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Multiplicity of weak solutions for a class of non-homogeneous anisotropic elliptic systems

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Abstract. We study the existence of infinitely many weak solutions for a new class of nonhomogeneous Neumann elliptic systems involving operators that extend both generalized Laplace operators and generalized mean curvature operators in the framework of anisotropic variable spaces.

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1. Introduction

In the recent years, the anisotropic variable exponent Sobolev space $W^{1,\vec{z}(\cdot)}(\Omega)$ have captured the attention of many mathematicians, physicists and engineers. The impulse for this mainly comes from their important applications in modelling real world problems in electrorheological, magneto-rheological fluids, elastic materials and image restoration (see for example [11, 20, 21]). Predominantly, the focus lies on boundary value problems featuring generalized Laplace operators or generalized mean curvature operators. An attractive proposal is to employ operators of greater generality, capable of producing both Laplace-style and mean curvature-style operators. This includes equations structured as follows:

$$\begin{cases} -\sum_{i=1}^{N} \partial_{x_i}\Big(\partial_3 A_i\Big(\cdot, u, \partial_{x_i} u\Big)\Big) = f(x,u) \quad \text{in } \Omega, \\ \sum_{i=1}^{N} \partial_3 A_i\Big(\cdot, u, \partial_{x_i} u\Big)\gamma_i = 0 \quad \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

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where $A_i : \Omega \times \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ for $i = 1, \dots, N$ are Carathéodory functions satisfy suitable conditions.

Moreover, on one hand the operator introduced in the previous equation has the potential to be transformed into the $\vec{z}(\cdot)$-Laplace anisotropic operator given by

$$\Delta_{\vec{z}(\cdot)} u = \sum_{i=1}^{N} \partial_{x_i} \left(|\partial_{x_i} u|^{z_i(x)-2} \partial_{x_i} u \right), \tag{1.2}$$

when

$$A_i(x, u, t) = \frac{1}{z_i(x)} |t|^{z_i(x)}$$

which fulfills the assumptions (H_1)-(H_4) in section 3. It is clear that by selecting $z_1(\cdot) = \cdots = z_N(\cdot) = z(\cdot)$, we find an operator known as the $z(\cdot)$-orthotropic operator, which possesses analogous characteristics to the variable exponent $z(\cdot)$-Laplace operator, the relation between the $\vec{z}(\cdot)$-Laplace anisotropic operator, the $z(\cdot)$-Laplacian, and the $z(\cdot)$-orthotropic operator is noteworthy. When $z_1, \dots, z_N$ are constant functions, we find the $\vec{z}$-Laplacian operator. Noting that $\vec{z}(\cdot)$-Laplacian operator acts as a versatile bridge between these different operational modes, facilitating the analysis of diverse situations. For some existing results for strongly nonlinear elliptic equations in the anisotropic variable exponent Sobolev spaces, see [4, 13, 26]. Notice that the general operator given by (1.2) can admit degenerate and singular points. It is no surprise to find that there are already papers treating problems with this kind of operator. To give some examples, we refer the reader to [8, 17, 18], where the authors were concerned with Dirichlet problems. We, on the other hand, are interested in a Neumann problem. We refer the reader to [1, 12].

On the other hand, the operator in (1.1) generalized the operator corresponding to the anisotropic variable mean curvature given by

$$\sum_{i=1}^{N} \partial_{x_i} \left((1 + |\partial_{x_i} u|^2)^{\frac{z_i(x)-2}{2}} \partial_{x_i} u \right), \tag{1.3}$$

when

$$A_i(x, u, t) = \frac{1}{z_i(x)} (1 + |t|^2)^{\frac{z_i(x)}{2}}$$

which satisfies the assumptions (H_1)-(H_4) in section 3.

Despite the fact that a specialized form of the operator described in (1.1) with $A_i(x, u, t) = a_i(x, t)$ was initially addressed by Boureanu in [9], it is essential to emphasize that the assumptions we have employed in our research are entirely unique. As a result, our outcomes are distinct, stemming from the utilization of a variational principle presented by Ricceri in [24].

In this paper, we are interested in the following problem:

$$(\mathcal{P})\begin{cases} -\sum_{i=1}^{N}\partial_{x_i}\Big(\partial_4 A_i\Big(\cdot,u,v,\partial_{x_i}u,\partial_{x_i}v\Big)\Big)+a_0(x,u,v)=\eta(x)f(u,v) & \text{in } \Omega,\\ -\sum_{i=1}^{N}\partial_{x_i}\Big(\partial_5 B_i\Big(\cdot,u,v,\partial_{x_i}u,\partial_{x_i}v\Big)\Big)+b_0(x,u,v)=\eta(x)g(u,v) & \text{in } \Omega,\\ \sum_{i=1}^{N}\partial_4 A_i\Big(\cdot,u,v,\partial_{x_i}u,\partial_{x_i}v\Big)\gamma_i=0 & \text{on } \partial\Omega,\\ \sum_{i=1}^{N}\partial_5 B_i\Big(\cdot,u,v,\partial_{x_i}u,\partial_{x_i}v\Big)\gamma_i=0 & \text{on } \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ be a rectangular-like domain, $\partial_4 A_i$ (resp $\partial_5 B_i$) stands for the partial derivative with respect to the fourth variable of A_i (resp the fifth variable of B_i), satisfying some conditions in Section 3.

Boureanu in [10] revolves around exploring the concept of weak solvability in the context of two distinct anisotropic systems characterized by variable exponents. The first system is situated within a rectangular like domain and is governed by no-flux boundary conditions, while the second system is located within a general bounded domain and is subject to zero Dirichlet boundary conditions. Both systems incorporate Leray-Lions type operators which is a particular case of the operator introduced in (1.1) and involve a function F exhibiting sublinear behavior at both zero and infinity. The operators we consider encompass a wide range of possibilities, including generalized Laplace operators, generalized orthotropic Laplace operators, Laplace-type operators stemming from capillary phenomena, and generalized mean curvature operators. The operators considered encompass a wide spectrum of possibilities, including generalized Laplace operators, generalized orthotropic Laplace operators, Laplace-type operators arising from capillary phenomena, and generalized mean curvature operators. The problem under consideration is characterized by carefully crafted hypotheses tailored to capture its unique intricacies, rendering it challenging to encapsulate within a single equation. The provided examples of function F illustrate the diversity inherent in our approach, and the multiplicity results are established through the application of critical point theory.

A large number of papers was devoted to the study the existence of solutions of elliptic systems under various assumptions and in different contexts for a review on classical results, see [2, 3, 6, 7, 20, 21, 25].

The main difficulties in this kind of problem are the framework of anisotropic Sobolev spaces and the fact that we have new class of non-homogeneous Neumann elliptic systems that make some difficulties in the application of Theorem 1.1.

We introduce the following theorem, which will be essential to establish the existence of weak solutions for our main problem.

Theorem 1.1. (*See* [*24*]*, Theorem 2.5*). *Let X be a reflexive real Banach space, and let $\Phi, \Psi : X \longrightarrow \mathbb{R}$ be two sequentially weakly lower semi-continuous and Gâteaux differentiable functionnals. Assume also that Ψ is (strongly) continuous and satisfies*

$\lim\limits_{\|u\|\to+\infty} \Psi(u) = +\infty$. *For each* $\rho > \inf_X \Psi$, *put*

$$\varphi(\rho) = \inf_{u\in\Psi^{-1}(]-\infty,\rho[)} \frac{\Phi(u) - \inf\limits_{v\in\overline{\Psi^{-1}(]-\infty,\rho[)}^{w}} \Phi(v)}{\rho - \Psi(u)}, \tag{1.4}$$

where $\overline{\Psi^{-1}(]-\infty,\rho[)}^{w}$ *is the closure of* $\Psi^{-1}(]-\infty,\rho[)$ *in the weak topology. Furthermore, set*

$$\gamma = \liminf_{\rho\to+\infty} \varphi(\rho), \tag{1.5}$$

and

$$\delta = \liminf_{\rho\to(\inf_X \Psi)^+} \varphi(\rho). \tag{1.6}$$

Then, the following conclusions hold:

(a) *For each* $\rho > \inf_X \Psi$ *and each* $t > \varphi(\rho)$, *the functional* $\Phi + t\Psi$ *has a critical point which lies in* $\Psi^{-1}(]-\infty,\rho[)$.

