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# Topological degree methods for a nonlinear elliptic systems with variable exponents

Samira Lecheheb and Abdelhak Fekrache

**Abstract.** In this paper, we consider the existence of a distributional solution for nonlinear elliptic system governed by $(p(x)$,$q(x))$-Laplacian operators. We show that the system has at least one solution by using the topological degree theory. Our results improve and generalize existing results with another technical approach.

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**Keywords:** $p(x)$-Laplacian, operator of $(S_+)$ type, variable exponent, topological degree.

## 1. Introduction

The main purpose of this paper is to obtain existence of distributional solution for the following nolinear elliptic system

$$\begin{cases} -\operatorname{div}(|\nabla u|^{p(x)-2}\nabla u) = f(x,w,\nabla w) \quad \text{in } \Omega, \\ -\operatorname{div}(|\nabla w|^{q(x)-2}\nabla w) = h(x,u,\nabla u) \quad \text{in } \Omega, \\ u = w = 0 \quad \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega$ is a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$, $p(\cdot), q(\cdot) \in C_+(\bar{\Omega})$. We assume also that $p(\cdot), q(\cdot)$ are log-Hölder continuous functions (see Lemma 2.10).

For it's various applications in various fields, the study of elliptic equations or systems with variable exponents became the most interesting and fascinating area of research (see [1, 11, 28, 29, 34] and so on).

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In the previous decades, the existence of the nontrivial solutions for elliptic equation involving $p$ and $p(x)$-Laplacian have been a large investigation. We refer the interested readers to [4, 9, 10, 14, 15, 16, 17, 18, 13, 20, 2, 25, 26, 27, 30, 23, 31, 24] and the references therein. Now let us briefly comment certain known results of them.

In [10], Chabrowski and Fu studied the $p(x)$-Laplacian problem

$$\begin{cases} -\text{div}(a(x)|\nabla u|^{p(x)-2}\nabla u) + b(x)|u|^{p(x)-2}u = f(x,u), \quad x \in \Omega, \\ u = 0 \quad \text{ on } \partial\Omega, \end{cases} \tag{1.2}$$

wheere $\Omega$ is a bounded domain in $\mathbb{R}^n$, $0 < a_0 \leq a(x) \in L^\infty(\Omega)$, $0 \leq b_0 \leq b(x) \in L^\infty(\Omega), p$ is Lipschitz continuous on $\bar{\Omega}$ and satisfies $1 < p_1 \leq p(x) \leq p_2 < n$. When $f(x,u)$ is assumed to satisfy their prototype cases, they obtained the existence of nontrivial and nonnegative solutions for problem (1.2).

Fan and Zhang [18] presents several sufficient conditions for the existence of solutions for the problem (1.2) with $a(x) \equiv 1$ and $b(x) = 0$. Especially, an existence criterion for infinite many pairs of solutions for the problem was obtained by them. By using the degree theory for $p(x)$ is a constant function with values in $(2, N)$, Kim and Hong [20] studied the problem

$$\begin{cases} -\Delta_p u = u + f(x,u,\nabla u), \quad x \text{ in } \Omega, \\ u = 0 \quad \text{ on } \partial\Omega, \end{cases} \tag{1.3}$$

where $\Omega$ is a bounded domain in $\mathbb{R}^N$ with smooth boundary. When $p(x)$ is a variable function, Ait Hammou et al [2] studied the problem on bounded domains. Under certain conditions, they established some results on the existence of solutions by the topological degree theory for a class of demicontinuous operators of generalized $(S_+)$ type.

Inspired by the works mentioned above, especially by [20, 2], we try to extend the results in [2] to the system (1.1). More precisely, the aim of this paper is to show the existence of solutions for (1.1) in the variational frame work by using the topological degree constructed by Kim and Hong [20]. This method may be one of the most effective tools in the study of nonlinear equations. For more details about the important stages in the history of this method, the reader can see [3, 6, 7, 8, 22].

The rest of this paper is organized as follows. In Section 2, we introduce some classes of mappings of generalized $(S_+)$ type, topological degree, some basic properties for variable exponent Sobolev spaces and we present several important properties of $p(x)-$Laplacian which will be later needed. In Section 3, we give our basic assumptions and we prove the main results of this paper. Finally, in Section 4, we present a discussion about our research results.

**Notation.** Throughout this paper, we shall denoted by "$\rightarrow$" and "$\rightharpoonup$" the strong and weak convergence. We use $B_R(a)$ to denote the open ball in the Banach space $X$ of radius $R > 0$ centered at $a$. The symbol "$\hookrightarrow$" means the continuous embedding.

