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Coupled system of sequential partial $\sigma(.,.)$ −Hilfer fractional differential equations with weighted double phase operator: Existence, Hyers-Ulam stability and controllability

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Abstract. In this paper, we are concerned by a sequential partial Hilfer fractional differential system with weighted double phase operator. First, we introduce the concept of Hyers-Ulam stability with respect to an operator L for an abstract equation of the form $u = LFu$ in Banach lattice by using the fixed point arguments and spectral theory. Then, we prove the controllability and apply the previous results obtained for abstract equation to prove existence and Hyers-Ulam stability of a coupled system of sequential fractional partial differential equations involving a weighted double phase operator. Finally, example illustrating the main results is constructed. This work contains several new ideas, and gives a unified approach applicable to many types of differential equations.

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1. Introduction

Fractional order and Hilfer fractional order differential equations involving a p-Laplacian operator are of great importance and are interesting class of problems. Such kinds of problems have been studied by many authors, see [3, 4, 5, 17]. At the same time, the studies of Hyers-Ulam stability have attracted a great deal of attention in the last ten years, (see [1, 2, 9, 10, 11, 12, 15, 13, 16]), and the references therein.

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In [19], the authors discussed the existence of positive solutions for the double phase differential equation

$$-D_{p,q}(u)(x) = f(x,u), \ x \in \Omega \subset \mathbb{R}^n,$$

with double phase differential operator $D_{p,q}(u) = \Delta_p u + a.\Delta_q u$.

In [14], existence and uniquness of solutions to sequential fractional differential equation

$$\lambda D^{\alpha} u(t) + D^{\beta} u(t) = f(t, u(t))$$

was investigated.

In [8], the authors worked on the existence and Hyers–Ulam stability for the following sequential fractional differential system:

$$\left[{}^{c}D_q^{\nu} + r.{}^{c}D_q^{\sigma}\right] u(t) = f\left(t, u(t), u(\alpha t), {}^{c}D_q^{\sigma}(\alpha t)\right), \ t \in (0,T)$$

where D^{ν}, D^{σ} are the Caputo fractional derivatives of orders $\nu \in (1,2]$ and $\sigma \in (0,1]$ respectively.

Motivated by the works mentioned above, in this paper, we give the existence, Hyers-Ulam stability and controllability results for the abstract equation $LFu = u$ and their application to the following coupled sequential partial Hilfer fractional differential system with weighted double phase partial differential operator:

$$\begin{cases} \left(\zeta_1(t).D_{0^+,t}^{\alpha+1,\omega,\sigma} + D_{0^+,t}^{\alpha,\omega,\sigma}\right)\left(\frac{\partial}{\partial x}\left(\phi\left(\theta_1(x)\frac{\partial u_1}{\partial x}\right)\right)\right)(t,x) + f_1(t,x,u_1,u_2) = 0, \\ \quad t,x > 0, \\ \left(\zeta_2(t).D_{0^+,t}^{\alpha+1,\omega,\sigma} + D_{0^+,t}^{\alpha,\omega,\sigma}\right)\left(\frac{\partial}{\partial x}\left(\phi\left(\theta_2(x)\frac{\partial u_2}{\partial x}\right)\right)\right)(t,x) + f_2(t,x,u_1,u_2) = 0, \\ \quad t,x > 0, \\ u_j(0,x) = u_j(t,0) = \lim\limits_{x\to+\infty}\frac{\partial u_j}{\partial x}(t,x) = 0, \ j \in \{1,2\}, \end{cases} \tag{1.1}$$

where $D_{0^+,t}^{\alpha,\omega,\sigma}$ is the partial $\sigma(.,.)-$Hilfer fractional derivative with respect to the variable t of order α and type $0 \le \omega \le 1$ with $0 < \alpha < 1$,

$$\phi = \phi_{p^-} + \phi_{p^+}, \ 1 < p^- < p^+$$

with

$$\phi_{p^{\nu}}(x) = |x|^{p^{\nu}-2}.x, \ \text{for } \nu \in \{-,+\},$$

and for $j \in \{1,2\}$,

$$\zeta_j(t) = a_j + t, \ a_j > 0,$$

The function $\sigma(t,x)$ is bounded and positive on $\mathbb{R}^+ \times \mathbb{R}^+$ having a continuous and positive derivative $\frac{\partial \sigma}{\partial t}(t,x) > 0$ with respect to the variable t on $(0,+\infty)$ with $\sigma(0,x) = 0$ for all $x \ge 0$ and such that

$$\left(\sigma^+\right)^{\alpha} \in L^1\left(\mathbb{R}^+\right) \ \text{and} \ \sigma^+(x) = \lim_{t\to+\infty}\sigma(t,x).$$

2. Abstract background

Let $(E, \|.\|)$ be a real Banach space. A nonempty subset P of E is said to be a cone if P is closed and convex, $P \cap (-P) = 0$ and for all $t \geq 0$, $tP \subset P$. In this situation, P induces a partial order in the Banach space E defined by $x \leq y$ if $y - x \in P$.