(b) *If* $\gamma < +\infty$, *then, for each* $t > \gamma$, *the following alternative holds: either* $\Phi + t\Psi$ *has a global minimum, or there exists a sequence* $(u_n)_n$ *of critical points of* $\Phi + t\Psi$ *such that* $\lim\limits_{n\to\infty} \Psi(u_n) = +\infty$.

(c) *If* $\delta < +\infty$, *then, for each* $t > \delta$, *the following alternative holds: either there exists a global minimum of* Ψ *which is a local minimum of* $\Phi + t\Psi$, *or there exists a sequence of pairwise distinct critical points of* $\Phi + t\Psi$ *which weakly converges to a global minimum of* Ψ.

This paper is organized as follows: In Section 2, we present some necessary preliminary knowledge on the anisotropic Sobolev spaces with variable exponents. We introduce in the Section 3, some assumptions for which our problem has a solutions and we prove the existence of infinitely many weak solutions for our Neumann elliptic problem.

2. Preliminaries results

In this section we summarize notation, definitions and properties of our framework. For more details we refer to [14]. Let Ω be a bounded domain in $\mathbb{R}^N$, we define:

$$\mathcal{C}_+(\overline{\Omega}) = \big\{\text{measurable function } p(\cdot) : \overline{\Omega} \longrightarrow \mathbb{R} \text{ such that } 1 < p^- \leq p^+ < \infty\big\},$$

where

$$p^- = \operatorname{ess\,inf}\big\{p(x) \,/\, x \in \overline{\Omega}\big\} \quad \text{and} \quad p^+ = \operatorname{ess\,sup}\big\{p(x) \,/\, x \in \overline{\Omega}\big\}.$$

We define the Lebesgue space with variable exponent $L^{p(\cdot)}(\Omega)$ as the set of all measurable functions $u : \Omega \longmapsto \mathbb{R}$ for which the convex modular

$$\rho_{p(\cdot)}(u) := \int_\Omega |u|^{p(x)} dx,$$

is finite, then

$$\|u\|_{L^{p(\cdot)}(\Omega)} = \|u\|_{p(\cdot)} = \inf\big\{\lambda > 0 : \rho_{p(\cdot)}(u/\lambda) \leq 1\big\},$$

defines a norm in $L^{p(\cdot)}(\Omega)$, called the Luxemburg norm. The space $(L^{p(\cdot)}(\Omega), \|\cdot\|_{p(\cdot)})$ is a separable Banach space. Moreover, the space $L^{p(\cdot)}(\Omega)$ is uniformly convex, hence reflexive, and its dual space is isomorphic to $L^{p'(\cdot)}(\Omega)$, where $\dfrac{1}{p(\cdot)} + \dfrac{1}{p'(\cdot)} = 1$. Finally, we have the Hölder type inequality:

$$\left| \int_\Omega uv \, dx \right| \leq \Big(\frac{1}{p^-} + \frac{1}{(p'^-)} \Big) \|u\|_{p(\cdot)} \|v\|_{p'(\cdot)}, \tag{2.1}$$

for all $u \in L^{p(\cdot)}(\Omega)$ and $v \in L^{p'(\cdot)}(\Omega)$.
An important role in manipulating the generalized Lebesgue spaces is played by the modular $\rho_{p(\cdot)}$ of the space $L^{p(\cdot)}(\Omega)$. We have the following result.

Proposition 2.1. *(See [14, 17].)* *If $u \in L^{p(\cdot)}(\Omega)$, then the following properties hold true:*

(i). $\|u\|_{p(\cdot)} < 1$*(respectively,* $= 1, > 1$*)* $\Leftrightarrow \rho_{p(\cdot)}(u) < 1$*(respectively,* $= 1, > 1$*)*,

(ii). $\|u\|_{p(\cdot)} > 1 \Rightarrow \|u\|_{p(\cdot)}^{p^-} < \rho_{p(\cdot)}(u) < \|u\|_{p(\cdot)}^{p^+}$,

(iii). $\|u\|_{p(\cdot)} < 1 \Rightarrow \|u\|_{p(\cdot)}^{p^+} < \rho_{p(\cdot)}(u) < \|u\|_{p(\cdot)}^{p^-}$.

We define the Sobolev space with variable exponent by:

$$W^{1,p(\cdot)}(\Omega) = \Big\{ u \in L^{p(\cdot)}(\Omega) \quad \text{and} \quad |\nabla u| \in L^{p(\cdot)}(\Omega) \Big\},$$

equipped with the following norm

$$\|u\|_{W^{1,p(\cdot)}(\Omega)} = \|u\|_{1,p(\cdot)} = \|u\|_{p(\cdot)} + \|\nabla u\|_{p(\cdot)}.$$

The space $(W^{1,p(\cdot)}(\Omega), \|\cdot\|_{1,p(\cdot)})$ is a separable and reflexive Banach space. We refer to [14] for the elementary properties of these spaces.

Remark 2.2. Recall that the definition of these spaces requires only the measurability of $p(\cdot)$. In this work, we do not need to use Sobolev and Poincaré inequalities. Note that the sharp Sobolev inequality is proved for $p(\cdot)$-log-Hölder continuous, while the Poincaré inequality requires only the continuity of $p(\cdot)$ (see [14]).

Now, we present the anisotropic Sobolev space with variable exponent which is used for the study of our main problem.
Let $p_1(\cdot), \dots, p_N(\cdot)$ be N variable exponents in $\mathcal{C}_+(\overline{\Omega})$. We denote

$$\vec{p}(\cdot) = \big\{ p_1(\cdot), \dots, p_N(\cdot) \big\}, \quad \text{and } D^i u = \frac{\partial u}{\partial x_i} \quad \text{for} \quad i = 1, \dots, N,$$

and for all $x \in \overline{\Omega}$ we put

$$p_M(\cdot) = \max \Big\{ p_1(\cdot), ..., p_N(\cdot) \Big\} \ \text{ and } \ p_m(\cdot) = \min \Big\{ p_1(\cdot), ..., p_N(\cdot) \Big\}.$$

We define

$$\underline{p} = \min \{ p_1^-, p_2^-, \dots, p_N^- \} \qquad \text{then} \qquad \underline{p} > 1, \tag{2.2}$$

and

$$\overline{p} = \max \{ p_1^+, p_2^+, \dots, p_N^+ \}. \tag{2.3}$$

The anisotropic variable exponent Sobolev space $W^{1,\vec{p}(\cdot)}(\Omega)$ is defined as follows

$$W^{1,\vec{p}(\cdot)}(\Omega)=\Big\{u\in L^{p_M(\cdot)}(\Omega)\quad\text{and}\quad D^iu\in L^{p_i(\cdot)}(\Omega),\quad i=1,2,\dots,N\Big\},$$

endowed with the norm

$$\|u\|_{W^{1,\vec{p}(\cdot)}(\Omega)}=\|u\|_{1,\vec{p}(\cdot)}=\|u\|_{L^{p_M(\cdot)}(\Omega)}+\sum_{i=1}^{N}\|D^iu\|_{L^{p_i(\cdot)}(\Omega)}. \tag{2.4}$$

(Cf. [5, 22, 23] for the constant exponent case). The space $\big(W^{1,\vec{p}(\cdot)}(\Omega),\|\cdot\|_{1,\vec{p}(\cdot)}\big)$ is a reflexive Banach space (cf [15]). The theory of such spaces was developed in [15, 16, 17, 19].

