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Around metric coincidence point theory

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Dedicated to Prof. Adrian Petruşel on the occasion of his 60^{th} *anniversary*

Abstract. Let (X,d) be a complete metric space, (Y,ρ) be a metric space and $f,g:X\to Y$ be two mappings. The problem is to give metric conditions which imply that, $C(f,g):=\{x\in X \mid f(x)=g(x)\}\neq\emptyset$. In this paper we give an abstract coincidence point result with respect to which some results such as of Peetre-Rus (I.A. Rus, *Teoria punctului fix în analiza funcţională*, Babeş-Bolyai Univ., Cluj-Napoca, 1973), A. Buică (A. Buică, *Principii de coincidenţă şi aplicaţii*, Presa Univ. Clujeană, Cluj-Napoca, 2001) and A.V. Arutyunov (A.V. Arutyunov, *Covering mappings in metric spaces and fixed points*, Dokl. Math., 76(2007), no.2, 665-668) appear as corollaries. In the case of multivalued mappings our result generalizes some results given by A.V. Arutyunov and by A. Petruşel (A. Petruşel, *A generalization of Peetre-Rus theorem*, Studia Univ. Babeş-Bolyai Math., 35(1990), 81-85). The impact on metric fixed point theory is also studied.

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1. Introduction

Let (X,d) be a complete metric space, (Y,ρ) be a metric space and $f,g:X\to Y$ be continuous mappings. The following results are well known:

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Peetre-Rus' Theorem ([42]). *We suppose that there exist two mappings* $\varphi, \psi : \mathbb{R}_+ \to \mathbb{R}_+$ *and* $M > 0$ *such that:*

(1) *there exists* $x \in X$ *such that,* $\rho(f(x), g(x)) \leq M$*;*
(2) *for each* $x \in X$ *with* $\rho(f(x), g(x)) \leq M$*, there exists* $x_1 \in X$ *such that,*

$$\rho(f(x_1), g(x_1)) \leq \varphi(\rho(f(x), g(x)))$$

and

$$d(x, x_1) \leq \psi(\rho(f(x), g(x)));$$

(3) φ *and* ψ *are increasing,* $\varphi(M) \leq M$*,* $\varphi^n(M) \to 0$ *as* $n \to \infty$ *and*

$$\sum_{i=0}^{\infty} \psi(\varphi^i(M)) < +\infty.$$

In these conditions, $C(f, g) := \{x \in X \mid f(x) = g(x)\} \neq \emptyset$*.*

Buică's Theorem ([15]; see also [2]). *We suppose that there exists,* $0 < l < 1$*,* $k > 0$ *and* $h : X \to X$ *such that:*

$$\rho(f(h(x)), g(h(x))) \leq l\rho(f(x), g(x)), \ \forall \ x \in X,$$

and

$$d(x, h(x)) \leq k\rho(f(x), g(x)), \ \forall \ x \in X.$$

Then we have that:

(*i*) $C(f, g) \neq \emptyset$*;*
(*ii*) *for each* $x_0 \in X$*,* $h^n(x_0) \to x^*(x_0)$ *as* $n \to \infty$ *and* $x^*(x_0) \in C(f, g)$*;*
(*iii*) $d(x_0, x^*(x_0)) \leq \frac{k}{1-l}\rho(f(x_0), g(x_0))$*,* $\forall \ x_0 \in X$*.*

Arutyunov's Theorem ([5]). *We suppose that:*

(1) f *is* α*-covering with* $\alpha > 0$*, i.e.,* $B_Y(f(x), \alpha r) \subset f(B_X(x, r))$*,* $\forall \ x \in X$*,* $\forall \ r > 0$*;*
(2) g *is L-Lipschitz with* $L < \alpha$*.*

Then for any $x_0 \in X$*, there exists* $x^*(x_0) \in X$ *such that:*

(*i*) $f(x^*(x_0)) = g(x^*(x_0))$*;*
(*ii*) $d(x_0, x^*(x_0)) \leq (\alpha - L)^{-1}\rho(f(x_0), g(x_0))$*,* $\forall \ x_0 \in X$*.*

In this paper we give an abstract result with respect to which the above results appear as corollaries. In the last section we present a similar result in the case of multivalued mappings, result which generalizes some results given by A.V. Arutyunov ([5]) and by A. Petruşel ([35]).

The impact of our results on metric fixed point theory is also analyzed.