The mapping $L : E \to E$ is said to be bounded if it maps bounded subsets in E into bounded subsets in E. L is said to be compact if it is continuous and maps bounded subsets in E into relatively compact subsets in E.

Definition 2.1. A normed lattice E is a vector space with a norm $\|.\|$ and a partial ordering $(\leq)$ under which it is a Riesz space and the following condition holds:
if $|x| \leq |y|$, then $\|x\| \leq \|y\|$, where

$$|u| = \sup\{u, -u\}.$$

If $(E, \|\|)$ is complete, it is called a Banach lattice.

Let us recall the definition and some properties of the resolvent:

Definition 2.2. [7, 18]Let $L : E \to E$ be a bounded and linear operator. The resolvent set of L is the set

$$\rho(L) = \{\lambda \in \mathbb{C} : \lambda I - L \text{ is invertible in } Q(E)\},$$

where $Q(E)$ is the unital Banach algebra defined by

$$Q(E) = \{f : E \to E : f \text{ is linear and bounded}\}$$

and $I : E \to E$ is the identity.

The resolvent of L is $r_L : \rho(L) \to Q(E)$ defined by

$$r_L(\lambda) = (\lambda I - L)^{-1} \in Q(E).$$

The spectrum of L, $\sigma(L) = \mathbb{C} \backslash \rho(L)$ is non-empty, compact and

$$r(L) = \max_{\lambda \in \sigma(L)} |\lambda| = \lim_{n \to \infty} \|L^n\|^{\frac{1}{n}},$$

called the spectral radius of L.

The serie's representation of the resolvent: If $|\lambda| > r(L)$, then $\lambda \in \rho(L)$ and $r_L(\lambda)$ is given by

$$r_L(\lambda) = \sum_{k=0}^{+\infty} \lambda^{-k-1} L^k.$$

Let $E^+ = \{u \in E, u \geq 0\}$ be the positive cone of a real Banach lattice $(E, \|.\|, \leq)$.

We consider an operator $T : E \to E$ defined by

$$Tu = LFu, \; u \in E$$

where $L : E \to E$ is a completely continuous operator and $F : E \to E$ is a continuous and bounded map.

Remark 2.3. T is completely continuous, because it is the composition of the completely continuous operator L and the bounded continuous map F.

We consider the equation

$$u = Tu. \tag{2.1}$$

Definition 2.4. Equation (2.1) is said to be Hyers-Ulam stable in E with respect to L (or L-Hyers-Ulam stable), if $T = LF$ and there exists $N > 0$, such that the following (p_N) property is satisfied:

$$\left\{ \begin{array}{c} \text{For all } \epsilon > 0 \text{ and all } (v, w) \in E \times \bar{B}(0, \epsilon) \setminus \{0\}, \\ \text{if } v = L(F(v) + w) \text{ then } T \text{ admits a fixed point } u \in G \text{ such that} \\ \|u - v\| \leq N.\epsilon. \end{array} \right. \tag{p_N}$$

The main tools of this work are the following Theorems:

Theorem 2.5. [6] *Let E be a Banach space, C be a nonempty bounded convex and closed subset of E, and $T : C \to C$ be a compact and continuous map. Then T has at least one fixed point in C.*

3. Main results

3.1. Existence and Hyers-Ulam stability of abstract equation

Throughout this paper, we assume that the following hypothesis hold:

$$\left\{ \begin{array}{c} \text{There exists an operator } L^{(k)} : E^+ \to E^+ \text{ such that, for all } u \in E \\ |L(u)| \leq L^{(k)}(|u|), \end{array} \right. \tag{3.1}$$

where $L^{(k)}$ is bounded, increasing, $k-$positively homogeneous and sub-additive on E, $k \in (0, 1]$, with $L^{(k)}(E^+ \setminus \{0\}) \subset E^+ \setminus \{0\}$.

$F : E \to E$ is a continuous mapping such that

$$\left\{ \begin{array}{c} \text{There exist } (g, h) \in E^+ \setminus \{0\} \times E^+ \text{ such that } \left\| L^{(k)}(g) \right\| < 1 \text{ and} \\ |F(u)| \leq g \|u\|^{\frac{1}{k}} + h, \text{ for all } u \in E. \end{array} \right. \tag{3.2}$$

Lemma 3.1. *Assume that If the hypothesis (3.1) and (3.2) hold true, and let Then T admits a fixed point u in $\bar{B}(0, r)$, $r > r_0$, where*

$$r_0 = \frac{\left\| L^{(k)}(h) \right\|}{1 - \left\| L^{(k)}(g) \right\|} \geq 0.$$