3. Essential assumptions and main results

Here and in the sequel, we assume that Ω is a rectangular like domain and let $p_1(\cdot),\dots,p_N(\cdot)$ and $q_1(\cdot),\dots,q_N(\cdot)$ be $2N$ variable exponents in $\mathcal{C}_+(\overline{\Omega})$ satisfying that so called log-Hölder continuity, there exists a positive constant $L>0$ such that

$$|p_i(x)-p_i(y)|\leq-\frac{L}{\log(x-y)},\ \text{for all } x,y\in\overline{\Omega}\text{ with }|x-y|\leq\frac{1}{2}, \tag{3.1}$$

$$|q_i(x)-q_i(y)|\leq-\frac{L}{\log(x-y)},\ \text{for all } x,y\in\overline{\Omega}\text{ with }|x-y|\leq\frac{1}{2}, \tag{3.2}$$

and we suppose also

$$\underline{p}>N\text{ and }\underline{q}>N. \tag{3.3}$$

The previous assumption gives the following result.

Proposition 3.1. *Since $W^{1,\vec{p}(\cdot)}(\Omega)$ (respectively $W^{1,\vec{q}(\cdot)}(\Omega)$) is continuously embedded in $W^{1,\underline{p}}(\Omega)$(respectively $W^{1,\underline{q}}(\Omega)$), and since $W^{1,\underline{p}}(\Omega)$ and $W^{1,\underline{q}}(\Omega)$ are compactly embedded in $C^0(\overline{\Omega})$ (the space of continuous functions), thus the spaces $W^{1,\vec{p}(\cdot)}(\Omega)$ and $W^{1,\vec{q}(\cdot)}(\Omega)$ are compactly embedded in $C^0(\overline{\Omega})$.*

Then we can set

$$C_1=\sup_{u\in W^{1,\vec{p}(\cdot)}(\Omega)\setminus\{0\}}\frac{\|u\|_\infty}{\|u\|_{1,\vec{p}(\cdot)}}. \tag{3.4}$$

$$C_2=\sup_{u\in W^{1,\vec{q}(\cdot)}(\Omega)\setminus\{0\}}\frac{\|u\|_\infty}{\|u\|_{1,\vec{q}(\cdot)}}. \tag{3.5}$$

We would like to highlight the relevance of the upcoming density result, as it is instrumental in assuring the sound definition of weak solutions pertaining to system $(\mathcal{P})$.

Theorem 3.2. (*See* [*10, 15*].) *Let Ω be a rectangular-like domain of $\mathbb{R}^N$. Under the assumptions* (3.1) *and* (3.2)*, it can be affirmed that $\mathcal{C}^\infty(\overline{\Omega})$ serves as a dense subset within both $W^{1,\vec{p}(\cdot)}(\Omega)$ and $W^{1,\vec{q}(\cdot)}(\Omega)$.*

We present now the characteristics of the functions $A_i,B_i:\Omega\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$, $i=1,\dots,N$, and $A_0,B_0:\Omega\times\mathbb{R}\times\mathbb{R}\longrightarrow\mathbb{R}$.

(H_1). For all $i = 0, \ldots, N$, A_i and B_i are continuous in x and of class $\mathcal{C}^1$ in (s,t), with $A_i(x,0,0,0,0) = B_i(x,0,0,0,0) = 0$ for a.e. $x \in \overline{\Omega}$.

(H_2). For all $i = 1, \ldots, N$, there are positive constants α_i and non-negative functions $c_i \in L^{p_i'(\cdot)}(\Omega)$ such that

$$\begin{aligned}&\Big|\partial_r A_i(x,s,t,r,\sigma)\Big| + \Big|\partial_\sigma B_i(x,s,t,r,\sigma)\Big| \\ &\leq \alpha_i\Big(c_i(x) + |s|^{p_i(x)} + |t|^{q_i(x)} + |r|^{p_i(x)-1} + |\sigma|^{q_i(x)-1}\Big), \qquad \text{for a.e. } x \in \overline{\Omega},\end{aligned} \tag{3.6}$$

and all $s,t,r,\sigma \in \mathbb{R}$, and, there are non-negative functions $\lambda_1, \lambda_2 \in L^1(\Omega)$ such that

$$\Big|A_0(x,s,t)\Big| \leq \lambda_1(x)\Big(|s|^{p_M(x)} + |t|^{q_M(x)}\Big), \qquad \text{for a.e. } x \in \overline{\Omega} \text{ and all } s,t \in \mathbb{R}.$$

$$\Big|B_0(x,s,t)\Big| \leq \lambda_2(x)\Big(|s|^{p_M(x)} + |t|^{q_M(x)}\Big), \qquad \text{for a.e. } x \in \overline{\Omega} \text{ and all } s,t \in \mathbb{R},$$

where $A_0(x,s,t) = \int_0^s a_0(x,\sigma,t)d\sigma$ and $B_0(x,s,t) = \int_0^t b_0(x,s,\sigma)d\sigma$.

(H_3). For all $i = 1, \ldots, N$ and for all $s,t,\sigma, r \neq r' \in \mathbb{R}$ and all $x \in \Omega$, one has

$$\sum_{i=1}^N \Big(\partial_s A_i\Big(x,s,t,r,\sigma\Big) - \partial_s A_i\Big(x,s,t,r',\sigma\Big)\Big)(r-r') > 0,$$

$$\sum_{i=1}^N \Big(\partial_t B_i\Big(x,s,t,r,\sigma\Big) - \partial_t B_i\Big(x,s,t,r,\sigma'\Big)\Big)(\sigma-\sigma') > 0$$

and,

$$\Big(\partial_s A_0(x,s,t) - \partial_s A_0(x,s,t')\Big)(t-t') > 0, \text{ for all } s, t \neq t' \in \mathbb{R}, \text{ and all } x \in \Omega.$$

$$\Big(\partial_s B_0(x,s,t) - \partial_s B_0(x,s,t')\Big)(t-t') > 0, \text{ for all } s, t \neq t' \in \mathbb{R}, \text{ and all } x \in \Omega.$$

(H_4). There are constants $\delta_0, \delta_1, \theta_0, \theta_1 > 0$ such that, for all $i = 1, \ldots, N$ we have

$$A_i(x,s,t,r,\sigma) \geq \delta_0|r|^{p_i(x)}, \qquad \text{for all } x \in \Omega \text{ and } s,t,r,\sigma \in \mathbb{R},$$

$$B_i(x,s,t,r,\sigma) \geq \delta_1|\sigma|^{q_i(x)}, \qquad \text{for all } x \in \Omega \text{ and } s,t,r,\sigma \in \mathbb{R},$$

$$A_0(x,s,t) \geq \theta_0|s|^{p_M(x)}, \qquad \text{for all } x \in \Omega \text{ and } s,t \in \mathbb{R},$$

and,

$$B_0(x,s,t) \geq \theta_1|t|^{q_M(x)}, \qquad \text{for all } x \in \Omega \text{ and } s,t \in \mathbb{R}.$$

(H_5). $\eta \in \mathcal{C}(\overline{\Omega})$ and $f, g \in \mathcal{C}(\mathbb{R}^2)$ such that the differential form $f(u,v)du + g(u,v)dv$ is exact.

