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Book reviews

Alexey R. Alimov and Igor' G. Tsar'kov, Geometric approximation theory,
Springer Monographs in Mathematics. Cham: Springer 2022, xxi+508 p.
ISBN: 978-3-030-90950-5/hbk; 978-3-030-90953-6/pbk; 978-3-030-90951-2/ebook).

The origins of abstract approximation theory can be traced back to the years 50s of the 19th century when P.L. Chebyshev considered the problem of uniform approximation of continuous functions by polynomials in connection with some technical problems (the construction of some mechanisms as "parallelograms" which transform a circular motion into a rectilinear one, devices used for steam engines). This proves that approximation theory had, and still have, important applications in various scientific and technical domains. Since then the domain developed in many directions by the contributions of many mathematicians and applied scientists.

The present book contains an encyclopedic presentations of a lot of topics in approximation theory in concrete as well as in general Banach spaces, starting with some classical and ending with some very recent results. The first chapter contains some preliminaries. Some classical results on best approximation in the space $C[a,b]$ are presented in the second chapter, including Chebyshev alternation theorem, de la Vallée Poussin and Haar theorems and Mairhuber theorem (the space $C(Q)$ contains a Chebyshev subspace of dimension $n \geq 2$ only if Q is homeomorphic to a subset of the unit circle). Applications are given to Remez's algorithm. Best approximation by rational functions in $C[a,b]$ and in L^p is treated in the 11th chapter.

Chapter 3, *Best approximation in Euclidean spaces* (meaning inner product spaces) contains Kolmogorov criterion on the characterization of best approximation elements and Phelps theorem on the convexity of sets with Lipschitz metric projection. The 4th chapter is dedicated to some notions (approximative compactness, bounded compactness as well as their generalizations, done by Blatter, to a regular mode of convergence) that are very efficient tools in proving existence results in best approximation.

The fifths chapter is concerned with solarity properties of sets and their role in the characterization of best approximation elements, continuity and differentiability properties of the metric projection. Notice that solarity is a recurrent topic of the book. Various types of suns and the relations between them are considered in Chapter 10, *Solarity of Chebyshev sets*, including recent important contributions of the authors.

An old and still unsolved problem in best approximation is that of the convexity of Chebyshev sets – *is any Chebyshev subset of a Hilbert space convex*? In Chapter 5, *Convexity of Chebyshev sets and suns*, the authors present five proofs (of Berdyshev-Klee-Vlasov, Asplund, Konyagin, Vlasov and Brosowski) on the convexity of Chebyshev sets in $\mathbb{R}^n$. Johnson's counterexample of a nonconvex Chebyshev set in an incomplete inner product space and a presentation of Klee caverns are included as well. Other counterexamples (Dunham's example of a Chebyshev set with an isolated point, Klee's example of a discrete Chebyshev set and Koshcheev example of a disconnected sun) are given in Chapter 7, *Connectedness and approximative properties of sets.*

Chapter 8 is concerned with the existence of Chebyshev subspaces in finite and infinite dimensional spaces, with emphasis on the space $L^1(\mu)$. The influence of some geometric properties of Banach spaces (Efimov-Stechkin property, uniform convexity and uniform smoothness) on the approximative properties of their subset is discussed in the 9th chapter.

Chapter 13, *Approximation of vector-valued functions*, contains some results of Zukhovickii, Stechkin, Tsar'kov, Garkavi, Koshcheev, a.o., on the extension of the results on best approximation in spaces of real-valued functions (characterization, Haar condition, Chebyshev systems, etc) to the case of the space $C(Q, X)$, where Q is a compact Hausdorff topological space and X a Banach space.

Chapter 14 is devoted to a detailed study of Jung constant defined as the radius of the smallest ball covering any set of diameter 1. This is a very important tool in the geometry of Banach spaces with applications to best approximation and to fixed point theory for nonexpansive mappings (the inverse of Jung constant is called the coefficient of normality of the corresponding Banach space) and for condensing mappings. Chapter 15 contains a detailed study of Chebyshev centers, a notion related to best approximation (simultaneous approximation) and having important practical applications as, for instance, to optimal location problem. One studies the existence and uniqueness of Chebyshev centers, continuity, stability and selections for the Chebyshev center map, algorithms for finding Chebyshev centers and applications.

Chapter 16 is concerned with several kinds of widths (Kolmogorov, Alexandrov, Fourier, Bernstein) which are strongly related to approximation theory, allowing to compare the efficiency of the approximation by various classes of approximating sets (algebraic or trigonometric polynomials, rational functions, etc).

The last chapter, Chapter 17, *Approximation properties of arbitrary sets in linear normed spaces. Almost Chebyshev sets and sets of almost uniqueness,* is concerned with genericity properties (in the sense of Baire category) and porosity results in best approximation problems and in the study of farthest points (existence and uniqueness), a direction of research initiated by S. B. Stechkin in 1963.

The book contains also three appendices: A. *Chebyshev systems of functions in the spaces C, C^n and L^p*, B. *Radon, Helly and Carathódory theorems. Decomposition theorem*, and C. *Some open problems.* Some open problems are also formulated throughout the main text.

The bibliography counts 632 items.

Written by two experts with substantial contributions to the domain, this book incorporates a lot of results, both classical but also new ones situated in the focus of current research (including authors' results). It can be warmly recommended to a large community of mathematicians interested in best approximation and its relations to Banach space geometry, but it can also be used for graduate courses in approximation theory.