Proof. Let $u \in \bar{B}(0, r)$, $r > r_0$. So,

$$\begin{aligned} |Tu| &= |LFu| \leq L^{(k)}(|Fu|) \leq L^{(k)}\left(\|u\|^{\frac{1}{k}}.g + h \right) \\ &\leq \|u\|.L^{(k)}(g) + L^{(k)}(h) \end{aligned}$$

this implies that

$$\|Tu\| \leq r. \left\| L^{(k)}(g) \right\| + \left\| L^{(k)}(h) \right\| = (r - r_0). \left\| L^{(k)}(g) \right\| + r_0 \leq r,$$

then $T\left(\bar{B}(0, r)\right) \subset \bar{B}(0, r)$. From Schauder fixed point theorem, we deduce that T has at least one fixed point $u \in \bar{B}(0, r)$. □

Lemma 3.2. *Assume that hypothesis (3.1) and (3.2) hold true.*
If $(v,w) \in E \times \bar{B}(0,\epsilon) \setminus \{0\}$, $\epsilon > 0$ *such that*

$$v = L(F(v) + w),$$

then $v \in \bar{B}(0, r_\epsilon)$, *with*

$$r_\epsilon = \frac{\left\|L^{(k)}(h)\right\| + \epsilon^k M}{1 - \left\|L^{(k)}(g)\right\|} \text{ and } M = \sup\left\{\left\|L^{(k)}(x)\right\|, x \in \bar{B}(0,1)\right\}.$$

Proof. Indeed, if $v = L(Fv + w)$, then

$$\begin{aligned} |v| &= |L(Fv+w)| \leq L^{(k)}(|Fv| + |w|) \leq L^{(k)}\left(\|v\|^{\frac{1}{k}}.g + h + |w|\right) \\ &\leq \|v\|.L^{(k)}(g) + L^{(k)}(h) + L^{(k)}(|w|). \end{aligned}$$

This leads

$$\|v\| \leq \|v\|.\left\|L^{(k)}(g)\right\| + \left\|L^{(k)}(h)\right\| + \left\|L^{(k)}(|w|)\right\|.$$

Thus

$$\|v\| \leq \frac{\left\|L^{(k)}(h)\right\| + \left\|L^{(k)}(|w|)\right\|}{1 - \left\|L^{(k)}(g)\right\|} \leq \frac{\left\|L^{(k)}(h)\right\| + \epsilon^k M}{1 - \left\|L^{(k)}(g)\right\|}.$$ □

Let $r_* = \max\left\{r_0, (r_0)^{\frac{1}{k}}\|g\| + \|h\|\right\} \geq 0$, where r_0 is the constant given in Lemma (3.1). We consider the following hypothesis:

There exist $\rho \in E^+ \setminus \{0\}$, $\lambda > 0$ and $r > r_*$ such that, for all $u, v \in \bar{B}(0,r)$,

$$|Fu - Fv| \leq \rho\|u - v\|, \tag{3.3}$$

and

$$|L(u) - L(v)| \leq \lambda L_+ |u - v|. \tag{3.4}$$

where L_+ is a linear, bounded and strictly positive operator on E.

Theorem 3.3. *Assume that hypothesis (3.1), (3.2), (3.3) and (3.4) hold true, and*

$$\lambda \in \left(0, \|L_+(\rho)\|^{-1}\right). \tag{3.5}$$

Then, equation (2.1) is L-Hyers-Ulam stable in E.

Proof. Suppose that

$$v = L(F(v) + w),$$

where $(v,w) \in E \times \bar{B}(0,\epsilon) \setminus \{0\}$, $\epsilon > 0$.
Let $r > r_* = \max\left\{r_0, (r_0)^{\frac{1}{k}}\|g\| + \|h\|\right\}$ be the constant given in the hypothesis (3.3).

We deduce from lemmas (3.1) and (3.2) that T admits a fixed point $u \in \bar{B}(0,r)$ and $v \in \bar{B}(0, r_\epsilon)$, with

$$r_\epsilon = \frac{\left\|L^{(k)}(h)\right\| + \epsilon^k M}{1 - \left\|L^{(k)}(g)\right\|} \text{ and } M = \sup\left\{\left\|L^{(k)}(x)\right\|, x \in \bar{B}(0,1)\right\}.$$

Now, let $x_0 > 0$ be the unique positive solution of the algebraic equation

$$\left(r_0 + \frac{M}{1-\left\|L^{(k)}(g)\right\|}.x^k\right)^{\frac{1}{k}} \|g\| + \|h\| + x - r = 0.$$

We distinguish the following three cases:

Case 1. If $r < (r_\epsilon)^{\frac{1}{k}} \|g\| + \|h\| + \epsilon$, then $\epsilon > x_0$. This leads

$$\|u - v\| \leq r + r_\epsilon \leq x_0^{-1}\left(2r + \frac{M.x_0^k}{1-\left\|L^{(k)}(g)\right\|}\right).\epsilon.$$