Remark 3.3. (H_5) implies that exists $H : \mathbb{R}^2 \longmapsto \mathbb{R}$ is the integral of the differential form $f(u,v)du + g(u,v)dv$ such that $H(0,0) = 0$.

Let X be the Cartesian product between Sobolev spaces $W^{1,\vec{p}(\cdot)}(\Omega)$ and $W^{1,\vec{q}(\cdot)}(\Omega)$ with the norm $\|(u,v)\|_X = \sqrt{\|u\|^2_{1,\vec{p}(\cdot)} + \|v\|^2_{1,\vec{q}(\cdot)}}$ or another equivalent to it.
We introduce the functionals $\Psi(\cdot,\cdot)$, $\Phi(\cdot,\cdot) : W^{1,\vec{p}(\cdot)}(\Omega) \times W^{1,\vec{q}(\cdot)}(\Omega) \longmapsto \mathbb{R}$ by

$$\begin{aligned}\Psi(u,v) &= \sum_{i=1}^{N} \int_\Omega A_i\Big(x,u,v,\partial_{x_i}u,\partial_{x_i}v\Big)\,dx + \sum_{i=1}^{N} \int_\Omega B_i\Big(x,u,v,\partial_{x_i}u,\partial_{x_i}v\Big)\,dx \\ &+ \int_\Omega A_0(x,u,v)dx + \int_\Omega B_0(x,u,v)dx, \end{aligned} \tag{3.7}$$

and

$$\Phi(u,v) = -\int_\Omega F(x,u(x),v(x))\,dx, \tag{3.8}$$

where $F : \overline{\Omega} \times \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ is defined as $F(x,u,v) = \eta(x)H(u,v)$.

Lemma 3.4. (*See [9]*). *The functionals $\Psi(\cdot,\cdot)$ and $\Phi(\cdot,\cdot)$ are well defined on X. In addition, $\Psi(\cdot,\cdot)$ and $\Phi(\cdot,\cdot)$ are of class $\mathcal{C}^1(X,\mathbb{R})$ and*

$$\begin{aligned}\Psi'(u,v)(w,\phi) &= \sum_{i=1}^{N} \int_\Omega \partial_4 A_i\Big(x,u,v,\partial_{x_i}u,\partial_{x_i}v\Big)\partial_{x_i}w dx \\ &+ \sum_{i=1}^{N} \int_\Omega \partial_5 B_i\Big(x,u,v,\partial_{x_i}u,\partial_{x_i}v\Big)\partial_{x_i}u\phi dx \\ &+ \int_\Omega \partial_2 A_0(x,u,v)w dx + \int_\Omega \partial_3 B_0(x,u,v)\phi dx, \end{aligned} \tag{3.9}$$

and

$$\Phi'(u,v)(w,\phi) = -\int_\Omega \eta(x)\Big(\partial_1 H(u(x),v(x))w(x) + \partial_2 H(u(x),v(x))\phi(x)\Big)\,dx, \tag{3.10}$$

for all $(u,v)(w,\phi) \in X$.

Lemma 3.5. (*See* [*9*]). *Under the hypothesis (H_1)-(H_5) and (3.3) the functionals $\Psi(\cdot,\cdot)$ and $\Phi(\cdot,\cdot)$ are weakly lower semi-continuous.*

Lemma 3.6. *Under the hypothesis (H_1)-(H_5) the functional $\Psi(\cdot,\cdot)$ is coercive, that is,*

$$\Psi(u,v) \longrightarrow +\infty \quad \text{as } \|(u,v)\|_X \longrightarrow +\infty \quad \text{for } (u,v) \in X.$$

Proof. Let $(u,v) \in X$. One has

$$\begin{aligned}\Psi(u,v) &= \sum_{i=1}^{N} \int_\Omega A_i\Big(x,u,v,\partial_{x_i}u,\partial_{x_i}v\Big)dx + \sum_{i=1}^{N} \int_\Omega B_i\Big(x,u,v,\partial_{x_i}u,\partial_{x_i}v\Big)\,dx \\ &+ \int_\Omega A_0(x,u,v)dx + \int_\Omega B_0(x,u,v)dx,\end{aligned}$$

then by using (H_4), we get

$$\begin{aligned}
\Psi(u,v) &\geq \sum_{i=1}^{N}\int_\Omega \delta_0\Big|\partial_{x_i}u\Big|^{p_i(x)}\,dx+\sum_{i=1}^{N}\int_\Omega \delta_1\Big|\partial_{x_i}v\Big|^{q_i(x)}\,dx+\theta_0\int_\Omega |u|^{p_M(x)}\,dx\\
&\quad +\theta_1\int_\Omega |u|^{q_M(x)}\,dx\\
&\geq \min\Big(\delta_0,\theta_0\Big)\Big[\sum_{i=1}^{N}\int_\Omega\Big|\partial_{x_i}u\Big|^{p_i(x)}\,dx+\int_\Omega |u|^{p_M(x)}\,dx\Big]\\
&\quad +\min\Big(\delta_1,\theta_1\Big)\Big[\sum_{i=1}^{N}\int_\Omega\Big|\partial_{x_i}v\Big|^{q_i(x)}\,dx+\int_\Omega |v|^{q_M(x)}\Big]dx\\
&\geq \min\Big(\delta_0,\theta_0\Big)\Big[\frac{1}{N^{\underline{p}-1}}\Big(\sum_{i=1}^{N}\Big\|\partial_{x_i}u\Big\|_{p_i(\cdot)}\Big)^{\underline{p}}+\|u\|^{\underline{p}}_{p_M(\cdot)}-N-1\Big]\\
&+\quad \min\Big(\delta_1,\theta_1\Big)\Big[\frac{1}{N^{\underline{q}-1}}\Big(\sum_{i=1}^{N}\Big\|\partial_{x_i}v\Big\|_{q_i(\cdot)}\Big)^{\underline{q}}+\|u\|^{\underline{q}}_{q_M(\cdot)}-N-1\Big]\\
&\geq \min\Big(\delta_0,\theta_0\Big)\Big[\frac{1}{(2N)^{\underline{p}-1}}\Big(\sum_{i=1}^{N}\Big\|\partial_{x_i}u\Big\|_{p_i(\cdot)}+\|u\|_{p_M(\cdot)}\Big)^{\underline{p}}-N-1\Big]\\
&+\quad \min\Big(\delta_1,\theta_1\Big)\Big[\frac{1}{(2N)^{\underline{q}-1}}\Big(\sum_{i=1}^{N}\Big\|\partial_{x_i}v\Big\|_{q_i(\cdot)}+\|u\|_{q_M(\cdot)}\Big)^{\underline{q}}-N-1\Big]\\
&= \frac{\min\Big(\delta_0,\theta_0\Big)}{(2N)^{\underline{p}-1}}\|u\|^{\underline{p}}_{1,\vec{p}(\cdot)}+\frac{\min\Big(\delta_1,\theta_1\Big)}{(2N)^{\underline{q}-1}}\|u\|^{\underline{q}}_{1,\vec{q}(\cdot)}-K_2\\
&\geq K_1\Big(\|u\|^{\underline{p}}_{1,\vec{q}(\cdot)}+\|v\|^{\underline{q}}_{1,\vec{q}(\cdot)}\Big)-K_2\\
&\geq K_1\|(u,v)\|_X-K_2,
\end{aligned}$$

where $K_1,K_2>0$ constants.
Thus, if $\|(u,v)\|_X\longrightarrow+\infty$ then $\Psi(u,v)\longrightarrow+\infty$. □

