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A fixed point approach to the semi-linear Stokes problem

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Abstract. The aim of this paper is to study the Dirichlet problem for semi-linear Stokes equations. The approach of this study is based on the operator method, using abstract results of nonlinear functional analysis. We first study the problem using Schauder's fixed point theorem and we prove the existence of a solution in case that the nonlinear term has a linear growth. Next we establish whether the existence of solutions can still be obtained without this linear growth restriction. Such a result is obtained by applying the Leray-Schauder fixed point theorem.

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1. Introduction

The field of fluid dynamics does not only engage the attention of mathematicians and physicists but also of astrophysicists, oceanographers and many others and this is due to the fact that it addresses real-world natural phenomena and tries to come up with mathematical models that help us to understand them.

An inertial fluid flow that is Newtonian, incompressible and homogeneous follows the Navier-Stokes equations, which are essentially derived from Newton's second law of motion applied to the fluid and the law of mass conservation in the context of constant density flow. If the velocity field is not time-dependent, then the flow is called steady, and it means that the fluid particles follow the streamlines, which do not change in time. Neglecting the nonlinear term in the Navier-Stokes system we get the Stokes system, which is in fact, the one that we are here interested in.

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The aim of this paper is to study the existence of solutions of the semi-linear Dirichlet problem for the steady Stokes system

$$\begin{cases} -\mu\Delta u + \nabla p = f(x, u(x)) & \text{in } \Omega \\ \text{div } u = 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.1}$$

2. Preliminaries

In this section we briefly recall without proof, some important results from functional analysis and some basic results regarding the Stokes system that are used in the forthcoming material. For additional details, we refer the reader to the following works [1, 2, 3, 4, 7, 8, 9, 10].

2.1. The Nemytskii operator

First we recall some properties of the Nemytskii superposition operator (see, e.g., [6]).

Definition 2.1 (Nemytskii operator). Let $\Omega \subset \mathbb{R}^N, N \geq 1$, be an open set and let $f : \Omega \times \mathbb{R}^n \to \mathbb{R}^m, n, m \geq 1$. By the Nemystkii operator associated to f we understand the operator N_f which, to each function $u : \Omega \to \mathbb{R}^n$, assigns $f \circ u$, that is

$$N_f u(x) = (f \circ u)(x) = f(x, u(x)), \text{ for } x \in \Omega.$$

Definition 2.2 (Carathéodory function). Let $\Omega \subset \mathbb{R}^N, N \geq 1$, be an open set. We say that $f : \Omega \times \mathbb{R}^n \to \mathbb{R}^m, n, m \geq 1$ is a Carathéodory function if it satisfies the following conditions:

(i) $x \mapsto f(x, y)$ is measurable in Ω for every $y \in \mathbb{R}^n$;
(ii) $y \mapsto f(x, y)$ is continuous on $\mathbb{R}^n$ for a.e. $x \in \Omega$.

Proposition 2.3 (see [7, Proposition 9.1]). *If f is a Carathéodory function, then the Nemystkii operator associated to the function f maps measurable functions into measurable functions.*

Theorem 2.4 (see [7, Theorem 9.1]). *Let $\Omega \subset \mathbb{R}^N$ be an open set, $f : \Omega \times \mathbb{R}^n \to \mathbb{R}^m$ and $1 \leq p, q < +\infty$. If f satisfies the Carathéodory conditions and there exists $a \in \mathbb{R}_+$ and $h \in L^q(\Omega; \mathbb{R}_+)$ such that*

$$||f(x, y)|| \leq a||y||^{\frac{p}{q}} + h(x)$$

for every $y \in \mathbb{R}^n$ and a.e. $x \in \Omega$, then the operator

$$N_f : L^p(\Omega; \mathbb{R}^n) \to L^q(\Omega; \mathbb{R}^m) \quad \text{given by } N_f(u) = f(\cdot, u)$$

is well defined, continuous and bounded. Moreover, the following inequality holds:

$$||N_f(u)||_{L^q} \leq a||u||_{L^p}^{\frac{p}{q}} + ||h||_{L^q} \quad \text{for all } u \in L^p(\Omega; \mathbb{R}^n).$$

2.2. Embedding results

The purpose of our work, namely the study of the existence of solutions for a semi-linear boundary valued problem, is achieved by looking for a weak solution which lead us to use continuous or compact embeddings of function spaces. In particular, we use the following embedding results due to Sobolev and Rellich-Kondrachov regarding the continuous and compact embeddings of Sobolev spaces into Lebesgues spaces.