## 2. Mathematical preliminaries

### 2.1. Classes of mappings and topological degree

For the reader's convenience, we bring in some necessary properties and definitions of the classes of mappings mentioned in the introduction which will be the key to proving the existence solution of system (1.1).

**Definition 2.1.** Let $X$ and $Y$ be two real separable, reflexive Banach spaces and $\Omega$ a nonempty subset of $X$. A mapping $F : \Omega \subset X \to Y$ is

1. demicontinuous, if for each $u \in \Omega$ and any sequence $(u_n)$ in $\Omega$, $u_n \to u$ implies $F(u_n) \rightharpoonup F(u)$.
2. bounded, if it takes any bounded set into a bounded set.
3. compact, if it is continuous and the image of any bounded set is relatively compact.

**Definition 2.2.** Let $X$ be a real separable reflexive Banach space with dual space $X^*$. An operator $F : \Omega \subset X \to X^*$ is said to be

1. of class $(S_+)$, if for any sequence $(u_n)$ in $\Omega$ with $u_n \rightharpoonup u$ and $\limsup \langle Fu_n, u_n - u\rangle \le 0$, we have $u_n \to u$.
2. quasimonotone, if for any sequence $(u_n)$ in $\Omega$ with $u_n \rightharpoonup u$, we have $\limsup \langle Fu_n, u_n - u\rangle \ge 0$.

**Definition 2.3.** Let $T : \Omega_1 \subset X \to X^*$ be a bounded mapping such that $\Omega \subset \Omega_1$. For any mapping $F : \Omega \subset X \to X$, we say that

1. $F$ satisfies condition $(S_+)_T$, if for any sequence $(u_n)$ in $\Omega$ with $u_n \rightharpoonup u$, $y_n := Tu_n \rightharpoonup y$ and $\limsup \langle Fu_n, y_n - y\rangle \le 0$, we have $u_n \to u$.
2. $F$ has the property $(QM)_T$, if for any sequence $(u_n)$ in $\Omega$ with $u_n \rightharpoonup u$, $y_n := Tu_n \rightharpoonup y$, we have $\limsup \langle Fu_n, y_n - y\rangle \ge 0$.

Now, let $\mathcal{O}$ be the collection of all bounded open set in $X$. For any $\Omega \subset X$, we consider the following classes of operators:

$$\begin{aligned}
\mathcal{F}_1(\Omega) &:= \{F : \Omega \to X^* | F \text{ is bounded, demicontinuous and of class } (S_+)\},\\
\mathcal{F}_{T,B}(\Omega) &:= \{F : \Omega \to X | F \text{ is bounded, demicontinuous and of class } (S_+)_T\},\\
\mathcal{F}_T(\Omega) &:= \{F : \Omega \to X | F \text{ is demicontinuous and of class } (S_+)_T\},\\
\mathcal{F}_B(X) &:= \{F \in \mathcal{F}_{T,B}(\overline{G}) | G \in \mathcal{O}, T \in \mathcal{F}_1(\overline{G})\},\\
\mathcal{F}(X) &:= \{F \in \mathcal{F}_T(\overline{G}) | G \in \mathcal{O}, T \in \mathcal{F}_1(\overline{G})\}.
\end{aligned}$$

Here, $T \in \mathcal{F}_1(\overline{G})$ is called an essential inner map to $F$.

**Lemma 2.4** ([5], **Lemmas** 2.2 **and** 2.4)**.** *Let $T \in \mathcal{F}_1(\overline{G}), G \in \mathcal{O}$, be continuous and $S : D_S \subset X^* \to X$ a bounded demicontinuous mapping such that $T(\overline{G}) \subset D_S$. Then the following statements are true:*

1. *If $S$ is quasimonotone, then $I + SoT \in \mathcal{F}_T(\overline{G})$, where $I$ denote the identity operator.*
2. *If $S$ of class $(S_+)$, then $SoT \in \mathcal{F}_T(\overline{G})$.*

**Definition 2.5.** Let $F, S \in \mathcal{F}_T(\overline{G})$ and let $G$ be a bounded open subset of a real reflexive Banach space $X$. The affine homotopy $H : [0,1] \times \overline{G} \to X$ given by

$$H(\lambda, u) := (1-\lambda)Fu + \lambda Su, \text{ for } (\lambda, u) \in [0,1] \times \overline{G}$$

is called an admissible affine homotopy with the continuous essential inner map $T$.