Now, we set $\eta_1=\left(\frac{C_1}{\theta_0\mathrm{meas}(\Omega)}\right)^{\overline{p}}$ and $\eta_2=\left(\frac{C_2}{\theta_1\mathrm{meas}(\Omega)}\right)^{\overline{q}}$,

$$\mu=\min\left\{\frac{1}{\eta_1^{\overline{p}}},\frac{1}{\eta_1^{\underline{p}}}\right\},\quad\text{and}\quad \nu=\min\left\{\frac{1}{\eta_2^{\overline{q}}},\frac{1}{\eta_2^{\underline{q}}}\right\}.$$

The sets $A(r),B(r),\ r>0$, below satisfied, play an important role in our exposition

$$A(r)=\left\{(\xi,\eta)\in\mathbb{R}^2 \text{ such that } \mu F_{\vec{p}(\cdot)}(\xi)+\nu F_{\vec{q}(\cdot)}(\eta)\leq r\right\}$$

and

$$B(r)=\left\{(\xi,\zeta)\in\mathbb{R}^2:\int_\Omega A_0(x,\xi,\zeta)dxD_{\vec{p}(\cdot)}(\xi)+\int_\Omega B_0(x,\xi,\zeta)dxD_{\vec{q}(\cdot)}(\zeta)\leq r\right\},$$

where $D_{\vec{r}(\cdot)}(t) = \max(|t|^{\overline{r}}, |t|^{\underline{r}})$ and $F_{\vec{r}(\cdot)}(t) = \min(|t|^{\overline{r}}, |t|^{\underline{r}})$ with $\vec{r}(\cdot) \in \{\vec{p}(\cdot), \vec{q}(\cdot)\}$ and $t \in \{\xi, \eta\}$.

Lemma 3.7. *For all $r > 0$, we have*

$$B(r) \subset A(r).$$

Proof. We observe that, by the definition of constants C_1 and C_2, we have

$$\|u\|_\infty \leq C_1\|u\|_{1,\vec{p}(\cdot)}, \forall u \in W^{1,\vec{p}(\cdot)}(\Omega),$$

and

$$\|v\|_\infty \leq C_2\|v\|_{1,\vec{q}(\cdot)}, \forall v \in W^{1,\vec{q}(\cdot)}(\Omega).$$

For $u \equiv v \equiv 1$, we get

$$1 \leq \eta_1^{\overline{p}}\theta_0 \text{meas}(\Omega) \leq \eta_1^{\overline{p}} \int_\Omega A_0(x, \xi, \eta)dx,$$

and,

$$1 \leq \eta_2^{\overline{q}}\theta_1 \text{meas}(\Omega) \leq \eta_2^{\overline{q}} \int_\Omega B_0(x, \xi, \eta)dx.$$

Thus, we obtain

$$\mu \leq \frac{1}{\eta_1^{\overline{p}}} \leq \int_\Omega A_0(x, \xi, \eta)dx, \quad \text{and } \nu \leq \frac{1}{\eta_2^{\overline{q}}} \leq \int_\Omega B_0(x, \xi, \eta)dx.$$

Since

$$F_{\vec{p}(\cdot)}(t) \leq D_{\vec{p}(\cdot)}(t), \quad \text{and } F_{\vec{q}(\cdot)}(t) \leq D_{\vec{q}(\cdot)}(t), \forall t \in \mathbb{R}.$$

Thus, the inequality

$$\mu F_{\vec{p}(\cdot)}(\xi) + \nu F_{\vec{q}(\cdot)}(\zeta) \leq \int_\Omega A_0(x, \xi, \eta)dx D_{\vec{p}(\cdot)}(\xi) + \int_\Omega B_0(x, \xi, \eta)dx D_{\vec{q}(\cdot)}(\zeta),$$

holds for every $(\xi, \zeta) \in \mathbb{R}^2$ and therefore the inclusion

$$B(r) \subset A(r), \forall r > 0,$$

holds. □

Definition 3.8. We say that $(u, v) \in X$ a weak solution to the problem $(\mathcal{P})$ if for all $(w, \phi) \in X$, we have

$$\begin{aligned}
&\sum_{i=1}^N \int_\Omega \partial_4 A_i\Big(x, u, v, \partial_{x_i}u, \partial_{x_i}v\Big)\partial_{x_i}w dx \\
&+ \sum_{i=1}^N \int_\Omega \partial_5 B_i\Big(x, u, v, \partial_{x_i}u, \partial_{x_i}v\Big)\partial_{x_i}\phi dx \\
&+ \int_\Omega \partial_u A_0(x, u, v)w \, dx + \int_\Omega \partial_v B_0(x, u, v)\phi dx \\
&= \int_\Omega \eta(x)\Big(\partial_u H(u(x), v(x))w(x) + \partial_v H(u(x), v(x))\phi(x)\Big) \, dx.
\end{aligned}$$

Remark 3.9. Note that the weak solutions of $(\mathcal{P})$ are precisely critical points of $\Psi + \Phi$.

Our first main result is the following theorem.

Theorem 3.10. *Suppose that* $\Psi(\cdot,\cdot)$ *and* $\Phi(\cdot,\cdot)$ *are as in (3.7) and (3.8) and* (H_1)*-*(H_5) *and (3.3) hold true. If there exist* $\rho_0 > 0$, $(\xi_0,\eta_0) \in \mathbb{R}^2$ *with* $(\xi_0,\zeta_0) \in Int(B(\rho_0))$ $\Big($*$Int(B)$ is the interior of B*$\Big)$ *and* $\max_{A(\rho_0)} H(\xi,\zeta) = H(\xi_0,\zeta_0)$*. Then, problem* $(\mathcal{P})$ *admits a weak solution* $(u,v) \in X$ *such that* $\Psi(u,v) < \rho_0$.

Proof. We apply the part (a) of Theorem 1.1 for showing that $\varphi(\rho_0) = 0$ (here φ is the function defined in the Theorem 1.1 and $t = 1$ is assumed).
First, we observe that, for all $(u,v) \in \Psi^{-1}(]-\infty,\rho_0[)$, one has

$$\begin{aligned} 0 \le \varphi(\rho_0) &= \inf_{\Psi^{-1}(]-\infty,\rho_0[)} \frac{\Phi(u,v) - \inf_{\overline{(\Psi^{-1}(]-\infty,\rho_0[))}^{w}} \Phi(u,v)}{\rho_0 - \Psi(u,v)} \\ &\le \frac{\Phi(u,v) - \inf_{\overline{\Psi^{-1}(]-\infty,\rho_0[)}^{w}} \Phi(u,v)}{\rho_0 - \Psi(u,v)}. \end{aligned} \tag{3.11}$$

Let $u_0(x) = \xi_0$, $v_0(x) = \zeta_0$, $\forall x \in \Omega$. Then $\nabla u_n = \nabla v_0 = 0$, and since $(\xi_0,\zeta_0) \in Int(B(\rho_0))$, one has

$$\Psi(u_0,v_0) = \int_\Omega \Big[A_0(x,\xi_0,\zeta_0) + B_0(x,\xi,\zeta_0)\Big]dx < \rho_0.$$

Then, for almost evrey $x \in \overline{\Omega}$ and $\forall (u,v) \in \overline{\Psi^{-1}(]-\infty,\rho_0[)}^{w}$, one has

$$\mu F_{\vec{p}(\cdot)}(u(x)) + \nu F_{\vec{q}(\cdot)}(v(x)) \le \Psi(u,v) \le \rho_0. \tag{3.12}$$