Let $1 \leq q \leq +\infty$. Then the critical exponent associated to q is denoted by q^* and is defined by

$$\begin{cases} \dfrac{1}{q^*} = \dfrac{1}{q} - \dfrac{1}{n}, & q < n \\ q^* = +\infty, & q \geq n, \end{cases}$$

where by n is denoted the dimension of the space.

Theorem 2.5 (Sobolev). *Let $\Omega \subset \mathbb{R}^n$ be an open set of class C^1 (or $\Omega = \mathbb{R}^n$). Then the following continuous embeddings hold:*

a) $H^1(\Omega) \subset L^q(\Omega)$ *for every $q \in [2, 2^*]$, where $n \geq 3$.*
b) $H^1(\Omega) \subset L^q(\Omega)$ *for every $q \in [2, +\infty)$, if $n = 2$.*

Theorem 2.6 (Rellich-Kondrachov). *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set of class C^1.*

a) *If $n \geq 3$, then the embedding $H^1(\Omega) \subset L^q(\Omega)$ is compact for $q \in [1, 2^*)$, where $2^* := 2n/(n-2)$.*
b) *If $n = 2$, then the embedding $H^1(\Omega) \subset L^q(\Omega)$ is compact for every $q \in [1, +\infty)$.*

We recall that a real number λ is said to be an eigenvalue of the Dirichlet problem for $-\Delta$ if the problem

$$\begin{cases} -\Delta u = \lambda u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

has nonzero weak solutions.

Theorem 2.7 (Poincaré's inequality). *Let Ω be a bounded open set of $\mathbb{R}^n$. Then there exists a constant C that depends on Ω such that*

$$||u||_{L^2(\Omega)} \leq C||\nabla u||_{L^2}, \quad \textit{for every } u \in H_0^1(\Omega).$$

Due to this result, the Sobolev space $H_0^1(\Omega)$ can be endowed with an equivalent norm

$$||u||_{H_0^1} := ||\nabla u||_{L^2} = \left(\int_\Omega ||\nabla u||^2 \right)^{1/2}$$

that comes from the scalar product in $H_0^1(\Omega)$

$$(u, v)_{H_0^1} = (\nabla u, \nabla v)_{L^2} = \int_\Omega \nabla u \cdot \nabla v.$$

Hence in terms of the new norm, Poincaré's inequality can be written as

$$||u||_{L^2} \leq C||u||_{H_0^1}, \quad u \in H_0^1(\Omega).$$

Since the first eigenvalue of the Dirichlet problem for $-\Delta$ is

$$\lambda_1 = \inf_{u \in H_0^1(\Omega)\setminus\{0\}} \frac{||u||_{H_0^1}^2}{||u||_{L^2}^2},$$

it follows that the smallest constant C for which the Poincaré's inequality holds, is in fact $\dfrac{1}{\sqrt{\lambda_1}}$. Therefore,

$$||u||_{L^2} \leq \frac{1}{\sqrt{\lambda_1}} ||u||_{H_0^1}, \quad \text{for all } u \in H_0^1(\Omega).$$

Moreover, the Poincaré's inequality also holds for the embedding $L^2(\Omega) \subset H^{-1}(\Omega)$ with the same constant, namely

$$||u||_{H^{-1}} \leq \frac{1}{\sqrt{\lambda_1}} ||u||_{L^2}, \quad \text{for all } u \in L^2(\Omega).$$

For a more detailed exposition of these results we refer the reader to [7, Chapter 3].