**Remark 2.6.** [5] The above affine homotopy satisfies condition $(S_+)$.

Now, we introduce the Berkovits topological degree for the class $\mathcal{F}_B(X)$. For more details see [5].

**Theorem 2.7.** *There exists a unique degree function*

$$\deg_B : \{(F, G, h) | G \in \mathcal{O}, T \in \mathcal{F}_1(\overline{G}), F \in \mathcal{F}_{T,B}(\overline{G}), h \notin F(\partial G)\} \to \mathbb{Z}$$

*that satisfies the following properties:*

1. *(Existence) If* $\deg_B(F, G, h) \neq 0$*, then the equation* $Fu = h$ *has a solution in* $G$*.*
2. *(Normalization) For any* $h \in G$*, we have* $\deg_B(I, G, h) = 1$*.*
3. *(Additivity) Let* $F \in \mathcal{F}_{T,B}(\overline{G})$*. If* $G_1$ *and* $G_2$ *are two disjoint open subsets of* $G$ *such that* $h \notin F(\overline{G} \setminus (G_1 \cup G_2))$*, then we have*
$$\deg_B(F, G, h) = \deg_B(F, G_1, h) + \deg_B(F, G2, h).$$
4. *(Homotopy invariance) If* $H : [0,1] \times \overline{G} \to X$ *is a bounded admissible affine homotopy with a common continuous essential inner map and* $h :: [0,1] \times X$ *is a continuous path in* $X$ *such that* $h(\lambda) \notin H(\lambda, \partial G)$ *for all* $\lambda \in [0,1]$*, then the value of* $\deg_B(H(\lambda, \cdot), G, h(\lambda))$ *is constant for all* $\lambda \in [0,1]$

### 2.2. Notation and preliminary results

In order to solve the problem (1.1), we need some necessary properties on variable exponent spaces $L^{p(x)}(\Omega)$ and $W_0^{1,p(x)}(\Omega)$. For a deeper treatment on these spaces, we refer to [12, 14, 15, 17, 19, 21], and the references therein.

In the sequel, we consider a bounded domain $\Omega \subset \mathbb{R}^N$, $N \geq 2$ with a Lipschitz boundary $\partial\Omega$ and the set

$$C_+(\overline{\Omega}) = \{g \in C(\overline{\Omega}) \mid \inf_{x \in \overline{\Omega}} g(x) > 1\},$$

$$g^- = \min_{x \in \overline{\Omega}} g(x), \quad g^+ = \max_{x \in \overline{\Omega}} g(x), \quad \text{for any } g \in C_+(\overline{\Omega}).$$

For any $p \in C_+(\overline{\Omega})$, we define the generalized Lebesgue space $L^{p(x)}(\Omega)$ by

$$L^{p(x)}(\Omega) = \left\{u \mid u : \Omega \to \mathbb{R} \text{ is a measurable function, } \rho_{p(x)}(u) < \infty\right\},$$

where

$$\rho_{p(x)}(u) = \int_\Omega |u(x)|^{p(x)} dx,$$

this space endowed with the Luxemburg norm

$$\|u\|_{p(x)} = \inf\{\lambda > 0 \mid \rho_{p(x)}(\frac{u}{\lambda}) \leq 1\},$$

and $(L^{p(x)}(\Omega), \|\cdot\|_{p(x)})$ becomes a Banach space.

**Lemma 2.8.** [21]

1. *The space $L^{p(x)}(\Omega)$ is a separable and reflexive Banach space.*
2. *The conjugate space of $L^{p(x)}(\Omega)$ is $L^{p'(x)}(\Omega)$, where $1/p(x) + 1/p'(x) = 1$. Then for any $u \in L^{p(x)}(\Omega)$ and $w \in L^{p'(x)}(\Omega)$, we have the following Hölder inequality*

$$\Big|\int_\Omega uwdx\Big| \leq \Big(\frac{1}{p^-} + \frac{1}{p'^-}\Big)\|u\|_{p(x)}\|w\|_{p'(x)} \leq 2\|u\|_{p(x)}\|w\|_{p'(x)}.$$

3. *If $p_1, p_2 \in C_+(\overline{\Omega})$, $p_1(x) \leq p_2(x)$ for any $x \in \overline{\Omega}$, then there exists the continuous embedding $L^{p_2(x)}(\Omega) \hookrightarrow L^{p_1(x)}(\Omega)$*