The first inequality in (3.12) is obtained by the Proposition 2.1, while the second inequality in (3.12) follows from the fact that $\overline{\Psi^{-1}(]-\infty,\rho_0[)}^{w} = \Psi^{-1}(]-\infty,\rho_0])$.
Thus, since $(u(x),v(x)) \in A(\rho_0)$ and $H(u(x),v(x)) \le H(\xi_0,\zeta_0)$, $\forall x \in \overline{\Omega}$.
Hence $-\Phi(u,v) \le -\Phi(u_0,v_0)$ $\forall (u,v) \in \overline{\Psi^{-1}(]-\infty,\rho_0[)}^{w}$. Because,

$$-\Phi(u_0,v_0) = \sup_{\overline{\Psi^{-1}(]-\infty,\rho_0[)}^{w}} (-\Phi(u,v)) = -\inf_{\overline{\Psi^{-1}(]-\infty,\rho_0[)}_{w}} \Phi(u,v),$$

and since $\Phi(u_0,v_0) < \rho_0$, it follows that

$$\Phi(u_0,v_0) - \inf_{\overline{\Psi^{-1}(]-\infty,\rho_0[)}^{w}} \Phi(u,v) = \Phi(u_0,v_0) - \Phi(u_0,v_0) = 0.$$

Then, by choosing $(u,v) = (u_0,v_0)$ in the inequality (3.11), one has $\varphi(\rho_0) = 0$.
The conclusion (a) of the Theorem 1.1 assures that there is a critical point of $\Psi+\Phi$. □

Now, we announce our second main result.

Theorem 3.11. *Suppose that* $\Psi(\cdot,\cdot)$ *and* $\Phi(\cdot,\cdot)$ *are as in (3.7) and (3.8) and* (H_1)*-*(H_5) *and (3.3) hold true. If there exist a sequences,* $(\rho_n)_n \subset \mathbb{R}^+$ *with* $\rho_n \to \infty$ *as* $n \to +\infty$ *and* $(\xi_n)_n, (\zeta_n)_n \subset \mathbb{R}$ *such that* $(\xi_n,\zeta_n) \in Int(B(\rho_n))$ *and*

$$\max_{A(\rho_n)} H(\xi,\eta) = H(\xi_n,\zeta_n),\ \forall n > 0$$

and if

$$\limsup_{(\xi,\zeta)\to+\infty} \frac{H(\xi,\zeta)\displaystyle\int_\Omega \eta(x)dx}{\Big[D_{\vec{p}(\cdot)}(\xi)+D_{\vec{p}(\cdot)}(\zeta)\Big]\displaystyle\int_\Omega A_0(x,\xi,\zeta)dx+\Big[D_{\vec{q}(\cdot)}(\xi)+D_{\vec{q}(\cdot)}(\zeta)\Big]\displaystyle\int_\Omega B_0(x,\xi,\zeta)dx}$$
$$> \big(\|\lambda_1\|_{L^1(\mathbb{R})}+\|\lambda_2\|_{L^1(\mathbb{R})}\big)\max_{(\xi,\zeta)\in\mathbb{R}^2}\left(\frac{1}{\displaystyle\int_\Omega A_0(x,\xi,\zeta)dx},\frac{1}{\displaystyle\int_\Omega B_0(x,\xi,\zeta)dx}\right).$$

Then, the problem $(\mathcal{P})$ admits an unbounded sequence of a weak solutions in X.

Proof. From the part (a). we know that $\varphi(\rho_n)=0,\ \forall n\in\mathbb{N}$.
Then, since

$$\lim_{n\to\infty}\rho_n=+\infty,$$

one has

$$\liminf_{\rho\to\infty}\varphi(\rho)\le\liminf_{n\to\infty}\varphi(\rho_n)=0<1=t.$$

Now, we fix h satisfying that

$$\limsup_{(\xi,\zeta)\to+\infty} \frac{H(\xi,\zeta)\displaystyle\int_\Omega \eta(x)dx}{\Big[D_{\vec{p}(\cdot)}(\xi)+D_{\vec{p}(\cdot)}(\zeta)\Big]\displaystyle\int_\Omega A_0(x,\xi,\zeta)dx+\Big[D_{\vec{q}(\cdot)}(\xi)+D_{\vec{q}(\cdot)}(\zeta)\Big]\displaystyle\int_\Omega B_0(x,\xi,\zeta)dx}$$
$$> h > \big(\|\lambda_1\|_{L^1(\mathbb{R})}+\|\lambda_2\|_{L^1(\mathbb{R})}\big)\max_{(\xi,\zeta)\in\mathbb{R}^2}\left(\frac{1}{\displaystyle\int_\Omega A_0(x,\xi,\zeta)dx},\frac{1}{\displaystyle\int_\Omega B_0(x,\xi,\zeta)dx}\right).$$

and we choose a sequence $(\varsigma_n,\tau_n)_n$ in $\mathbb{R}^2$ such that $\sqrt{\varsigma_n^2+\tau_n^2}\ge n$ and $\forall n\in\mathbb{N}$ one has

$$\begin{aligned}&H(\varsigma_n,\tau_n)\int_\Omega \eta(x)dx\\ &> h\Big[D_{\vec{p}(\cdot)}(\varsigma_n)+D_{\vec{p}(\cdot)}(\tau_n)\Big]\int_\Omega A_0(x,\varsigma_n,\tau_n)dx\\ &+\Big[D_{\vec{q}(\cdot)}(\varsigma_n)+D_{\vec{q}(\cdot)}(\tau_n)\Big]\int_\Omega B_0(x,\varsigma_n,\tau_n)dx.\end{aligned}$$

If we denote by u_n and v_n the constant functions on Ω which take the ς_n and τ_n values respectively, by using assumptions (H_2) we have

$$
\begin{aligned}
&\Phi(u_n, v_n) + \Psi(u_n, v_n)\\
&= \Phi(\varsigma_n, \tau_n) + \Psi(\varsigma_n, \tau_n)\\
&= -H(\varsigma_n, \tau_n)\int_\Omega \eta(x)dx + \int_\Omega A_0(x, \varsigma_n, \tau_n)dx + \int_\Omega B_0(x, \varsigma_n, \tau_n)dx\\
&\le -hD_{\vec{p}(\cdot)}(\varsigma_n)\int_\Omega A_0(x, \varsigma_n, \tau_n)dx - hD_{\vec{p}(\cdot)}(\tau_n)\int_\Omega A_0(x, \varsigma_n, \tau_n)dx\\
&-hD_{\vec{q}(\cdot)}(\varsigma_n)\int_\Omega B_0(x, \varsigma_n, \tau_n)dx - hD_{\vec{q}(\cdot)}(\tau_n)\int_\Omega B_0(x, \varsigma_n, \tau_n)dx\\
&+\|\lambda_1\|_{L^1(\mathbb{R})}\Big[D_{\vec{p}(\cdot)}(\varsigma_n) + D_{\vec{q}(\cdot)}(\tau_n)\Big] + \|\lambda_2\|_{L^1(\mathbb{R})}\Big[D_{\vec{p}(\cdot)}(\varsigma_n) + D_{\vec{q}(\cdot)}(\tau_n)\Big]\\
&= \left(\|\lambda_1\|_{L^1(\mathbb{R})} + \|\lambda_2\|_{L^1(\mathbb{R})} - h\int_\Omega A_0(x, \varsigma_n, \tau_n)dx\right) D_{\vec{p}(\cdot)}(\varsigma_n)\\
&+ \left(\|\lambda_1\|_{L^1(\mathbb{R})} + \|\lambda_2\|_{L^1(\mathbb{R})} - h\int_\Omega B_0(x, \varsigma_n, \tau_n)dx\right) D_{\vec{q}(\cdot)}(\tau_n)\\
&-hD_{\vec{p}(\cdot)}(\tau_n)\int_\Omega A_0(x, \varsigma_n, \tau_n)dx - hD_{\vec{q}(\cdot)}(\varsigma_n)\int_\Omega B_0(x, \varsigma_n, \tau_n)dx\\
&< 0, \quad \forall n \in \mathbb{N}.
\end{aligned}
$$

Since $(\sqrt{\varsigma_n^2 + \tau_n^2})_n$ is unbounded, at least one of the two sequences $(\varsigma_n)_n$ or $(\tau_n)_n$ admits one divergent subsequence.
Hence $(D_{\vec{p}(\cdot)}(\tau_n))_n$ and $(D_{\vec{q}(\cdot)}(\tau_n))_n$ admit one divergent subsequence, thus, the functional $\Phi + \Psi$ is unbounded from below.
The conclusion (b) of the Theorem 1.1 assures that there is a sequence $(x_n, y_n)_n$ of critical points of $\Phi + \Psi$ such that $\lim_{n\to+\infty}\Psi(x_n, y_n) = +\infty$.
Moreover, since Ψ is bounded on each bounded subset of X, the sequence $(x_n, y_n)_n$ must be unbounded in X. □

The following result is a practicable form of Theorem 3.11.