The paper has the following structure:

2. Preliminaries
    2.1. Comparison functions
    2.2. Pre-weakly Picard mappings
    2.3. Covering mappings
    2.4. Conditions, on a functional on metric space, weaker then continuity
3. Basic coincidence point results in metric spaces
4. Ulam-Hyers stability of a coincidence point equation

5. Well-posedness of the coincidence point problem
6. The case of multivalued mappings

2. Preliminaries

2.1. Comparison functions

For $M \in]0, +\infty]$, a function $\varphi : [0, M[\to [0, M[$ is called:

(a) a *comparison function on* $[0, M[$ if φ is increasing and $\varphi^n(t) \to 0$ as $n \to \infty$, $\forall\, t \in [0, M[$;

(b) a *strong comparison function on* $[0, M[$ if φ is a comparison function on $[0, M[$ and

$$\sum_{i=0}^{\infty} \varphi^i(t) < +\infty, \ \forall\, t \in [0, M[.$$

We remark that if φ is a comparison function on $[0, M[$ then, $\varphi(t) < t$, $\forall\, t \in]0, M[$ and $\varphi(0) = 0$, i.e., φ is a Picard function.

Now, let $\varphi : [0, M[\to [0, M[$ and $\psi : [0, M[\to \mathbb{R}_+$ be two functions. By definition, the pair (φ, ψ) is a *comparison pair on* $[0, M[$ if:

(1) φ is a comparison function on $[0, M[$;

(2) ψ is increasing, $\psi(0) = 0$ and ψ is continuous in 0;

(3) $\sum_{i=0}^{\infty} \psi(\varphi^i(t)) < +\infty$, $\forall\, t \in [0, M[$.

Example 2.1. For each $M \in]0, +\infty]$, if $\varphi(t) := lt$, where $0 < l < 1$ and $\psi(t) := kt$, where $k > 0$ and $t \in [0, M[$, then the pair (φ, ψ) is a comparison pair on $[0, M[$. In this case, $\sum_{i=0}^{\infty} \psi(\varphi^i(t)) = \frac{kt}{1-l}$, $\forall\, t \in [0, M[$.

Example 2.2. If $\varphi : [0, M[\to [0, M[$ is a strong comparison function on $[0, M[$ and $\psi(t) := kt$, $\forall\, t \in [0, M[$, with $k > 0$, then the pair (φ, ψ) is a comparison pair on $[0, M[$.

2.2. Pre-weakly Picard mappings

Let (X, d) be a metric space. By definition, a mapping $f : X \to X$ is a *pre-weakly Picard mapping* (*pre-WPM*) if the sequence $(f^n(x))_{n\in\mathbb{N}}$ is a convergent sequence for all $x \in X$.

If $f : X \to X$ is pre-WPM, then we consider the mapping $f^\infty : X \to X$, defined by $f^\infty(x) := \lim_{n\to\infty} f^n(x)$.

By definition, if $f : X \to X$ is pre-WPM with

$$f^\infty(x) \in F_f := \{x \in X \mid f(x) = x\}, \ \forall\, x \in X,$$

then f is a *weakly Picard mapping* (*WPM*).

Example 2.3. If (X, d) is a complete metric space and $f : X \to X$ is a *graphic contraction* (i.e., $d(f^2(x), f(x)) \le ld(x, f(x))$, $\forall\, x \in X$ with $0 < l < 1$) then f is a pre-WPM (see [13] and [47]).

Example 2.4. If (X,d) is a complete metric space and $f : X \to X$ is a *Caristi mapping* (i.e., $d(x,f(x)) \le \theta(x) - \theta(f(x))$, $\forall\, x \in X$, with the functional $\theta : X \to \mathbb{R}_+$) then f is a pre-WPM (see [13] and [47]).

Example 2.5. Let $f : \mathbb{R} \to \mathbb{R}$ be a function defined by

$$f(x) := \begin{cases} \frac{1}{2}x, & \text{for } x \in \mathbb{R} \setminus \mathbb{Q} \\ \frac{1}{2}(x+1), & \text{for } x \in \mathbb{Q} \end{cases}.$$

In this case:

(a) f is pre-WPM;

(b) $f^\infty(x) = \begin{cases} 0, & \text{for } x \in \mathbb{R}\setminus\mathbb{Q} \\ 1, & \text{for } x \in \mathbb{Q} \end{cases}$;

(c) $F_f = \{1\}$;

(d) $f^\infty(0) = 1$ and $f^\infty(1) = 1$.

In the sequel of our paper we need the following result.

Invariant partition lemma. *Let (X,d) be a metric space. If $f : X \to X$ is pre-WPM, then there exists a partition of X,*

$$X = \bigcup_{u \in f^\infty(X)} X_u$$

such that $f(X_u) \subset X_u$.