**Lemma 2.9.** [19, 33] *If $u, u_n \in L^{p(x)}(\Omega)$, then the following assertions hold true:*

1. $\|u\|_{p(x)} < 1\ (= 1, > 1) \Leftrightarrow \rho_{p(x)}(u) < 1\ (= 1, > 1)$.
2. $\|u\|_{p(x)} < 1 \Rightarrow \|u\|_{p(x)}^{p^+} \leq \rho_{p(x)}(u) \leq \|u\|_{p(x)}^{p^-}$.
3. $\|u\|_{p(x)} > 1 \Rightarrow \|u\|_{p(x)}^{p^-} \leq \rho_{p(x)}(u) \leq \|u\|_{p(x)}^{p^+}$.
4. $\lim_{n\to\infty} \|u_n - u\|_{p(x)} = 0 \Leftrightarrow \lim_{n\to\infty} \rho_{p(x)}(u_n - u) = 0$.
5. $\|u\|_{p(x)} \leq \rho_{p(x)}(u) + 1$.
6. $\rho_{p(x)}(u) \leq \|u\|_{p(x)}^{p^-} + \|u\|_{p(x)}^{p^+}$.

Now, we define the usual Sobolev space with variable exponent $W^{1,p(x)}(\Omega)$ as

$$W^{1,p(x)}(\Omega) = \{u \in L^{p(x)}(\Omega) \mid |\nabla u| \in L^{p(x)}(\Omega)\},$$

whose norm is defined as

$$\|u\|_{W^{1,p(x)}} = \|u\|_{p(x)} + \|\nabla u\|_{p(x)}. \tag{2.1}$$

Let $W_0^{1,p(x)}(\Omega)$ denote the subspace of $W^{1,p(x)}(\Omega)$ which is the closure of $C_0^\infty(\Omega)$ with respect to the norm (2.1).

**Lemma 2.10.** [12, 19, 21]

1. *The two spaces $W_0^{1,p(x)}(\Omega)$ and $W^{1,p(x)}(\Omega)$ are a Banach spaces separable and reflexive.*
2. *If $p(x)$ satisfies the log-Hölder continuity condition, i.e., there is a constant $\alpha > 0$ such that for every $x, y \in \Omega, x \notin y$ with $|x - y| \leq \frac{1}{2}$ one has*

$$|p(x) - p(y)| \leq \frac{\alpha}{-\log|x - y|},$$

*then there exists a constant $C > 0$, such that*

$$\|u\|_{p(x)} \leq C\|\nabla u\|_{p(x)}, \quad \forall u \in W_0^{1,p(x)}(\Omega).$$

3. *If $p \in C_+(\overline{\Omega})$ for any $x \in \overline{\Omega}$, then the imbedding $W_0^{1,p(x)}(\Omega) \hookrightarrow L^{p(x)}(\Omega)$ is compact.*

**Remark 2.11.** By (2) of Lemma 2.10, we know that $\|\nabla u\|_{p(x)}$ and $\|u\|$ are equivalent norms on $W_0^{1,p(x)}(\Omega)$.

The dual space of $W_0^{1,p(x)}(\Omega)$ is $W^{-1,p'(x)}(\Omega)$, which endowed with the norm

$$\|w\|_{-1,p'(x)} = \inf\Big\{\|w_0\|_{p'(x)} + \sum_{i=1}^{N} \|w_i\|_{p'(x)}\Big\},$$

where the infinimum is taken on all possible decompositions $w = w_0 - divF$ with $w_0 \in L^{p'(x)}(\Omega)$ and $F = (w_1, \cdots, w_N) \in (L^{p'(x)}(\Omega))^N$.

Let us define $V = W_0^{1,p(x)}(\Omega) \times W_0^{1,q(x)}(\Omega)$ endowed with the norm $\|(u,w)\|_V = \max(\|u\|_{1,p(x)}, \|w\|_{1,q(x)})$ where $\|u\|_{1,p(x)} = \|\nabla u\|_{p(x)}$ and $(V, \|\cdot\|)$ is a Banach space, separable and reflexive.