Corollary 3.12. *Let $(a_n)_n$ and $(b_n)_n$ be two sequences in $\mathbb{R}^+$ satisfying*

$$b_n < a_n \quad \forall n \in \mathbb{N}, \lim_{n\to+\infty} b_n = +\infty, \qquad \lim_{n\to+\infty}\frac{a_n}{b_n} = +\infty,$$

and let

$$A_n = \Big\{(\xi, \zeta) \in \mathbb{R}^2 : \mu F_{\vec{p}(\cdot)}(\xi) + \nu F_{\vec{q}(\cdot)}(\eta) \le a_n\Big\},$$

$$B_n = \Big\{(\xi, \zeta) \in \mathbb{R}^2 : \int_\Omega A_0(x, \xi, \zeta)dx D_{\vec{p}(\cdot)}(\xi) + \int_\Omega B_0(x, \xi, \zeta)dx D_{\vec{q}(\cdot)}(\zeta) \le b_n\Big\},$$

be such that $\sup_{A_n\setminus IntB_n} H \leq 0$ *for all* $n \in \mathbb{N}$.
Finally, let us assume that

$$\limsup_{(\xi,\zeta)\to+\infty} \frac{H(\xi,\zeta)\displaystyle\int_\Omega \eta(x)dx}{\Big[D_{\vec{p}(\cdot)}(\xi)+D_{\vec{p}(\cdot)}(\zeta)\Big]\displaystyle\int_\Omega A_0(x,\xi,\zeta)dx+\Big[D_{\vec{q}(\cdot)}(\xi)+D_{\vec{q}(\cdot)}(\zeta)\Big]\displaystyle\int_\Omega B_0(x,\xi,\zeta)dx}$$

$$> \left(\|\lambda_1\|_{L^1(\mathbb{R})}+\|\lambda_2\|_{L^1(\mathbb{R})}\right)\max_{(\xi,\zeta)\in\mathbb{R}^2}\left(\frac{1}{\displaystyle\int_\Omega A_0(x,\xi,\zeta)dx},\frac{1}{\displaystyle\int_\Omega B_0(x,\xi,\zeta)dx}\right).$$

Then, Problem $(\mathcal{P})$ *admits an unbounded sequence of weak solutions in* X.

Proof. Since $b_n < a_n$ it follows that $B_n \subseteq A_n$.
Let

$$\gamma = \min\{\mu,\nu\} > 0$$

$$\delta = \left(\|\lambda_1\|_{L^1(\mathbb{R})}+\|\lambda_2\|_{L^1(\mathbb{R})}\right)\max_{(\xi,\zeta)\in\mathbb{R}^2}\left(\frac{1}{\displaystyle\int_\Omega A_0(x,\xi,\zeta)dx},\frac{1}{\displaystyle\int_\Omega B_0(x,\xi,\zeta)dx}\right) > 0.$$

Then $\frac{\delta}{\gamma} > 0$ and in virtue of $\lim_{n\to+\infty}\frac{a_n}{b_n} = +\infty$, then we get $\frac{\delta}{\gamma} < \frac{a_n}{b_n}$ for $n \in \mathbb{N}$ large enough.
Let $\rho_n = \gamma a_n$. Then $\{\rho_n\}_n \subset \mathbb{R}^+$ is a divergent sequence and for n large enough, the following inclusions hold

$$IntB_n \subseteq B_n \subseteq B(\rho_n) \subseteq A(\rho_n) \subseteq A_n.$$

Then, since H is negative in the set $A_n \setminus \mathrm{Int}B_n$ for all $n \in \mathbb{N}$, we have

$$\max_{\mathrm{Int}B_n} H = \max_{A_n} H,$$

in particular, we have $\max_{\mathrm{Int}B_n} H = \max_{A(\rho_n)} H$ for $n \in \mathbb{N}$ large enough, i.e. there exist at least one sequence $(\xi_n,\zeta_n)_n \subset \mathrm{Int}B_n$ such that for n large enough, we have

$$\max_{A(\rho_n)} H(\xi,\zeta) = H(\xi_n,\zeta_n).$$

Thus, the sequences $(\xi_n)_n$, $(\zeta_n)_n$ and $(\rho_n)_n$ have got the properties required in Theorem 3.10(b).
This completes the proof. □

Our third main result reads as follows.

Theorem 3.13. *Suppose that* $\Psi(\cdot,\cdot)$ *and* $\Phi(\cdot,\cdot)$ *are as in (3.7) and (3.8) and* (H_1)-(H_5) *and (3.3) hold true. If there exist sequence,* $(\rho_n)_n \subset \mathbb{R}^+$ *with* $\rho_n \longrightarrow 0$ *as* $n \longrightarrow +\infty$ *and* $(\xi_n)_n, (\zeta_n)_n \subset \mathbb{R}$ *such that* $(\xi_n,\zeta_n) \in Int(B(\rho_n))$ *and* $\max_{A(\rho_n)} H(\xi,\zeta) =$

$H(\xi_n, \zeta_n)$, $\forall n > 0$ *and if*

$$\limsup_{(\xi,\zeta)\to(0,0)} \frac{H(\xi,\zeta)\displaystyle\int_\Omega \eta(x)dx}{\Big[D_{\vec{p}(\cdot)}(\xi)+D_{\vec{p}(\cdot)}(\zeta)\Big]\displaystyle\int_\Omega A_0(x,\xi,\zeta)dx+\Big[D_{\vec{q}(\cdot)}(\xi)+D_{\vec{q}(\cdot)}(\zeta)\Big]\displaystyle\int_\Omega B_0(x,\xi,\zeta)dx}$$

$$> \big(\|\lambda_1\|_{L^1(\mathbb{R})}+\|\lambda_2\|_{L^1(\mathbb{R})}\big)\max_{(\xi,\zeta)\in\mathbb{R}^2}\left(\frac{1}{\displaystyle\int_\Omega A_0(x,\xi,\zeta)dx},\frac{1}{\displaystyle\int_\Omega B_0(x,\xi,\zeta)dx}\right).$$

Then the problem $(\mathcal{P})$ *admits a sequence of non trivial weak solutions which strongly converges to* (u,v) *in* X*.*

Proof. We apply the part (c) of Theorem 1.1. As before, from the (a). we know that $\varphi(\rho_n)=0$, $\forall n\in\mathbb{N}$.
Therefore after observing that $\inf_X \Psi = \Psi(u,v)=0$, since $\lim_{n\to\infty}\rho_n=0$, we have

$$\delta=\liminf_{\rho\to0^+}\varphi(\rho)\le\liminf_{n\to+\infty}\varphi(\rho_n)=0<1=t.$$