Remark 2.8. Since we are concerned with n-dimensional vector-valued functions, we shall use the notations

$$L^p(\Omega) := (L^p(\Omega))^n, \qquad H^m(\Omega) := (H^m(\Omega))^n, \qquad H_0^m(\Omega) := (H_0^m(\Omega))^n.$$

2.3. The variational form of the Stokes system

Let us consider the Dirichlet problem for the steady non-homogeneous Stokes system

$$\begin{cases} -\mu\Delta u + \nabla p = f & \text{in } \Omega \\ \text{div } u = 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.1}$$

Here $\Omega \subset \mathbb{R}^n$ is a bounded open set, $\mu > 0$ is a constant representing the kinematic viscosity, $u : \Omega \to \mathbb{R}^n$ is the velocity field, p is the pressure and $f \in L^2(\Omega)$ is the external force. In this subsection we give the variational formulation of problem (2.1). For a very detailed way of getting to the variational form of the Stokes equation we refer the reader to [10].

We define the Hilbert space

$$V := \{v \in H_0^1(\Omega) : \text{ div } v = 0\},$$

endowed with the scalar product

$$(u, v)_V = \int_\Omega \nabla u \cdot \nabla v, \quad \text{for } u, v \in V$$

and the corresponding norm

$$\|u\|_V = \left(\int_\Omega |\nabla u|^2 \right)^{\frac{1}{2}}.$$

We can now state the variational formulation of problem (2.1):
Given $f \in L^2(\Omega)$ find $u \in V$ such that

$$\mu(u, v)_V = (f, v)_{L^2}, \quad \text{for all } v \in V. \tag{2.2}$$

Definition 2.9 (Weak solution). Let $f \in L^2(\Omega)$. By the *weak solution* of the Stokes problem (2.1) we mean a function $u_f \in V$ that satisfies (2.2).

One has the following embeddings:

$$V \subset H_0^1(\Omega) \subset L^2(\Omega) \subset H^{-1}(\Omega) \subset V'.$$

Then, by the Riesz's representation theorem, we can extend (2.2) so that for any $f \in V'$ there exist a unique $u_f \in V$ such that

$$\mu(u_f, v)_V = (f, v), \quad \text{for all } v \in V. \tag{2.3}$$

Notice that the notation (f, v), for $f \in V'$ and $v \in V$, stands for the value at v of the linear functional f.

Definition 2.10 (Solution operator). The operator $S : V' \to V$ defined by $Sf := u_f$ for any $f \in V'$ is called the *solution operator*.

If in (2.3) we take in particular $v := u_f$ we obtain

$$\|u_f\|_V^2 = \frac{1}{\mu}(f, u_f) \leq \frac{1}{\mu}\|f\|_{V'}\|u_f\|_V.$$

Hence we have $\|u_f\|_V \leq \mu^{-1}\|f\|_{V'}$, that is

$$\|Sf\|_V \leq \frac{1}{\mu}\|f\|_{V'}.$$

Thus the linear operator S is continuous from V' to V.

Remark 2.11. Note that the existence of the pressure p is guaranteed as a consequence of De Rham's Lemma.

3. Main results

Let us now turn back to the semi-linear problem (1.1), where $\Omega \subset \mathbb{R}^n, n \geq 2$ is a bounded open set, $f : \Omega \times \mathbb{R}^n \to \mathbb{R}^n$, $p : \Omega \to \mathbb{R}$.
We seek weak solutions, i.e., functions $u \in V$ such that

$$f(\cdot, u(\cdot)) \in H^{-1}(\Omega)$$

and

$$\mu(u, v)_V = (f(\cdot, u), v) \;\text{ for all } v \in V.$$

The system (1.1) can be written as an equivalent fixed point equation

$$u = T(u), \quad u \in V,$$

where

$$T := S \circ F,$$

where $F : V \to V'$, $F(u) = f(\cdot, u(\cdot))$.

3.1. Application of Schauder's fixed point theorem

In this section we find sufficient conditions that assure the existence of a solution of problem (1.1), having in mind Schauder's fixed point theorem on the space V.