Proof. For $u \in f^\infty(X)$, we take $X_u := \{x \in X \mid f^n(x) \to u \text{ as } n \to \infty\}$. □

2.3. Covering mappings

If (X,d) is a metric space, then we denote by

$B_X(x,r) := \{u \in X \mid d(x,u) < r\}$ - the open ball of radius $r \in \mathbb{R}_+^*$, centered at $x \in X$;

$\tilde{B}_X(x,r) := \{u \in X \mid d(x,u) \le r\}$ - the closed ball of radius $r \in \mathbb{R}_+$, centered at $x \in X$.

Let (X,d) and (Y,ρ) be two metric spaces, $f : X \to Y$ be a mapping and $\alpha \in \mathbb{R}_+^*$. By definition, f is an *α-covering mapping* if,

$$\tilde{B}_Y(f(x), \alpha r) \subset f(\tilde{B}_X(x,r)),\ \forall\, x \in X,\ \forall\, r \in \mathbb{R}_+. \tag{CV}$$

The condition (CV) is equivalent with each of the following ones:

(CV_1) For all $r \in \mathbb{R}_+$ the following implication holds,
$x \in X$, $y \in Y$ and $\rho(f(x),y) \le \alpha r \Rightarrow$ there exists $x_1 \in X$ such that, $f(x_1) = y$ and $d(x,x_1) \le r$;

(CV_2) For all $r \in \mathbb{R}_+$, the following implication holds,
$x \in X$, $y \in Y$ and $\rho(f(x),y) \le r \Rightarrow$ there exists $x_1 \in X$ such that, $f(x_1) = y$ and $d(x,x_1) \le \frac{r}{\alpha}$.

It is clear that each covering mapping is surjective.

For more considerations on covering mappings (also named *open with linear rate*) and its relations with metric regularity see [2]-[8], [17], [54], [55], [14], [25], [18], [16].

2.4. Conditions, on a functional on metric space, weaker than continuity

Let (X,d) be a metric space and $F : X \to \mathbb{R}$ be a functional. By definition (Angrisani [3], Kirk-Saliga [26], Aamri-Chaira [1]), the functional F is called a *regular-global-inf (r.g.i.)* if for each $x \in X$, $F(x) > \inf\limits_X F := \inf\{F(u) \mid u \in X\}$ implies that there exist $\varepsilon > 0$ such that, $\varepsilon < F(x) - \inf\limits_X F$, and a neighborhood $V(x)$ of x, such that, $F(y) > F(x) - \varepsilon$, for each $y \in V(x)$.

From this definition it follows that:

(1) ([26]) The functional F is an r.g.i. on X if and only if for any sequence $(x_n)_{n\in\mathbb{N}} \subset X$, we have the following implication:
$$x_n \to x^* \text{ and } F(x_n) \to \inf_X F \Rightarrow F(x^*) = \inf_X F;$$

(2) If $F : X \to \mathbb{R}_+$ is an r.g.i. on X, $(x_n)_{n\in\mathbb{N}} \subset X$ with $x_n \to x^*$ and $F(x_n) \to 0$, then $F(x^*) = 0$.

We also have:

(3) If $F : X \to \mathbb{R}_+$ is a lower semicontinuous (l.s.c.) functional and $(x_n)_{n\in\mathbb{N}} \subset X$, then the following implication holds:
$$x_n \to x^* \text{ and } F(x_n) \to 0 \Rightarrow F(x^*) = 0.$$

3. Basic coincidence point results in metric spaces

Let (X,d) and (Y,ρ) be two metric spaces and $f,g : X \to Y$ be two mappings. For $M \in]0,+\infty]$ we denote by $X_M := \{x \in X \mid \rho(f(x),g(x)) < M\}$. We remark that:
$$X_\infty = X \text{ and } C(f,g) \subset X_M,\ \forall\ M \in]0,\infty].$$

More general, if $\lambda : Y \times Y \to \mathbb{R}_+$ is a functional, we denote by
$$X_M := \{x \in X \mid \lambda(f(x),g(x)) < M\}.$$

For some M, X_M may be $\emptyset$. For example, for $f,g : \mathbb{R} \to \mathbb{R}$,
$$f(x) = x,\ g(x) = x+1,\ X_M = \emptyset,$$
for $M \le 1$ and $X_M = \mathbb{R}$, for $M \in]1,+\infty]$.