Now, we fix h satisfying

$$\limsup_{(\xi,\zeta)\to(0,0)} \frac{H(\xi,\zeta)\displaystyle\int_\Omega \eta(x)dx}{\Big[D_{\vec{p}(\cdot)}(\xi)+D_{\vec{p}(\cdot)}(\zeta)\Big]\displaystyle\int_\Omega A_0(x,\xi,\zeta)dx+\Big[D_{\vec{q}(\cdot)}(\xi)+D_{\vec{q}(\cdot)}(\zeta)\Big]\displaystyle\int_\Omega B_0(x,\xi,\zeta)dx}$$

$$> h > \big(\|\lambda_1\|_{L^1(\mathbb{R})}+\|\lambda_2\|_{L^1(\mathbb{R})}\big)\max_{(\xi,\zeta)\in\mathbb{R}^2}\left(\frac{1}{\displaystyle\int_\Omega A_0(x,\xi,\zeta)dx},\frac{1}{\displaystyle\int_\Omega B_0(x,\xi,\zeta)dx}\right).$$

and choose a sequence $((\varsigma_n,\tau_n))_n$ in $\mathbb{R}^2\setminus\{(0,0)\}$ such that $\sqrt{\varsigma_n^2+\tau_n^2}\le\frac{1}{n}$ and for all $n\in\mathbb{N}$, one has

$$\begin{aligned}H(\varsigma_n,\tau_n)\int_\Omega\eta(x)dx > h\bigg(&\Big[D_{\vec{p}(\cdot)}(\varsigma_n)+D_{\vec{p}(\cdot)}(\tau_n)\Big]\int_\Omega A_0(x,\varsigma_n,\tau_n)dx\\ &+\Big[D_{\vec{q}(\cdot)}(\varsigma_n)+D_{\vec{q}(\cdot)}(\tau_n)\Big]\int_\Omega B_0(x,\varsigma_n,\tau_n)dx\bigg).\end{aligned}$$

Once more if we denote by u_n and v_n the constant functions on Ω which equal ς_n and ς_n respectively.
Then, from Proposition 2.1 the sequence $((u_n,v_n))_n$ strongly converges to (u,v) in X and one has

$$\Phi(u_n,v_n)+\Psi(u_n,v_n)=\Phi(\varsigma_n,\tau_n)+\Psi(\varsigma_n,\tau_n)$$

$$
\begin{aligned}
&= -H(\varsigma_n,\tau_n)\int_\Omega \eta(x)dx + \int_\Omega A_0(x,\varsigma_n,\tau_n)dx + \int_\Omega B_0(x,\varsigma_n,\tau_n)dx \\
&\leq -hD_{\vec{p}(\cdot)}(\varsigma_n)\int_\Omega A_0(x,\varsigma_n,\tau_n)dx - hD_{\vec{p}(\cdot)}(\tau_n)\int_\Omega A_0(x,\varsigma_n,\tau_n)dx \\
&-hD_{\vec{q}(\cdot)}(\varsigma_n)\int_\Omega B_0(x,\varsigma_n,\tau_n)dx - hD_{\vec{q}(\cdot)}(\tau_n)\int_\Omega B_0(x,\varsigma_n,\tau_n)dx \\
&+\|\lambda_1\|_{L^1(\mathbb{R})}\Big[D_{\vec{p}(\cdot)}(\varsigma_n) + D_{\vec{q}(\cdot)}(\tau_n)\Big] + \|\lambda_2\|_{L^1(\mathbb{R})}\Big[D_{\vec{p}(\cdot)}(\varsigma_n) + D_{\vec{q}(\cdot)}(\tau_n)\Big] \\
&= \left(\|\lambda_1\|_{L^1(\mathbb{R})} + \|\lambda_2\|_{L^1(\mathbb{R})} - h\int_\Omega A_0(x,\varsigma_n,\tau_n)dx\right) D_{\vec{p}(\cdot)}(\varsigma_n) \\
&+ \left(\|\lambda_1\|_{L^1(\mathbb{R})} + \|\lambda_2\|_{L^1(\mathbb{R})} - h\int_\Omega B_0(x,\varsigma_n,\tau_n)dx\right) D_{\vec{q}(\cdot)}(\tau_n) \\
&-hD_{\vec{p}(\cdot)}(\tau_n)\int_\Omega A_0(x,\varsigma_n,\tau_n)dx - hD_{\vec{q}(\cdot)}(\varsigma_n)\int_\Omega B_0(x,\varsigma_n,\tau_n)dx \\
&< 0, \quad \forall n \in \mathbb{N}.
\end{aligned}
$$

Since $\Phi(u,v) + \Psi(u,v) = 0$ in virtue of the last inequality (u,v) can't be a local minimum of $\Phi + \Psi$.
Then, since (u,v) is the only global minimum of Ψ, the conclusion (c) of the Theorem 1.1 assures that there is a sequence of pairwise distinct critical points of $\Phi+\Psi$ such that $\lim_{n\to\infty}\Psi(x_n,y_n) = 0$ with $x_n, y_n \rightharpoonup 0$, thus $(x_n,y_n)_n$ must be in norm infinitesimal. □

As an immediate consequence of Theorem 3.13 we get the following corollary.

Corollary 3.14. *Let $(a_n)_n$ and $(b_n)_n$ be two sequences in $\mathbb{R}^+$ satisfying*

$$
b_n < a_n \quad \forall n \in \mathbb{N}, \lim_{n\to+\infty} a_n = 0, \qquad \lim_{n\to+\infty}\frac{a_n}{b_n} = +\infty,
$$

and let

$$
A_n = \Big\{(\xi,\zeta) \in \mathbb{R}^2 : \mu F_{\vec{p}(\cdot)}(\xi) + \nu F_{\vec{q}(\cdot)}(\eta) \leq a_n\Big\},
$$

$$
B_n = \Big\{(\xi,\zeta) \in \mathbb{R}^2 : \int_\Omega A_0(x,\xi,\zeta)dx D_{\vec{p}(\cdot)}(\xi) + \int_\Omega B_0(x,\xi,\zeta)dx D_{\vec{q}(\cdot)}(\zeta) \leq b_n\Big\},
$$

be such that $\sup_{A_n\setminus IntB_n} H \leq 0$ for all $n \in \mathbb{N}$.
Finally, let us assume that

$$
\limsup_{(\xi,\zeta)\to(0,0)} \frac{H(\xi,\zeta)\displaystyle\int_\Omega \eta(x)dx}{\Big[D_{\vec{p}(\cdot)}(\xi) + D_{\vec{p}(\cdot)}(\zeta)\Big]\displaystyle\int_\Omega A_0(x,\xi,\zeta)dx + \Big[D_{\vec{q}(\cdot)}(\xi) + D_{\vec{q}(\cdot)}(\zeta)\Big]\displaystyle\int_\Omega B_0(x,\xi,\zeta)dx}
$$

$$
> \big(\|\lambda_1\|_{L^1(\mathbb{R})} + \|\lambda_2\|_{L^1(\mathbb{R})}\big) \max_{(\xi,\zeta)\in\mathbb{R}^2}\left(\frac{1}{\displaystyle\int_\Omega A_0(x,\xi,\zeta)dx}, \frac{1}{\displaystyle\int_\Omega B_0(x,\xi,\zeta)dx}\right).
$$

Then, problem $(\mathcal{P})$ admits a sequence of non-zero weak solutions which strongly converges to (u,v) in X.

Proof. Likewise, by applying Theorem 1.1 part (c), we get the Corollary 3.14, whose proof will be omitted. □

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