Case 2. If $r < r_\epsilon$, then $\epsilon > \mu$, with

$$\mu = \left[\frac{(r - r_0)\left(1-\left\|L^{(k)}(g)\right\|\right)}{M}\right]^{\frac{1}{k}},$$

and so,

$$\|u - v\| \leq 2r + \frac{\epsilon^k M}{1-\left\|L^{(k)}(g)\right\|} \leq \mu^{-1}\left(2r + \frac{M.\mu^k}{1-\left\|L^{(k)}(g)\right\|}\right).\epsilon.$$

Case 3. If $\max\left\{r_\epsilon, (r_\epsilon)^{\frac{1}{k}} \|g\| + \|h\| + \epsilon\right\} \leq r$, then $(Fu, (Fv) + w) \in \bar{B}(0, r) \times \bar{B}(0, r)$, and from hypothesis (3.4), it follows that

$$|L(Fu) - L(Fv + w)| \leq \lambda L_+ |Fu - Fv - w|.$$

And by using (3.3), we obtain

$$\begin{aligned} |u - v| &\leq \lambda L_+ |Fu - Fv - w| \\ &\leq \lambda L_+ |Fu - Fv| + \lambda L_+ (|w|) \\ &\leq \lambda. \|u - v\| L_+(\rho) + \lambda \epsilon L_+\left(\frac{|w|}{\|w\|}\right) \end{aligned}$$

thus

$$\|u - v\| \leq \left(\frac{\lambda \|L_+\|}{1 - \lambda. \|L_+(\rho)\|}\right).\epsilon.$$

Consequently,

$$\|u - v\| \leq N.\epsilon$$

where

$$N = \max\left\{\gamma_1'\left(2r + \frac{M.\gamma_2'}{1-\left\|L^{(k)}(g)\right\|}\right), \left(\frac{\lambda \|L_+\|}{1 - \lambda. \|L_+(\rho)\|}\right)\right\},$$

with

$$\gamma_1' = \max\left\{x_0^{-1}, \mu^{-1}\right\} \text{ and } \gamma_2' = \max\left(x_0^k, \mu^k\right).$$

Proving our claim. □

Now, we replace the hypothesis (3.3) and (3.4) by the following conditions: There exists $\lambda_0 > 0$ and $r > r_*$ such that, for all $u, v \in \bar{B}(0, r)$,

$$|F(u) - F(v)| \leq \lambda_0 |u - v|, \tag{3.6}$$

and

$$|L(u) - L(v)| \leq L_0 |u - v|, \tag{3.7}$$

where $L_0 : E \to E$ is a linear, compact and strictly positive operator.

Theorem 3.4. *Assume that hypothesis (3.1), (3.2), (3.6) and (3.7) hold, and*

$$r(L_0) < \lambda_0^{-1}. \tag{3.8}$$

Then equation (2.1) is L-Hyers-Ulam stable in E.

Proof. Suppose that $v = L(F(v) + w)$, $(v, w) \in E \times \bar{B}(0, \epsilon) \setminus \{0\}$, $\epsilon > 0$. Let $r > r_* = \max\left\{r_0, (r_0)^{\frac{1}{k}} \|g\| + \|h\|\right\}$ is the constant given in the hypothesis (3.6). It follows from lemmas (3.1) and (3.2), that $v \in \bar{B}(0, r_\epsilon)$ and T admits a fixed point $u \in \bar{B}(0, r)$, with

$$r_\epsilon = \frac{\left\|L^{(k)}(h)\right\| + \epsilon^k M}{1 - \left\|L^{(k)}(g)\right\|} \text{ and } M = \sup\left\{\left\|L^{(k)}(x)\right\|, x \in \bar{B}(0, 1)\right\}.$$

We have seen in the proof of theorem (3.3) that, if

$$r \leq \max\left\{r_\epsilon, (r_\epsilon)^{\frac{1}{k}} \|g\| + \|h\| + \epsilon\right\},$$

then $\epsilon \geq \max\{\mu, x_0\}$, where $x_0 > 0$ is the positive solution of the algebraic equation

$$\left(r_0 + \frac{M}{1 - \left\|L^{(k)}(g)\right\|}.x^k\right)^{\frac{1}{k}} \|g\| + \|h\| + x - r = 0$$

In this case, we have

$$\|u - v\| \leq \gamma_1' \left(2r + \frac{M.\gamma_2'}{1 - \left\|L^{(k)}(g)\right\|}\right).\epsilon,$$

where

$$\gamma_1' = \max\left\{x_0^{-1}, \mu^{-1}\right\} \text{ and } \gamma_2' = \max\left(x_0^k, \mu^k\right).$$