### 2.3. Properties of $(p(x), q(x))$-Laplacian operators

In the present subsection, we discuss the properties of $(p(x), q(x))$-Laplacian operators

$$-\Delta_{p(x)} u = -\text{div}(|\nabla u|^{p(x)-2}\nabla u),$$

and

$$-\Delta_{q(x)} w = -\text{div}(|\nabla w|^{q(x)-2}\nabla w).$$

We consider the following functional:

$$\mathcal{J}(u,w) = \int_\Omega \frac{|\nabla u|^{p(x)}}{p(x)} dx + \int_\Omega \frac{|\nabla w|^{q(x)}}{q(x)} dx.$$

It is well known that $\mathcal{J} \in C^1(V, \mathbb{R})$ and for any $(\varphi, \phi) \in V$

$$\langle \mathcal{J}'(u,w), (\varphi,\phi)\rangle$$
$$= \int_\Omega |\nabla u|^{p(x)-2}\nabla u \nabla\varphi dx + \int_\Omega |\nabla w|^{q(x)-2}\nabla w \nabla\phi dx, \quad \forall u, w \in V.$$

Denote $M = \mathcal{J}' : V \to V^*$.

**Theorem 2.12.** [18]

1. *$M : V \to V^*$ is a mapping of type $(S_+)$.*
2. *$M : V \to V^*$ is a continuous, bounded and strictly monotone operator.*
3. *$M : V \to V^*$ is a homeomorphism.*

The proof of the above theorem can be found in [18].

## 3. Hypotheses and the main results

### 3.1. Hypotheses

Let $\Omega$ be a bounded domain in $\mathbb{R}^N$ $(N \geq 2)$ with a Lipschitz boundary $\partial\Omega$. Let $p, q \in C_+(\bar{\Omega})$, $1 < p^- \leq p(x) \leq p^+ < \infty$, $1 < q^- \leq q(x) \leq q^+ < \infty$ and $f, h : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ are a real-valued functions such that

$(A_1)$. (Continuity) $f, h$ are the Carathéodory functions ( i.e., $f(x, \cdot, \cdot)$ is continuous in $(s_1, s_2)$ for almost every $x \in \Omega$ and $f(\cdot, s_1, s_2)$ is measurable in $x$ for each $(s_1, s_2) \in \mathbb{R} \times \mathbb{R}^N$)

$(A_2)$. (Growth) There exist a positive constants $c_1, c_2$, $b \in L^{p'(x)}(\Omega), d \in L^{q'(x)}(\Omega)$ and $1 < \alpha^- \leq \alpha(x) \leq \alpha^+ < p^-$, $1 < \beta^- \leq \beta(x) \leq \beta^+ < q^-$, such that

$$|f(x, s_1, s_2)| \leq c_1(b(x) + |s_1|^{\alpha(x)-1} + |s_2|^{\alpha(x)-1}),$$
$$|h(x, \xi_1, \xi_2)| \leq c_2(d(x) + |\xi_1|^{\beta(x)-1} + |\xi_2|^{\beta(x)-1}).$$

**3.2. Main results**

The main tool that we shall use to prove the existence of weak solutions of the problem (1.1) is the degree theory introduced in section 2.

**Definition 3.1.** We say that $(u, w) \in V$ is a distributional solution of the system (1.1) if for any $(\varphi, \phi) \in V$ we have

$$\begin{aligned} &\int_\Omega |\nabla u|^{p(x)-2} \nabla u \nabla \varphi dx + \int_\Omega |\nabla w|^{q(x)-2} \nabla w \nabla \phi dx \\ &= \int_\Omega f(x, w, \nabla w) \varphi dx + \int_\Omega h(x, u, \nabla u) \phi dx \end{aligned} \tag{3.1}$$

**Lemma 3.2.** *Assume that* $(A_1)$ *and* $(A_2)$ *hold. Then the operator* $T : V \to V^*$ *given by*

$$\begin{cases} (u, w) \in V, \\ \langle T(u, w), (\varphi, \phi) \rangle = -\displaystyle\int_\Omega f(x, w, \nabla w) \varphi dx - \int_\Omega h(x, u, \nabla u) \phi dx, \ \forall (\varphi, \phi) \in V \end{cases}$$

*is compact.*

*Proof.* First, let $\chi : W_0^{1,p(x)} \to L^{p'(x)}(\Omega)$, $\pi : W_0^{1,q(x)} \to L^{q'(x)}(\Omega)$ be two operators defined by

$$\chi u(x) = -h(x, u, \nabla u) \ \text{ for } u \in W_0^{1,p(x)} \ \text{ and } x \in \Omega,$$

and

$$\pi w(x) = -f(x, w, \nabla w) \ \text{ for } w \in W_0^{1,q(x)} \ \text{ and } x \in \Omega.$$

We divide the proof into three steps.