Notice that a two volume preliminary version of the book was published in Russian (Ontoprint, Moskva, 2017 and 2018), but the present one is entirely rewritten, updated and enlarged. (A review of the Russian edition was published in Stud. Univ. Babeş-Bolyai, Mathematica **63** (2018), no. 4.).

S. Cobzaş

Saeed Zakeri, A Course in Complex Analysis, Princeton University Press, 2021, xii+428 pages, hardback, ISBN: 9780691207582, ebook, ISBN: 9780691218502.

The book under review is an excellent introduction to Complex Analysis.

The author managed to put together in a harmonious way a large variety of classical results of the theory. Here is a list with the most important topics and results with complete self-contained proofs in the book: the Cauchy-Riemann equations, Cauchy's theorems and their homology versions, Liouville's theorem and its hyperbolic version, the identity theorem, the open mapping theorem, the maximum principle for holomorphic and harmonic functions, the residue theorem, the argument principle, Möbius maps and their dynamics, conformal metrics, the Schwarz-Pick lemma and Ahlfor's generalization, Montel's theorem and its generalization, the convergence results of Weierstrass, Hurwitz and Vitali, Marty's theorem, the Riemann mapping theorem, Koebe's distorsion bounds for the class of schlicht functions, the Carathéodory extension theorem, the solution of the Dirichlet problem on the disk with the Poisson kernel, the Fatou theorem, harmonic measures and Blaschke products, Weierstrass' factorization theorem, Jensen's formula, Mittag-Leffler's theorem, elliptic functions, Runge's theorem, Schönflies' theorem, conformal models of finitely connected domains, natural boundaries, Ostrowski's theorem, the monodromy theorem, the Schwarz reflection principle for analytic arcs, the Hausdorff measure and holomorphic removability, the Schwarz-Christoffel formula, Bloch's theorem, Schottky's theorem, Picard's theorems, Zalcman's rescaling theorem, branched coverings, the Riemann-Hurwitz formula, the modular group, the uniformization theorem for spherical domains, the characterization of hyperbolic domains, holomorphic covering maps of topological annuli.

Each chapter ends with a generous list of problems. Even though the book doesn't include the solutions, the problems have short solutions and are not too hard, but sufficiently challenging to motivate the reader to go again through the theory, and thus to understand better the key ideas of each chapter.

All the arguments are very rigorous and presented in depth, without burdening the reader with unnecessary details. The exposition is clear and intuitive with lots of suggestive examples. Moreover, the coloring of the definitions and the beautiful pictures make the study of the book a pleasant experience. Some pictures are so well designed that they represent proofs without words (a nice example is the picture that illustrates the jumping principle for the winding number). Furthermore, the historical marginal notes and the pictures of the mathematicians that obtained the results are very welcome.

As a minor drawback, we believe that the section dedicated to the covering properties of the exponential map is superfluous, taking into account the section about covering spaces, because the ideas in the particular case are pretty much the same as in the general setting.

The book is dedicated to graduate students and advanced undergraduate students. The main prerequisite is a basic background knowledge of Real Analysis, Topology and Measure Theory. In order to truly appreciate the geometric viewpoint and to enjoy the intuition behind some analytic results, we believe the reader should have some knowledge of Differential Geometry of curves and surfaces (in particular, tangent vectors, curvature of curves/surfaces, conformal maps and geodesics).

We encourage the reader to take a look also at the website of the book, where the author provides, for each chapter, additional comments, explanations, problems and an errata: http://qcpages.qc.cuny.edu/ zakeri/CAbook/ACCA.html

Mihai Iancu

Shahriar Shahriari, An Invitation to Combinatorics, Cambridge Mathematical Textbooks, xv + 613 p. 2022. ISBN 978-1-108-47654-6/hbk; 978-1-108-56870-8/ebook.

Combinatorics is a branch of mathematics that deals with counting problems and some other related concepts. Knowledge of the basic principles of combinatorics could greatly simplify the task of counting. The present book attempts at an accessible, amicable and conversational exposition of the art and the science of counting.

The first three chapters, 1. *Induction and recurrence relations*, 2. *The Pigeonhole Principle and Ramsey Theory,* and 3. *Counting, probability, balls and boxes*, are concerned with the foundational or fundamental concepts of combinatorics. These include induction, recurrence relations, the pigeonhole principle, multisets, graphs, Ramsey theory, Schur, Van der Waerden and graph Ramsey numbers, besides the fundamental principles of counting, such as the addition principle and the multiplication principle.

The next four chapters, 4. *Permutations and combinations*, 5. *Binomial and multinomial coefficients*, 6. *Stirling numbers,* and 7. *Integer partitions,* capitalize on the foundational concepts and introduce various techniques and special kinds of numbers that simplify the task of counting. These include permutations, falling factorials, combinations, binomial coefficients, lattice paths, Ming-Catalan numbers, Stirling numbers (both of the first and of the second kind), partitions of integers and pentagonal number theorem.

The last four chapters, 8. *The Inclusion-Exclusion Principle*, 9. *Generating functions*, 10. *Graph theory,* and 11. *Posets, matchings, and Boolean lattices*, are concerned with some advanced combinatorics concepts such as the inclusion-exclusion principle, combinations of multisets, restricted permutations, generating functions, basics of graph theory, posets (partially ordered sets), total orders and the matching problem.

The book also contains ten collaborative mini-projects meant for groups of three or four students to work and explore things collaboratively. There is a great emphasis on problem-solving and guided discovery.

The book has been written in a conversational style making it both accessible and engaging for the readers. The book is an excellent invitation to the world of combinatorial thinking.

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