Now, we assume that $\max\left\{r_\epsilon, (r_\epsilon)^{\frac{1}{k}} \|g\| + \|h\| + \epsilon\right\} \leq r$. Then $(Fu, (Fv) + w) \in \bar{B}(0, r) \times \bar{B}(0, r)$, and by using hypothesis (3.4), it follows that

$$|L(Fu) - L(Fv + w)| \leq L_0 |Fu - Fv - w|. \tag{3.9}$$

By using (3.6), inequality (3.9) leads

$$\begin{aligned} |u - v| &\leq L_0 |Fu - Fv - w| \\ &\leq L_0 |Fu - Fv| + L_0 (|w|) \\ &\leq \lambda_0.L_0 (|u - v|) + \epsilon.\pi_w, \end{aligned}$$

where

$$\pi_w = L_0\left(\frac{|w|}{\|w\|}\right) \in E^+\backslash\{0\}.$$

Then

$$\begin{aligned} z &= |u-v| \leq \lambda_0.L_0(z) + \epsilon L_0\left(\frac{|w|}{\|w\|}\right) \\ &\leq \lambda_0.L_0(z) + \epsilon\pi_w \\ &\leq \lambda_0.L_0(\lambda_0.L_0(z) + \epsilon\pi_w) + \epsilon\pi_w \\ &\leq \lambda_0^3.L_0^3(z) + \epsilon.\left(\lambda_0^2.L_0^2(\pi_w) + \lambda_0.L_0(\pi_w) + \pi_w\right) \\ &\leq \lambda_0^n.L_0^n(z) + \epsilon.\sum_{k=0}^{n-1}\lambda_0^k L_0^k(\pi_w) \in E^+\backslash\{0\}, \text{ for all } n \in \mathbb{N}^*. \end{aligned}$$

As $\lambda_0.r(L_0) = \lambda_0.\lim_{n\to\infty}\sqrt[n]{\|L_0^n\|} < 1$ then $\lim_{n\to\infty}\lambda_0^n.L_0^n(z) = 0$, $\lambda_0^{-1} \in \rho(L_0)$ and $(I - \lambda_0.L_0)$ is invertible. The serie's representation of the resolvent r_{L_0} at λ_0^{-1} is given by

$$r_{L_0}\left(\lambda_0^{-1}\right) = \left(\lambda_0^{-1}I - L_0\right)^{-1} = \sum_{k=0}^{+\infty}(\lambda_0)^{k+1}L_0^k.$$

Then

$$\sum_{k=0}^{+\infty}\lambda_0^k L_0^k(\pi_w) = (I - \lambda_0.L_0)^{-1}(\pi_w) \in E^+\backslash\{0\}.$$

Thus,

$$\|u-v\| \leq \left\|(I-\lambda_0.L_0)^{-1}(\pi_w)\right\|.\epsilon \leq \left\|(I-\lambda_0.L_0)^{-1}\right\|\|L_0\|.\epsilon.$$

Consequently,

$$\|u-v\| \leq N.\epsilon$$

where

$$N = \max\left\{\gamma_1'\left(2r_0 + \frac{M.\gamma_2'}{1-\left\|L^{(k)}(g)\right\|}\right), \left\|(I-\lambda_0.L_0)^{-1}\right\|\|L_0\|\right\}.$$

Proving our claim. □

3.2. Existence and Hyers-Ulam stability of coupled system IVS

In this section, we use the results obtained in the previous section to prove existence and Hyers-Ulam stability of the coupled system of sequential time $\sigma-$Hilfer fractional differential equations (1.1), where $D_{0^+,t}^{\alpha,\omega,\sigma}$ is the $\sigma-$Hilfer fractional derivative with respect to the variable t of order α and type $0 \leq \omega \leq 1$ with $0 < \alpha < 1$,

$$\phi = \phi_{p^-} + \phi_{p^+}\ ,\ 1 < p^- < p^+$$

with

$$\phi_{p^\nu}(x) = |x|^{p^\nu - 2}.x, \text{ for } \nu \in \{-,+\},$$

and for $j \in \{1,2\}$,

$$\zeta_j(t) = a_j + t,\ a_j > 0.$$

We suppose that the following conditions hold,

$$\begin{cases} f_j \in C\left(\mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^2, \mathbb{R}\right), \ \frac{1}{\theta_j} \in L^1\left(\mathbb{R}^+, \mathbb{R}^+\right) \\ \text{and} \\ \sigma^+ \in L^\alpha \ \left(\mathbb{R}^+\right), \end{cases} \tag{3.10}$$

with

$$0 < \sigma^+ (x) = \sup \{\sigma (t, x), \ t \geq 0\} < \infty, \ \forall x \geq 0.$$

Next, we recall the definitions of $\sigma-$Hilfer fractional orders integrals and derivatives of order α and type $0 \leq \omega \leq 1$, where $J \subset \mathbb{R}^n$ and $\sigma : I \times J \to \mathbb{R}^+$ is the positive function on $I \times J \subset \mathbb{R}^+ \times \mathbb{R}^+$ having a continuous and positive derivative $\dfrac{\partial \sigma}{\partial t}(t, x) > 0$ with respect to the variable t on $(0, +\infty)$ with $\sigma(0, x) = 0$ for all $x \geq 0$.