Our basic result, in the case of singlevalued mappings, is the following one:

Theorem 3.1. *Let (X,d) be a complete metric space, (Y,ρ) be a metric space, $f,g : X \to Y$ be two mappings, $M \in]0,+\infty]$ and $\lambda : Y \times Y \to \mathbb{R}_+$ be a functional. We suppose that:*

(1) *$X_M := \{x \in X \mid \lambda(f(x),g(x)) < M\} \neq \emptyset$;*
(2) *The coincidence point λ-displacement, $\lambda_{f,g} : X_M \to \mathbb{R}_+$, $\lambda_{f,g}(x) := \lambda(f(x),g(x))$ is l.s.c.;*
(3) *There exists a comparison pair, (φ,ψ), on $[0,M[$ with respect to which, for each, $x \in X_M$ there exists $x_1 \in X_M$ such that:*
    (a) *$\lambda(f(x_1),g(x_1)) \le \varphi(\lambda(f(x),g(x)))$;*
    (b) *$d(x,x_1) \le \psi(\lambda(f(x),g(x)))$.*

Then there exists a pre-WPM, $h : X_M \to X_M$ such that:

(*i*) $\lambda(f(h^\infty(x)), g(h^\infty(x))) = 0$, $\forall\, x \in X_M$;

(*ii*) $d(x, h^\infty(x)) \leq \sum_{i=0}^{\infty} \psi(\varphi^i(\lambda(f(x), g(x))))$, $\forall\, x \in X_M$;

(*iii*) *If in addition,*

$$u, v \in Y,\ \lambda(u, v) = 0 \ \Rightarrow\ u = v,$$

then, $h^\infty(x) \in C(f, g)$, $\forall\, x \in X_M$, *i.e.,* $C(f, g) \neq \emptyset$.

Proof. From (a) and (b) there exists $h : X_M \to X_M$ such that,

$$\lambda(f(h(x)), g(h(x))) \leq \varphi(\lambda(f(x), g(x))),\ \forall\, x \in X_M,$$

and

$$d(x, h(x)) \leq \psi(\lambda(f(x), g(x))),\ \forall\, x \in X_M.$$

These imply that,

$$\lambda(f(h^n(x)), h^{n+1}(x)) \leq \varphi^n(\lambda(f(x), g(x))) \to 0 \text{ as } n \to \infty,\ \forall\, x \in X_M$$

and

$$d(h^n(x), h^{n+1}(x)) \leq \psi(\varphi^n(\lambda(f(x), g(x)))),\ \forall\, x \in X_M.$$

Since (X, d) is a complete metric space and (φ, ψ) is a comparison pair on $[0, M[$, it follows that h is pre-WPM.

On the other hand, from (2) we have,

$$0 \leq \lambda(f(h^\infty(x)), g(h^\infty(x))) \leq \underline{\lim}_{n\to\infty} \lambda(f(h^n(x)), g(h^n(x))) = 0.$$

It is clear that, from the above considerations, we have (i), (ii) and (iii). □

Remark 3.2. If f and g are continuous, $M < +\infty$, $\lambda := \rho$, then from Theorem 3.1 we have Peetre-Rus' theorem.

Remark 3.3. If f and g are continuous, $M = +\infty$, $\lambda := \rho$, $\varphi(t) := lt$, where $0 < l < 1$ and $\psi(t) := kt$, with $k > 0$, $\forall\, t \in [0, M[$, then from Theorem 3.1 we have Buică's theorem.

Remark 3.4. Let f and g be as in Arutyunov's theorem. Since f is α-covering, for $r := \frac{t}{\alpha}$ we have that, if $x \in X$, $y \in Y$ with $\rho(f(x), y) \leq t$, there exists $x_1 \in X$ such that, $f(x_1) = y$ and $d(x_1, x) \leq \frac{t}{\alpha}$. So, $\rho(f(x), f(x_1)) \leq t$.

If we take, $t := \rho(f(x), g(x))$ and $y := g(x) = f(x_1)$, we have

$$\rho(f(x_1), g(x_1)) \leq \frac{L}{\alpha}\rho(f(x), g(x)) \text{ and } d(x, x_1) \leq \frac{1}{\alpha}\rho(f(x), g(x)).$$

If we take in Theorem 3.1, f continuous and α-covering, g L-Lipschitz with $L < \alpha$, $M := +\infty$, $\lambda := \rho$, $\varphi(t) := \frac{L}{\alpha}t$ and $\psi(t) := \frac{t}{\alpha}$, we have Arutyunov's theorem. Moreover, from the above proof, we have:

Theorem 3.5. *Let (X, d) be a complete metric space, (Y, ρ) be a metric space, $f : X \to Y$ be continuous, $g : X \to Y$ be L-Lipschitz and $\alpha > 0$ with $L < \alpha$. We suppose that the following implication holds:*

$x \in X$, $y \in Y$, $\rho(f(x), y) \leq \rho(f(x), g(x)) \Rightarrow$ *there exists* $x_1 \in X$ *such that,* $f(x_1) = y$ *and* $d(x, x_1) \leq \frac{1}{\alpha}\rho(f(x), g(x))$.