**Step 1.** We show that $\chi$ and $\pi$ are bounded.

For each $u \in W_0^{1,p(x)}(\Omega)$, we have by (5), (6) of Lemma 2.9 and the assumption $(A_2)$ that

$$\begin{aligned} \|\chi u\|_{p'(x)} &\leq \rho_{p'(x)}(\chi u) + 1 \\ &= \int_\Omega |h(x, u(x), \nabla u(x))|^{p'(x)} + 1 \\ &\leq \text{const}\Big( \int_\Omega \Big( |d| + |u|^{\beta(x)-1} + |\nabla u|^{\beta(x)-1} \Big)^{p'(x)} dx \Big) \\ &\leq \text{const}\Big( \rho_{p'(x)}(d) + \rho_{\gamma(x)}(u) + \rho_{\gamma(x)}(\nabla u) \Big) + 1 \\ &\leq \text{const}\Big( \|d\|_{p'(x)}^{p'^-} + \|d\|_{p'(x)}^{p'^+} + \|u\|_{\gamma(x)}^{\gamma^-} + \|u\|_{\gamma(x)}^{\gamma^+} + \|\nabla u\|_{\gamma(x)}^{\gamma^-} \\ &\quad + \|\nabla u\|_{\gamma(x)}^{\gamma^+} \Big) + 1, \end{aligned}$$

where

$$\gamma(x) = (\beta(x) - 1)p'(x) < p(x).$$

By (2) of Lemma 2.10 and the continuous embedding $L^{p(x)} \hookrightarrow L^{\gamma(x)}$, we get

$$\|\chi u\|_{p'(x)} \leq \text{ const}\Big(\|d\|_{p'(x)}^{p'^-} + \|d\|_{p'(x)}^{p'^+} + \|u\|_{1,p(x)}^{\gamma^-} + \|u\|_{1,p(x)}^{\gamma^+}\Big) + 1,$$

which implies that $\chi$ is bounded on $W_0^{1,p(x)}$.
Similarly, we can show that $\pi$ is bounded on $W_0^{1,q(x)}$.

**Step 2.** We show that $\chi$ and $\pi$ are continuous.

Let $(u_n, w_n)$ converge to $(u, w)$ in $V$. Then

$$u_n \to u \text{ and } \nabla u_n \to \nabla u \text{ in } W_0^{1,p(x)},$$

$$w_n \to w \text{ and } \nabla w_n \to \nabla w \text{ in } W_0^{1,q(x)}.$$

Hence there exist two subsequences denote again by $(u_n)$, $(w_n)$ and measurable functions $g_1$ (resp. $g_2$) in $L^{p(x)}(\Omega)$ (resp. in $L^{q(x)}(\Omega)$) and $g_1^*$ (resp.$g_2^*$) in $(L^{p(x)}(\Omega))^N$ (resp. in $(L^{q(x)}(\Omega))^N$), such that

$$u_n(x) \to u(x) \text{ and } \nabla u_n(x) \to \nabla u(x),$$

$$w_n(x) \to w(x) \text{ and } \nabla w_n(x) \to \nabla w(x),$$

$$|u_n(x)| \leq g_1(x),\ |\nabla u_n(x)| \leq |g_1^*(x)|$$

and

$$|w_n(x)| \leq g_2(x),\ |\nabla w_n(x)| \leq |g_2^*(x)|,$$

for almost all $x \in \Omega$ and all $n \in N$. From $(A_1)$ and $(A_2)$, we have

$$h(x, u_n(x), \nabla u_n(x)) \to h(x, u(x), \nabla u(x)) \text{ for almost all } x \in \Omega,$$

and

$$|h(x, u_n(x), \nabla u_n(x))| \leq \text{ const}\Big(d(x) + |g_1(x)|^{\beta(x)-1} + |g_1^*(x)|^{\beta(x)-1}\Big),$$

for almost all $x \in \Omega$ and all $n \in N$ and

$$d + |g_1|^{\beta(x)-1} + |g_1^*|^{\beta(x)-1} \in L^{p'(x)}(\Omega).$$

Taking into account the equality

$$\rho_{p'(x)}(\chi u_n - \chi u) = \int_\Omega |h(x, u_n(x), \nabla u_n(x)) - h(x, u(x), \nabla u(x))|^{p'(x)} dx,$$

the equivalence (4) of Lemma 2.9 and the Lebesgue dominated convergence theorem imply that

$$\chi u_n \to \chi u \text{ in } L^{p'(x)}(\Omega),$$

which shows that the entire sequence $(\chi u_n)$ is continuous.
Similarly, we obtain that the entire sequence $(\pi w_n)$ is continuous.