Definition 3.5. [17] Let $a \in \mathbb{R}^+$, $\alpha > 0$ and $J \subset \mathbb{R}^n$. Then the $\sigma-$left-sided fractional integral of a function u with respect to t on $\mathbb{R}^+$ is defined by

$$I^{\alpha,\sigma}_{a^+,t} u(t,x) = \frac{1}{\Gamma(\alpha)} \int_a^t \frac{\partial \sigma}{\partial t}(t,x) \left(\sigma(t,x) - \sigma(\tau,x)\right)^{\alpha-1} u(\tau,x) d\tau.$$

In the case $\alpha = 0$, this integral is interpreted as the identity operator $I^{0,\sigma}_{a^+} u = u$.

Definition 3.6. [17] Let $\alpha \in (n-1, n)$ with $n \in \mathbb{N}$, u and σ two functions such that $t \mapsto u(t,.) \in C^n(\mathbb{R}^+, \mathbb{R})$ and $t \mapsto \sigma(t,.) \in C^n(\mathbb{R}^+, \mathbb{R})$. The σ-Hilfer fractional derivative $D^{\alpha,\omega,\sigma}_{a^+,t}$ of u with respect to t of order $n - 1 < \alpha < n$ and type $0 \leq \omega \leq 1$ is defined by

$$D^{\alpha,\omega,\sigma}_{a^+,t} u(t,x) = I^{\omega(n-\alpha),\sigma}_{a^+,t} \left(\frac{1}{\sigma'_t(t,x)} \frac{\partial}{\partial t}\right)^n I^{(1-\omega)(n-\alpha),\sigma}_{a^+,t} u(t,x),$$

where $\sigma'_t(t,x) = \dfrac{\partial \sigma}{\partial t}(t,x)$.

Let's also recall the following important result ([17]):

Theorem 3.7. *If $t \mapsto u(t,x) \in C^n(\mathbb{R}^+)$, $n-1 < \beta < \alpha < n$, $0 \leq \omega \leq 1$ and $\xi = \alpha + \omega(n - \alpha)$, then*

$$I^{\alpha,\sigma}_{a^+,t} . D^{\alpha,\omega,\sigma}_{a^+,t} u(t,x)$$

$$= u(t,x) - \sum_{k=1}^{n} \frac{(\sigma(t,x) - \sigma(a,x))^{\xi-k}}{\Gamma(\xi - k + 1)} \left(\frac{1}{\sigma'_t(t,x)} \frac{\partial}{\partial t}\right)^{n-k} I^{(1-\omega)(n-\alpha),\sigma}_{a^+,t} u(a,x).$$

Moreover,

$$I^{\alpha,\sigma}_{a^+,t} I^{\beta,\sigma}_{a^+,t}(u) = I^{\alpha+\beta,\sigma}_{a^+,t}, \quad D^{\alpha,\omega,\sigma}_{a^+,t}\left(D^{\beta,\omega,\sigma}_{a^+,t} u\right) = D^{\alpha+\beta,\omega,\sigma}_{a^+,t} u,$$

$$D^{1,\omega,\sigma}_{a^+,t} u = D^1_t u = \frac{\partial u}{\partial t} \ \textit{and} \ D^{\alpha,\omega,\sigma}_{a^+,t} I^{\alpha,\sigma}_{a^+,t}(u) = u.$$

Remark 3.8. In this paper, we assume that $\sigma : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ is continuous having a positive and continuous derivative $\frac{\partial \sigma}{\partial t}(t,x)$ on $\mathbb{R}^+ \times \mathbb{R}^+$ such that $\sigma(0,x) = 0$, for all $x \in \mathbb{R}^+$. If $\alpha \in (0,1)$, then $n = 1$ and for $t, x > 0$

$$I_{0^+,t}^{\alpha,\sigma} . D_{0^+,t}^{\alpha,\omega,\sigma} u(t,x) = u(t,x) - \frac{(\sigma(t,x))^{\xi-1}}{\Gamma(\xi)} \left(I_{0^+,t}^{(1-\omega)(1-\alpha),\sigma} u \right) (0^+, x).$$

Moreover, if u is continuous, then

$$\lim_{t \to 0^+} \left(I_{0^+,t}^{(1-\omega)(1-\alpha),\sigma} u \right)(t,x) = 0, \ \forall x \geq 0$$

and so $I_{0^+,t}^{\alpha,\sigma} . D_{0^+,t}^{\alpha,\omega,\sigma} u(t,x) = u(t,x)$.