Then there exists a pre-WPM, $h : X \to X$ *such that:*

(*i*) $h^\infty(x) \in C(f,g)$, $\forall\, x \in X$;
(*ii*) $d(x, h^\infty(x)) \le (\alpha - L)^{-1}\rho(f(x), g(x))$.

Remark 3.6. Let us consider in Theorem 3.1, $M := +\infty$, $\lambda := \rho$, $\varphi(t) := lt$, where $0 < l < 1$, $\psi(t) := kt$, with $k > 0$, $\forall\, t \in [0, M[$. In this case Theorem 3.1 takes the following form:

Theorem 3.7. *Let (X,d) be a complete metric space, (Y,ρ) be a metric space, $f, g : X \to Y$ be two mappings. We suppose that:*

($2'$) *The coincidence point displacement, $\rho_{f,g} : X \to \mathbb{R}_+$, $x \mapsto \rho(f(x), g(x))$ is l.s.c.;*
($3'$) *There exist $0 < l < 1$ and $k > 0$ w.r.t. which for each $x \in X$ there exists $x_1 \in X$ such that:*
  (a') $\rho(f(x_1), g(x_1)) \le l\rho(f(x), g(x))$;
  (b') $d(x, x_1) \le k\rho(f(x), g(x))$.

Then there exists a pre-WPM, $h : X \to X$, such that:

(i') $h^\infty(x) \in C(f,g)$, $\forall\, x \in X$, *i.e.,* $C(f,g) \neq \emptyset$;
(ii') $d(x, h^\infty(x)) \le \frac{k}{1-l}\rho(f(x), g(x))$, $\forall\, x \in X$.

Remark 3.8. If in Theorem 3.7 we take, $Y := X$ and $g := 1_X$ we have the following result:

Theorem 3.9. *Let (X,d) be a complete metric space, ρ be a metric on X and $f : X \to X$ be a mapping. We suppose that:*

($2''$) *The fixed point displacement, $\rho_f : (X,d) \to \mathbb{R}_+$, $\rho_f(x) := \rho(x, f(x))$, is l.s.c.;*
($3''$) *There exist $0 < l < 1$ and $k > 0$ w.r.t. which for each $x \in X$ there exists $x_1 \in X$ such that:*
  (a'') $\rho(x_1, f(x_1)) \le l\rho(x, f(x))$;
  (b'') $d(x, x_1) \le k\rho(x, f(x))$.

Then there exists a pre-WPM, $h : (X,d) \to (X,d)$ such that:

(i'') $h^\infty(x) \in F_f$, $\forall\, x \in X$, *i.e.,* $F_f \neq \emptyset$;
(ii'') $d(x, h^\infty(x)) \le \frac{k}{1-l}\rho(x, f(x))$, $\forall\, x \in X$.

Remark 3.10. If in Theorem 3.9 we take, $\rho := d$ and f and l-graphic contraction, then we have:

Theorem 3.11. *Let (X,d) be a complete metric space and $f : X \to X$ be an l-graphic contraction. If the fixed point displacement $d_f : X \to \mathbb{R}_+$, $x \mapsto d(x, f(x))$ is l.s.c., then f is a WPM and*

$$d(x, f^\infty(x)) \le \frac{1}{1-l} d(x, f(x)),\ \forall\, x \in X.$$

Proof. We take, $h(x) := f(x)$. □

Remark 3.12. If in Theorem 3.9 we take, $\rho := d$ and f an l-contraction, then we have the following variant of contraction principle:

Theorem 3.13. *Let (X,d) be a complete metric space and $f : X \to X$ be an l-contraction. Then we have that:*

(*i*) $F_f = F_{f^n} = \{x^*\}$, $\forall\ n \in \mathbb{N}^*$;
(*ii*) $f^n(x) \to x^*$ *as* $n \to \infty$, $\forall\ x \in X$;
(*iii*) $d(x,x^*) \leq \frac{1}{1-l} d(x,f(x))$, $\forall\ x \in X$.

The above consideration give rise to the following questions:

Problem 3.14. To translate Arutyunov's theorem in terms of metric regularity.

References: [17], [14], [25], [18], [16].

Problem 3.15. Which metric conditions on $f : X \to X$ imply that:
(*i*) f is a graphic contraction ?
(*ii*) $d_f : X \to \mathbb{R}_+$, $d_f(x) := d(x,f(x))$ is l.s.c. ?

References: [47], [49], [27], [32], [28], [1], [21], [26], [12], [44].