**Step 3.** As the embedding $\mathcal{I} : V \to U$ is compact, it is known that the adjoint operator $\mathcal{I}^* : U^* \to V^*$is also compact. So the compositions $\mathcal{I}^* o\chi$ and $\mathcal{I}^* o\pi : V \to V^*$ are compact, which completes the proof. □

Let us now mention our main result in this paper:

**Theorem 3.3.** *Under conditions* $(A_1)$ *and* $(A_2)$*, problem* (1.1) *has a distributional solution* $(u, w)$ *in* $V$*.*

*Proof.* Let $T$ be an operator from $V$ into its dual $V^*$ as defined in Lemma 3.2, ant let $M : V \to V^*$, as in subsection 2.3, given by

$$\begin{cases} (u, w) \in V, \\ \langle M(u, w), (\varphi, \phi)\rangle = \displaystyle\int_\Omega |\nabla u|^{p(x)-2}\nabla u \nabla\varphi dx + \int_\Omega |\nabla w|^{q(x)-2}\nabla w \nabla\phi dx, \end{cases}$$

for all $(\varphi, \phi) \in V$. Then $(u, w) \in V$ is a distributional solution of (1.1) if and only if

$$M(u, w) = -T(u, w). \tag{3.2}$$

Thanks to Lemma 3.2, the operator $T$ is bounded, continuous and quasimonotone. On the other hand, according to the properties of the operator $M$ seen in Theorem 2.12 and by using the Minty-Browder Theorem (see [32], Theorem 26$A$), the inverse operator $N = M^{-1} : V^* \to V$ is bounded, continuous and satisfies condition $(S_+)$.

Therefore, equation (3.2) is equivalent to

$$(u, w) = N(\varphi, \phi) \text{ and } (\varphi, \phi) + ToN(\varphi, \phi) = 0. \tag{3.3}$$

To solve (3.3), we shall using the degree theory introduced in subsection 2.1. For this, we first show that the set

$$\Sigma = \{(\varphi, \phi) \in V^* | (\varphi, \phi) + \lambda ToN(\varphi, \phi) = 0 \text{ for some } \lambda \in [0, 1]\}$$

is bounded. Indeed, let $(\varphi, \phi) \in \Sigma$ and take $(u, w) = N(\varphi, \phi)$, then

$$\|N(\varphi, \phi)\|_V = \|(u, w)\|_V = \max(\|\nabla u\|_{p(x)}, \|\nabla w\|_{q(x)}).$$

If $\|\nabla u\|_{p(x)} \le 1$ and $\|\nabla w\|_{q(x)} \le 1$, then $\|N(\varphi, \phi)\|_V$ is bounded.

If $\|\nabla u\|_{p(x)} > 1$ and $\|\nabla w\|_{q(x)} > 1$, then by using the assumption $(A_2)$, (3), (6) of Lemma 2.9, (2) of Lemma 2.8 and the Young inequality, we obtain the estimate

$$\begin{aligned}
\|N(\varphi,\phi)\|_V^{\min(p^-,q^-)} &= \|(u,w)\|_V^{\min(p^-,q^-)}\\
&\leq \rho_{p(x)}(\nabla u)+\rho_{q(x)}(\nabla w)\\
&= \langle M(u,w),(u,w)\rangle\\
&= \langle(\varphi,\phi),N(\varphi,\phi)\rangle\\
&= -\lambda\langle ToN(\varphi,\phi),N(\varphi,\phi)\rangle\\
&= \lambda\Big(\int_\Omega f(x,w,\nabla w)udx+\int_\Omega h(x,u,\nabla u)wdx\Big)\\
&\leq \text{const}\ \Big(\|b\|_{p'(x)}\|u\|_{p(x)}+\frac{1}{\alpha'^-}\rho_{\alpha(x)}(w)+\frac{1}{\alpha^-}\rho_{\alpha(x)}(u)\\
&\quad+\frac{1}{\alpha'^-}\rho_{\alpha(x)}(\nabla w)+\frac{1}{\alpha^-}\rho_{\alpha(x)}(u)+\|d\|_{q'(x)}\|w\|_{q(x)}\\
&\quad+\frac{1}{\beta'^-}\rho_{\beta(x)}(u)+\frac{1}{\beta^-}\rho_{\beta(x)}(w)+\frac{1}{\beta'^-}\rho_{\beta(x)}(\nabla u)\\
&\quad+\frac{1}{\beta^-}\rho_{\beta(x)}(w)\Big)\\
&\leq \text{const}\ \Big(\|u\|_{p(x)}+\|w\|_{\alpha(x)}^{\alpha^+}+\|u\|_{\alpha(x)}^{\alpha^+}+\|\nabla w\|_{\alpha(x)}^{\alpha^+}\\
&\quad+\|w\|_{q(x)}+\|u\|_{\beta(x)}^{\beta^+}+\|w\|_{\beta(x)}^{\beta^+}+\|\nabla u\|_{\beta(x)}^{\beta^+}\Big).
\end{aligned}$$