First we show that T *is a completely continuous operator*. In order to do so we would like to have the representation of $F = I \circ N_f \circ P$, where

- $P : V \to L^2(\Omega),\ Pu = u;$
- $N_f : L^2(\Omega) \to L^2(\Omega),\ N_f(w) = f(\cdot, w(\cdot));$
- $I : L^2(\Omega) \to V',\ I(v) = (v, \cdot)_{L^2}.$

Let us observe that by the Theorem 2.6, the embedding $H_0^1(\Omega) \subset L^2(\Omega)$ is continuous. Then it follows that P is a continuous linear operator, hence bounded. Also, since the embedding $L^2(\Omega) \subset H^{-1}(\Omega)$ is compact it follows that operator I is a completely continuous linear operator. It remains to see whether the operator $N_f : L^2(\Omega) \to L^2(\Omega)$ is well-defined. For this purpose let us assume that f is a Carathéodory function. Hence, for any $w \in L^2(\Omega)$, $N_f(w)$ is also measurable. We impose a linear growth condition on f, that is

$$||f(x,u)|| \leq a||u|| + k(x), \quad \text{for all } u \in \mathbb{R}^n \text{ and a.e. } x \in \Omega, \tag{3.1}$$

for some $k \in L^2(\Omega, \mathbb{R}_+)$ and $a \in \mathbb{R}_+$. Then, we have

$$||N_f(w)(x)|| \leq a||w(x)|| + k(x), \quad \text{for a.e. } x \in \Omega.$$

Hence, by these assumptions over f, it follows that N_f is well-defined, continuous and bounded.

Due to the boundedness of the operators P and N_f, it follows that $N_f \circ P$ is bounded too. Therefore, since I is completely continuous, it follows that the operator F is completely continuous from V to V'. Next, by the linearity and continuity of the solution operator S we have that $T = S \circ F$ is completely continuous from V to itself.

Secondly, we show that T *is a self-map of a closed ball of* V. To this purpose, let $u \in V$. Notice that for every $h \in H^{-1}(\Omega)$ one has

$$||h||_{V'} \leq ||h||_{H^{-1}}.$$

Indeed, since $V \subset H_0^1(\Omega)$ we have that

$$||h||_{V'} = \sup_{v \in V} \frac{|(h,v)|}{||v||_V} \leq \sup_{v \in H_0^1(\Omega)} \frac{|(h,v)|}{||v||_{H_0^1}} = ||h||_{H^{-1}}.$$

Then, since the operator S is linear and continuous and also by the Poincaré´s inequality we have

$$\begin{aligned}\|T(u)\|_V = \|S \circ F(u)\|_V &\leq \frac{1}{\mu}\|F(u)\|_{V'} \leq \frac{1}{\mu}||F(u)||_{H^{-1}} \\ &\leq \frac{1}{\mu\sqrt{\lambda_1}}\|F(u)\|_{L^2} = \frac{1}{\sqrt{\mu\lambda_1}}\|f(\cdot, u(\cdot))\|_{L^2}.\end{aligned}$$

By the growth condition (3.1), we deduce that

$$||T(u)||_V \leq \frac{a}{\mu\sqrt{\lambda_1}}||u||_{L^2} + \frac{1}{\mu\sqrt{\lambda_1}} \cdot ||k||_{L^2}.$$

Since $u \in V$, we can apply again the Poincaré inequality and we obtain that

$$||T(u)||_V \leq \frac{a}{\mu\lambda_1}||u||_V + \frac{1}{\mu\sqrt{\lambda_1}} \cdot ||k||_{L^2}.$$

In the end, we assume that $\frac{a}{\mu\lambda_1} < 1$ so that there exists a radius $r > 0$ such that if $||u||_V \leq r$ then $||T(u)||_V \leq r$. Indeed, from $\frac{a}{\mu\lambda_1} < 1$ and $||u||_V \leq r$ we have that

$$||T(u)||_V \leq \frac{a}{\mu\lambda_1}r + \frac{1}{\mu\lambda_1}||k||_{L^2} \leq r \quad \text{for } r > 0 \text{ large enough.}$$