Problem 3.16. Let (X,d) be a metric space and $f : X \to X$ be a mapping. The problem is to compare the following conditions on f:
(1) the graphic of f is closed;
(2) f is orbitally continuous;
(3) the fixed point displacement of f, $d_f : X \to X$, $d_f(x) := d(x,f(x))$ is l.s.c.

References: [1], [47], [27], [32], [21], [26], [3].

4. Ulam-Hyers stability of coincidence point equations

Let (X,d) and (Y,ρ) be two metric spaces, $f,g : X \to Y$ be two mappings and $\lambda : Y \times Y \to \mathbb{R}_+$ be such that the following implications hold,

$$u,v \in Y,\ \lambda(u,v) = 0 \ \Leftrightarrow\ u = v.$$

Let us consider the coincidence point equation,

$$f(x) = g(x) \tag{0}$$

and for each $\varepsilon > 0$, the ε-coincidence point inequation, with respect to λ,

$$\lambda(f(x),g(x)) \leq \varepsilon \tag{ε}$$

We denote by, $C_{\varepsilon,\lambda}(f,g) := \{x \in X \mid \lambda(f(x),g(x)) \leq \varepsilon\}$, the solution set of (ε).

By definition, the equation (0) is Ulam-Hyers stable, with respect to the functional λ, if there exists $c > 0$ such that for each $\varepsilon > 0$ we have that: for each $u^* \in C_{\varepsilon,\lambda}(f,g)$ there exists $x^* \in C(f,g)$ with, $d(u^*,x^*) \leq c\varepsilon$.

Our result is the following.

Theorem 4.1. *Let $f,g : X \to Y$ be as in Theorem 3.1. If in addition, $M := +\infty$ and the comparison pair, (φ,ψ) is such that there exists $c > 0$ for which, $\sum_{i=0}^{\infty} \psi(\varphi^i(t)) \leq ct$, for all $t \geq 0$, then the equation (0) is Ulam-Hyers stable with respect to the functional λ.*

Proof. For $u^* \in C_{\varepsilon,\lambda}(f,g)$ we take $x^* := h^\infty(u^*)$. □

Remark 4.2. If we take $\lambda := \rho$ then we have the Ulam-Hyers stability with respect to ρ, in the conditions of Buică's theorem and in the conditions of Arutyunov's theorem.

Remark 4.3. For more considerations on Ulam-Hyers stability of fixed point equations and of coincidence point equations see: [45], [30] and the references therein.

5. Well-posedness of the coincidence point problem

Let (X, d) and (Y, ρ) be two metric spaces and $f, g : X \to Y$ with $C(f, g) \neq \emptyset$ and $r : X \to C(f, g)$ be a set retraction. By definition, the coincidence point problem for the pair (f, g) is well-posed with respect to r and to the functional $\lambda : Y \times Y \to \mathbb{R}_+$ if for each $x^* \in C(f, g)$ and each $(x_n)_{n\in\mathbb{N}} \subset r^{-1}(x^*)$ the following implication holds:

$$\lambda(f(x_n), g(x_n)) \to 0 \text{ as } n \to \infty \;\Rightarrow\; x_n \to x^* \text{ as } n \to \infty.$$

We have the following result:

Theorem 5.1. *Let $f, g : (X, d) \to (Y, \rho)$, $\lambda : Y \times Y \to \mathbb{R}_+$ and $h : X \to X$ as in Theorem 3.1. If in addition, $M := +\infty$ and $\lambda(u, v) = 0 \Leftrightarrow u = v$, then the coincidence point problem for the pair (f, g) is well-posed with respect to h^∞ and to λ, if the pair (φ, ψ) is such that $\sum_{i=0}^{\infty} \psi(\varphi^i(t)) \to 0$ as $t \to 0$.*

Proof. First, we remark that $h^\infty : X \to C(f, g)$ is a set retraction. Let $(x_n)_{n\in\mathbb{N}} \subset (h^\infty)^{-1}(x^*)$ with $x^* \in C(f, g)$. Then,

$$d(x_n, x^*) = d(x_n, h^\infty(x_n)) \leq \sum_{i=0}^{\infty} \psi(\varphi^i(\lambda(f(x_n), g(x_n)))) \to 0 \text{ as } n \to \infty. \qquad \square$$

For more consideration on well-posedness of fixed point problem and of coincidence point problem, see: [47], [22], [46], [32].

6. The case of multivalued mappings

Throughout this section we follow the notations and terminology in [39]. See also: [37], [38], [49], [9], [29], [11], [16].

