}{\Phi(u_{\varepsilon_0})} - \frac{\sup\{J(u):\Phi_\mu(u)\leq\rho\}}{\rho}}.$$

Taking into account Lemmas, 3.2 and 3.3, all the assumptions of Theorem 2.2 are verified. Thus there exists an open interval $I_\mu \subset [0, \overline{a}]$ and a number $\sigma_\mu > 0$ such that for each $\lambda \in I_\mu$, the equation $\mathcal{E}'_{\lambda,\mu}(u) = \Phi'_\mu(u) - \lambda J'(u)$ admits at least three solutions in $W_{g,G}^{2,2}(M)$ having $W_g^{2,2}(M)$-norms less than σ_μ. This concludes the proof. □

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