By (2) of Lemma 2.10 and the continuous embedding $L^{p(x)}\hookrightarrow L^{\alpha(x)}$ and $L^{q(x)}\hookrightarrow L^{\beta(x)}$, we get

$$\|N(\varphi,\phi)\|_V^{\min(p^-,q^-)}\leq \text{const}\ (\|N(\varphi,\phi)\|_V+\|N(\varphi,\phi)\|_V^{\max(\alpha^+,\beta^+)}).$$

If $\|\nabla u\|_{p(x)}>1$ and $\|\nabla w\|_{q(x)}\leq 1$ (resp. if $\|\nabla u\|_{p(x)}\leq 1$ and $\|\nabla w\|_{q(x)}>1$), we can also get that $\|N(\varphi,\phi)\|_V$ is bounded.
Consequently $\{N(\varphi,\phi)|(\varphi,\phi)\in\Sigma\}$ is bounded.

Since the operator $T$ is bounded, it is obvious from (3.3) that the set $\Sigma$ is bounded in $V^*$. Hence, we can choose a positive constant $R$ such that

$$\|(\varphi,\phi)\|_{V^*}<R \text{ for all } (\varphi,\phi)\in\Sigma.$$

It follows that

$$(\varphi,\phi)+\lambda ToN(\varphi,\phi)\neq 0 \text{ for all } (\varphi,\phi)\in\partial B_R(0) \text{ and all } \lambda\in[0,1],$$

where $B_R(0)$ is the ball of radius $R$ and center 0 in $V^*$.
By Lemma 2.4, we have

$$I+ToN\in\mathcal{F}_T(\overline{B_R(0)}) \text{ and } I=MoN\in\mathcal{F}_T(\overline{B_R(0)}).$$

Since the operators $I,T$ and $N$ are bounded, $I+ToN$ is also bounded. We conclude that

$$I+ToN\in\mathcal{F}_{T,B}(\overline{B_R(0)}) \text{ and } I\in\mathcal{F}_{T,B}(\overline{B_R(0)}).$$

Now, we consider an affine homotopy $H:[0,1]\times\overline{B_R(0)}\to V^*$ given by

$$H(\lambda,\varphi,\phi):=(\varphi,\phi)+\lambda ToN(\varphi,\phi) \text{ for } (\lambda,\varphi,\phi)\in[0,1]\times\overline{B_R(0)}.$$

All those properties allow us to apply the homotopy invariance and normalization property of the degree $\deg_B$ stated in Theorem 2.7 and obtain

$$\deg_B(I + ToN, B_R(0), 0) = \deg_B(I, B_R(0), 0) = 1,$$

consequently, there exists a point $(\varphi, \phi) \in B_R(0)$ such that

$$(\varphi, \phi) + ToN(\varphi, \phi) = 0.$$

This implies that $(u, w) = N(\varphi, \phi)$ is a distributional solution of (1.1). The proof is complete. □

## 4. Conclusion

In this paper, we have studied the existence of distributional solutions for a nonlinear elliptic systems with variable exponents. By using the topological degree theory, we showed that system (1.1) has at least one solutions when the functions $f$ and $h$ satisfying some suitable conditions. This study can be extend in the futur works to more general boundary value problems involving fractional derivatives models.

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Samira Lecheheb
Laboratory of Applied Mathematics and History and Didactics of Mathematics "LAMAHIS",
Department of Mathematics,
University of 20 August 1955, Skikda, Algeria
e-mail: lecheheb.samira24@gmail.com; s.lecheheb@univ-skikda.dz

Abdelhak Fekrache
Faculty of Sciences, University of 20 August 1955,
Skikda, Algeria
e-mail: abde.fek@gmail.com