The basic result of this section is the following:

Theorem 6.1. *Let (X, d) be a complete metric space, (Y, ρ) be a metric space, $T, S : X \to P_{cl}(Y)$ be two multivalued mappings, $M \in]0, +\infty]$, (φ, ψ) be a comparison pair on $[0, M[$ and $\Lambda : P_{cl}(Y) \times P_{cl}(Y) \to \mathbb{R}_+ \cup \{+\infty\}$ be a functional. We suppose that:*

(1) $X_M := \{x \in X \mid \Lambda(T(x), S(x)) < M\} \neq \emptyset$;

(2) *The Λ-coincidence point displacement functional, $\Lambda_{T,S} : X_M \to \mathbb{R}_+$,*

$$\Lambda_{T,S}(x) := \Lambda(T(x), S(x))$$

is l.s.c.;

(3) *For each $x \in X_M$ there exists $x_1 \in X_M$ such that:*

   (a) $\Lambda(T(x_1), S(x_1)) \leq \varphi(\Lambda(T(x), S(x)))$;

(b) $d(x, x_1) \leq \psi(\Lambda(T(x), S(x)))$.

Then there exists a pre-WPM, $h : X_M \to X_M$ such that:

(i) $\Lambda(T(h^\infty(x)), S(h^\infty(x))) = 0, \ \forall\ x \in X_M$;

(ii) $d(x, h^\infty(x)) \leq \sum_{i=0}^{\infty} \psi(\varphi^i(\Lambda(T(x), S(x)))), \ \forall\ x \in X_M$;

(iii) *If in addition, for $A, B \in P_{cl}(Y)$, $\Lambda(A, B) = 0$ implies that:*

(iii_1) $A \cap B \neq \emptyset$,
then, $C(T, S) := \{x \in X \mid T(x) \cap S(x) \neq \emptyset\} \neq \emptyset$;

(iii_2) $A = B$,
then, $C(T, S) \neq \emptyset$ *and* $T(h^\infty(x)) = S(h^\infty(x)), \ \forall\ x \in X_M$;

(iii_3) $A = B = \{y^*\}$,
then $C(T, S) \neq \emptyset$ *and* $T(h^\infty(x)) = S(h^\infty(x)) = \{y^*_x\}$.

Proof. If we take, $h(x) := x_1$, then we have that:

$$\Lambda(T(h(x)), S(h(x))) \leq \varphi(\Lambda(T(x), S(x))), \ \forall\ x \in X_M,$$

and

$$d(x, h(x)) \leq \psi(\Lambda(T(x), S(x))), \ \forall\ x \in X_M.$$

These imply that,

$$\Lambda(T(h^n(x)), S(h^n(x))) \to 0 \text{ as } n \to \infty,$$

and h is a pre-WPM, and

$$d(x, h^\infty(x)) \leq \sum_{i=0}^{\infty} \psi(\varphi^i(\Lambda(T(x), S(x)))), \ \forall\ x \in X_M.$$

Since, $\Lambda_{T,S}$ is l.s.c., it follows that,

$$0 \leq \Lambda(T(h^\infty(x)), S(h^\infty(x))) \leq \underline{\lim}_{n\to\infty} \Lambda(T(h^n(x)), S(h^n(x)))$$
$$= \lim_{n\to\infty} \Lambda(T(h^n(x)), S(h^n(x))) = 0.$$

So, we have the conclusions (i), (ii) and (iii). □

Remark 6.2. If we take in Theorem 6.1, $\Lambda := H_\rho$, the Pompeiu-Hausdorff metric on $P_{cl}(Y)$, then we have:

Theorem 6.3. *Let $\Lambda := H_\rho$ in Theorem 6.1. Then we have the following conclusions:*

There exists a pre-WPM, $h : X_M \to X_M$ such that:

(i′) $T(h^\infty(x)) = S(h^\infty(x)), \ \forall\ x \in X_M$;

(ii′) $d(x, h^\infty(x)) \leq \sum_{i=0}^{\infty} \psi(\varphi^i(H_\rho(T(x), S(x)))), \ \forall\ x \in X_M$.

Remark 6.4. Let $\Lambda := e$, in Theorem 6.1., the excess functional. In this case, $A, B \in P_{cl}(Y)$, $e(A, B) := \sup\{\rho(a, B) \mid a \in A\} = 0 \Rightarrow A \subset B$. So, we have the following result:

Theorem 6.5. *If in Theorem 6.1., we take $\Lambda := e$, then the conclusions of this theorem take the following form:*

There exists a pre-WPM, $h : X_M \to X_M$ such that:

(*i''*) $T(h^\infty(x)) \subset S(h^\infty(x))$, *i.e.*, $C(T,S) \neq \emptyset$;

(*ii''*) $d(x,h^\infty(x)) \leq \sum_{i=0}^{\infty} \psi(\varphi^i(e(T(x),S(x))))$, $\forall\, x \in X_M$.