Definition 3.9. We say that IVS (1.1) has the Hyers-Ulam stability in a Banach space $E = G \times G$ if there exits a constant $N > 0$ such that for every $\epsilon > 0$, $v = (v_1, v_2) \in E$, if

$$\begin{cases} \left| \left(\zeta_1(t) . D_{0^+,t}^{\alpha+1,\omega,\sigma} + D_{0^+,t}^{\alpha,\omega,\sigma} \right) \left(\frac{\partial}{\partial x} \left(\phi \left(\theta_1(x) \frac{\partial v_1}{\partial x} \right) \right) \right)(t,x) + f_1(t,x,v_1,v_2) \right| \leq \epsilon, \\ t, x > 0, \\ \\ \left| \left(\zeta_2(t) . D_{0^+,t}^{\alpha+1,\omega,\sigma} + D_{0^+,t}^{\alpha,\omega,\sigma} \right) \left(\frac{\partial}{\partial x} \left(\phi \left(\theta_2(x) \frac{\partial v_2}{\partial x} \right) \right) \right)(t,x) + f_2(t,x,v_1,v_2) \right| \leq \epsilon, \\ t, x > 0, \\ \\ v_j(0,x) = v_j(t,0) = \lim\limits_{x \to +\infty} \frac{\partial v_j}{\partial x}(t,x) = 0, \ j \in \{1,2\}, \end{cases} \tag{3.11}$$

then there exists a solution $u \in E$ of IVS (1.1), such that

$$\|u - v\| \leq N.\epsilon. \tag{3.12}$$

We call such N a Hyers-Ulam stability constant.

Let $E = G \times G$ be a real Banach space with

$$G = \left\{ u \in C(\mathbb{R}^+ \times \mathbb{R}^+, \mathbb{R}) : \sup_{t,x \geq 0} |u(t,x)| < \infty \right\}$$

equipped with the norm $\|(u,v)\| = \max(\|u\|_0, \|v\|_0)$ where

$$\|u\|_0 = \sup_{t,x \in \mathbb{R}^+} (|u(t,x)|).$$

Remark 3.10. E is a Banach lattice under the partial ordering $(\leq)$ defined by

$$(u_1, u_2) \leq (v_1, v_2) \Leftrightarrow u_1(x) \leq v_1(x) \text{ and } u_2(x) \leq v_2(x) \text{ for all } x \geq 0.$$

under which it is a Riesz space and $|(u,v)| = (|u|, |v|)$.
Moreover, $E^+ = \{(u,v) \in E, (u,v) \geq 0\}$ is the positive cone of $(E, \|.\|, \leq)$.

We consider the operator $T : E \to E$ defined by

$$T(u_1, u_2) = LF(u_1, u_2), \quad (u_1, u_2) \in E$$

where

$$L(u_1, u_2) = (L_1(u_1, u_2), L_2(u_1, u_2)) \text{ and } F(u_1, u_2) = (F_1(u_1, u_2), F_2(u_1, u_2)),$$

such that for $j \in \{1, 2\}$

$$\begin{aligned} L_j(u_1, u_2)(t, x) &= \int_0^x \frac{1}{\theta_j(z)} \psi\left(\int_z^{+\infty} I_{0^+,t}^{\alpha,\sigma}\left(\frac{1}{\zeta_j(t)} \int_0^t (u_j)(\tau, s)\, d\tau\right)(t, s)\, ds\right) dz, \\ F_j(u_1, u_2)(t, x) &= f_j(t, x, u_1(t, x), u_2(t, x)), \end{aligned}$$

where $\psi = \phi^{-1} : \mathbb{R} \to \mathbb{R}$ is the inverse function of sum of p_i-Laplacian operators

$$\phi = \sum_{i=1}^{i=N} \phi_{p_i},$$

with $\phi_{p_i}(x) = |x|^{p_i - 2}.x$ and ψ_{p_i} is the inverse function of ϕ_{p_i}.

We denote

$$T = (T_1, T_2)$$

with

$$T_j = L_j F, \; j \in \{1, 2\}.$$

Remark 3.11. Let $p^- = \min\{p_1, p_2 ... p_N\}$ and $p^+ = \max\{p_1, p_2 ... p_N\}$. For all $x \geq 0$, $i \in \{1, 2 ... N\}$

$$\phi_{p_i}(x) \leq \phi(x) \leq N.\phi^+(x)$$

where

$$\phi^+(x) = \begin{cases} \phi_{p^+}(x) & if\ x \geq 1 \\ \phi_{p^-}(x) & if\ x \leq 1 \end{cases}$$

and so, we conclude that

$$\psi^+\left(\frac{x}{N}\right) \leq \psi(x) \leq \psi_{p_i}(x) \tag{3.13}$$

where

$$\psi^+\left(\frac{x}{N}\right) = \begin{cases} \psi_{p^+}\left(\frac{x}{N}\right) & if\ x \geq 1 \\ \psi_{p^-}\left(\frac{x}{N}\right) & if\ x \leq 1. \end{cases}$$

Moreover, for $x \geq y \geq 0$,

$$\begin{cases} \psi_p(x + y) \leq \psi_p(x) + \psi_p(y), & \text{if } p \geq 2, \\ \psi_p(x + y) \leq (2)^{\frac{2-p}{p-1}}.[\psi_p(x) + \psi_p(y)], & \text{if } p < 2. \end{cases} \tag{3.14}$$

Remark 3.12. The condition (3.10) makes that the operator L_j is completely continuous and F_j is bounded for each $j \in \{1, 2\}$, and so, T is completely continuous.