Hence, $||T(u)||_V \leq r$.
Therefore, based on Schauder's fixed point theorem we can state the following result:

Theorem 3.1. *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set and $f : \Omega \times \mathbb{R}^n \to \mathbb{R}^n$ be a mapping such that:*

(a) *f is a Carathéodory function;*
(b) *there is a positive constant a and $k \in L^2(\Omega, \mathbb{R}_+)$ such that*

$$||f(x, u)|| \leq a||u|| + k(x) \quad \textit{for all } u \in \mathbb{R}^n \textit{ and a.e. } x \in \Omega.$$

Also, assume that $\frac{a}{\mu\lambda_1} < 1$. Then the semi-linear Stokes problem (1.1) has at least one solution (u, p) with $u \in V$.

3.2. Application of Lerray-Schauder's fixed point theorem

In this section we consider more generally that the right hand side of the problem (1.1) is of the form $f_0 + f_1(\cdot, u(\cdot))$, where $f_0 \in H^{-1}(\Omega)$ and $f_1 : \Omega \times \mathbb{R}^n \to \mathbb{R}^n$. As before, the problem can be written as an equivalent fixed point equation

$$u = T(u),$$

where this time

$$T = S \circ (F + f_0),$$

with

$$F : V \to V', \quad F(u) = f_1(\cdot, u(\cdot)).$$

We are now interested if one can still obtain the existence of the solution of the new problem without a linear growth restriction. We shall see this is possible due to the Lerray-Schauder's fixed point theorem (see [5]).

We first guarantee the *complete continuity* of the operator T. The idea we follow here is similar to the one in the previous section: since the operator S is linear and bounded, in order for $T = S \circ F$ to be completely continuous, we need that the operator F is completely continuous. To this end, we write the operator F as $F = I \circ N_{f_1} \circ P$, where

- $P : V \to L^{2^*}(\Omega), \quad Pu = u;$
- $N_{f_1} : L^{2^*}(\Omega) \to L^q(\Omega), \quad N_{f_1}(\omega) = f_1(\cdot, \omega);$
- $I : L^q(\Omega) \to V', \quad I(v) = (v, \cdot)_{L^2}, \quad$ for some $q \in ((2^*)', +\infty)$.

Due to Theorem 2.6 it follows that the embedding $(V \subset) H_0^1(\Omega) \subset L^{2^*}(\Omega)$ is continuous. Therefore the operator P is a continuous linear operator, hence bounded. Since $H_0^1(\Omega) \subset L^p(\Omega)$ is compact for $p \in [1, 2^*)$, passing to duals, we get that $L^q(\Omega) \subset H^{-1}(\Omega)$ is also compact for $q \in ((2^*)', +\infty)$, where $(2^*)' = 2n/(n+2)$ is the conjugate of 2^*. Therefore, if $q > (2^*)'$, the inclusion operator I is completely continuous. We now show the operator N_{f_1} is well-defined. For this purpose we will make use of Theorem 2.4. In view of this result, we need to impose a growth condition on the function f_1, namely, for some $a \in \mathbb{R}_+$ and $\overline{h} \in L^q(\Omega)$ to have

$$||f_1(x, \omega)|| \leq a||\omega||^{\frac{2^*}{q}} + \overline{h}(x),$$

for every $\omega \in \mathbb{R}^n$ and a.e. $x \in \Omega$. To this aim, it sufficies to have

$$||f_1(x, \omega)|| \leq a||\omega||^{\alpha} + h(x). \tag{3.2}$$

for some $\alpha \in [1, 2^*/q]$ and $h \in L^q(\Omega)$. Note that from $\alpha \leq 2^*/q$ it follows that $q \leq 2^*/\alpha$. Together with the condition $q > (2^*)'$, this shows that

$$(2^*)' < q \leq \frac{2^*}{\alpha},$$

and so, such a q exists if

$$\alpha < \frac{2^*}{(2^*)'} = \frac{n+2}{n-2}.$$

Thus, the condition (3.2) holds for $\alpha \in [1, (n+2)/(n-2))$; hence, we can let $h \in L^{2^*/\alpha}(\Omega)$.