Remark 6.6. If we take $\Lambda := D$, where for $A, B \in P_{cl}(Y)$,

$$D(A,B) := \inf\{\rho(a,b) \mid a \in A,\ b \in B\},$$

then we have a theorem given by A. Petruşel in [35] and A.V. Arutyunov in [5].

Remark 6.7. If we take in Theorem 6.1, $Y := X$, $S := d$, $S(x) := \{x\}$, $\forall\, x \in X$, then Theorem 6.1 takes the following form:

Theorem 6.8. *Let* (X,d) *be a complete metric space,* $T : X \to P_{cl}(X)$, $M \in]0,+\infty]$, *and* (φ,ψ) *be a comparison pair on* $[0,M[$, $\Lambda : X \times P_{cl}(X) \to \mathbb{R}_+ \cup \{+\infty\}$ *be a functional. We suppose that:*

(1) $X_M := \{x \in X \mid \Lambda(x,T(x)) < M\} \neq \emptyset$;
(2) $\Lambda_T : X_M \to \mathbb{R}_+$, $\Lambda_T(x) = \Lambda(x,T(x))$, *the* Λ*-fixed point displacement is l.s.c.;*
(3) *For each* $x \in X_M$, *there exists* $x_1 \in X_M$ *such that:*
   (*a*) $\Lambda(x_1,T(x_1)) \leq \varphi(\Lambda(x,T(x)))$;
   (*b*) $d(x,x_1) \leq \psi(\Lambda(x,T(x)))$.

Then there exists a pre-WPM, $h : X_M \to X_M$ *such that:*

(*i*) $\Lambda(h^\infty(x),T(h^\infty(x))) = 0$, $\forall\, x \in X_M$;

(*ii*) $d(x,h^\infty(x)) \leq \sum_{i=0}^{\infty} \psi(\varphi^i(\Lambda(x,T(x))))$, $\forall\, x \in X_M$.

Remark 6.9. It is clear that, if in Theorem 6.8 we take:

- $\Lambda := H_d$, then $T(h^\infty(x)) = \{h^\infty(x)\}$, i.e., $h^\infty(x)$ is a strict fixed point of T, $\forall\, x \in X_M$;
- $\Lambda := D$, then $h^\infty(x) \in T(h^\infty(x))$, $\forall\, x \in X_M$, i.e., $h^\infty(x)$ is a fixed point of T, $\forall\, x \in X_M$.

Remark 6.10. In [19] Y. Feng and S. Liu, have given the following fixed point result:

Let (X,d) be a complete metric space, $T : X \to P_{cl}(X)$ be a multivalued mapping. For a positive constant $b \in]0,1[$, set

$$I_b^x := \{y \in T(x) \mid bd(x,y) \leq d(x,T(x))\}.$$

If there exists a constant $c \in]0,1[$ such that for any $x \in X$, there is $y \in I_b^x$ satisfying

$$d(y,T(y)) \leq cd(x,y),$$

then T has a fixed point in X provided $c < b$ and the fixed point displacement, d_T is l.s.c. We remark that we are in the conditions of Theorem 6.8, with, $M := +\infty$, $\varphi(t) := \frac{c}{b}t$, $\psi(t) := \frac{t}{b}$ and $\Lambda := D$.

From the considerations presented in this section, the following questions follow:

Problem 6.11. Let (X,d) and (Y,ρ) be two metric spaces and $T,S : X \to P_{cl}(Y)$ be two multivalued mapping. Let $P : P_{cl}(Y) \times P_{cl}(Y) \to \mathbb{R}_+ \cup \{+\infty\}$ be a metric $(H_\rho, H_\rho^+, \ldots)$. In which conditions the $P_{T,S}$-coincidence displacement, $P_{T,S} : X \to \mathbb{R}_+$, $P_{T,S}(x) := P(T(x), S(x))$, is l.s.c. ?

References: [37], [39], [27], [9], [29], [4].

Problem 6.12. To use Theorem 6.1 in studying the Ulam-Hyers stability of a coincidence equation.

References: [45], [49], [30], [50].

Problem 6.13. To use Theorem 6.1 to study the well-posedness of coincidence point problem.

References: [39], [40], [49], [9], [29], [11], [53], [50].

Problem 6.14. Which metric fixed point theorems appear as consequences of Theorem 6.8 ?

References: [17], [22], [37], [48], [49], [10], [41], [51], [11], [52], [36], [12], [44].

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