Then from Theorem 2.4 it follows that the Nemytskii operator N_{f_1} is well defined, continuous and bounded. Therefore, the operator F is well-defined and completely continuous. Hence the operator T is completely continuous.

Finally, we carry on with the *a priori bounds* of solutions, that is to show there is a positive constant $R > 0$ such that $||u||_V < R$ for any solution $u \in V$ of the equation $\lambda T(u) = u$ and any $\lambda \in (0, 1)$. Let $u \in V$ be any solution of the equation $\lambda T(u) = u$ for some $\lambda \in (0, 1)$. Thus, u is a weak solution of the problem

$$\begin{cases} -\mu\Delta u = -\nabla p + \lambda f_0(x) + \lambda f_1(x, u) & \text{in } \Omega \\ \text{div } u = 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.3}$$

Therefore

$$(u, v)_V = \frac{\lambda}{\mu}(f_0(\cdot) + f_1(\cdot, u(\cdot)), v), \quad \text{for any } v \in V.$$

If we take in particular $v = u$ we obtain

$$||u||_V^2 = \frac{\lambda}{\mu}(f_0 + f_1(\cdot, u), u) = \frac{\lambda}{\mu}(f_0, u) + \frac{\lambda}{\mu}(f_1(\cdot, u), u).$$

Note that since $f_1(\cdot, u(\cdot)) \in L^{(2^*)'}(\Omega)$, one has

$$(f_1(\cdot, u), u) = \int_\Omega u(x) \cdot f_1(x, u(x)).$$

Let us now assume that there exists a positive constant k such that

$$y \cdot f_1(x, y) \leq k||y||^2, \quad \text{for all } y \in \mathbb{R}^n \text{ and a.e. } x \in \Omega.$$

Then $(f_1(\cdot, u), u) \leq k||u||^2_{L^2}$ and using Poincaré's inequality we obtain

$$\begin{aligned} ||u||^2_V &= \frac{\lambda}{\mu}(f_0, u) + \frac{\lambda}{\mu}\int_\Omega u(x) \cdot f_1(x, u(x)) \\ &\leq \frac{\lambda}{\mu}\left(||f_0||_{H^{-1}}||u||_V + k||u||^2_{L^2}\right) \\ &< \frac{1}{\mu}||f_0||_{H^{-1}}||u||_V + \frac{k}{\mu}||u||^2_{L^2} \\ &\leq \frac{1}{\mu}||f_0||_{H^{-1}}||u||_V + \frac{k}{\mu\lambda_1}||u||^2_V. \end{aligned}$$

Hence, we have

$$||u||_V \left(1 - \frac{k}{\mu\lambda_1}\right) \leq \frac{1}{\mu}||f_0||_{H^{-1}}.$$

Assuming that $k < \mu\lambda_1$ it follows that

$$||u||_V < \frac{\lambda_1}{\mu\lambda_1 - k}||f_0||_{H^{-1}} := R.$$

Therefore, we can state the following result:

Theorem 3.2. *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set, $f = f_0 + f_1$ with $f_0 \in H^{-1}(\Omega)$ and f_1 a function such that*

(a) *f_1 is a Carathéodory function;*

(b) *there is a positive constant a and $\alpha \in [1, (n+2)/(n-2))$ and a function $h \in L^{2^*/\alpha}(\Omega)$ such that*

$$||f_1(x, u)|| \leq a||u||^\alpha + h(x),$$

for any $u \in \mathbb{R}^n$ and a.e. $x \in \Omega$;

(c) *there is a positive constant $k < \mu\lambda_1$ such that the condition*

$$y \cdot f_1(x, y) \leq k||y||^2$$

holds for any $y \in \mathbb{R}^n$ and a.e. $x \in \Omega$.

Then the problem (3.3) has at least one solution (u, p) with $u \in V$ and

$$||u||_V \leq \frac{\lambda_1}{\mu\lambda_1 - k}||f_0||_{H^{-1}}.$$

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