Lemma 3.13. *Let $h_1, h_2 \in C(\mathbb{R}^+ \times \mathbb{R}^+, \mathbb{R}^+)$ be continuous and bounded functions. $(u_1, u_2) \in C^1(\mathbb{R}^+ \times \mathbb{R}^+) \times C^1(\mathbb{R}^+ \times \mathbb{R}^+)$ is solution of IVS (3.15)*

$$\begin{cases} \left(\zeta_1(t).D_{0^+,t}^{\alpha+1,\omega,\sigma} + D_{0^+,t}^{\alpha,\omega,\sigma}\right)\left(\frac{\partial}{\partial x}\left(\phi\left(\theta_1(x)\frac{\partial u_1}{\partial x}\right)\right)\right)(t,x) + h_1(t,x) = 0, \\ \quad t, x > 0, \\ \left(\zeta_2(t).D_{0^+,t}^{\alpha+1,\omega,\sigma} + D_{0^+,t}^{\alpha,\omega,\sigma}\right)\left(\frac{\partial}{\partial x}\left(\phi\left(\theta_2(x)\frac{\partial u_2}{\partial x}\right)\right)\right)(t,x) + h_2(t,x) = 0, \\ \quad t, x > 0, \\ u_j(0,x) = u_j(t,0) = \lim_{x\to+\infty}\frac{\partial u_j}{\partial x}(t,x) = 0, \ j \in \{1,2\}, \end{cases} \tag{3.15}$$

if and only if

$$u_j(t,x) = \int_0^x \frac{1}{\theta_j(z)}\psi\left(\int_z^{+\infty} I_{0^+,t}^{\alpha,\sigma}\left(\frac{1}{\zeta_j(t)}\int_0^t h_j(\tau,s)\,d\tau\right)(t,s)\,ds\right)dz, \ \textit{for } j \in \{1,2\}.$$

(u_1, u_2) is fixed point of T (i.e $T(u_1,u_2) = (u_1,u_2)$).

Proof. First, assume that $(u_1, u_2) \in E$ is a solution of IVS (3.15), then for each $j \in \{1,2\}$, The function u_j satisfies equation

$$D_t^1\left((a_j + t).D_{0^+,t}^{\alpha,\omega,\sigma}\left[\frac{\partial}{\partial x}\left(\phi\left(\theta_j(x)\frac{\partial u_j}{\partial x}\right)\right)\right]\right)(t,x) = -h_j(t,x),$$

where $\phi = \phi_{p^-} + \phi_{p^+}$. Integrating, we have

$$D_{0^+,t}^{\alpha,\omega,\sigma}\left[\frac{\partial}{\partial x}\left(\phi\left(\theta_j(x)\frac{\partial u_j}{\partial x}\right)\right)\right](t,x) = \frac{-1}{a_j + t}\int_0^t h_j(\tau,x)\,d\tau, \ t > 0. \tag{3.16}$$

Applying $I_{0^+,t}^{\alpha,\sigma}$ on both sides of equation (3.16) and using Lemma (3.7) and initial condition $\frac{\partial u_j}{\partial x}(0,x) = 0$, we obtain

$$\frac{\partial}{\partial x}\left(\phi\left(\theta_j(x)\frac{\partial u_j}{\partial x}\right)\right)(t,x) = -I_{0^+,t}^{\alpha,\sigma}\left(\frac{1}{\zeta_j(t)}\int_0^t h_j(\tau,x)\,d\tau\right)(t,x)$$

By integrating on $[x, +\infty[$ and using the boundary conditions

$$u_j(t,0) = \lim_{x\to+\infty}\frac{\partial u_j}{\partial x}(t,x) = 0,$$

we have

$$\phi\left(\theta_j(x)\frac{\partial u_j}{\partial x}\right) = \int_x^{+\infty} I_{0^+,t}^{\alpha,\sigma}\left(\frac{1}{\zeta_j(t)}\int_0^t h_j(\tau,s)\,d\tau\right)(t,s)\,ds$$

and so

$$u_j(t,x) = \int_0^x \frac{1}{\theta_j(z)}\psi\left(\int_z^{+\infty} I_{0^+,t}^{\alpha,\sigma}\left(\frac{1}{\zeta_j(t)}\int_0^t h_j(\tau,s)\,d\tau\right)(t,s)\,ds\right)dz.$$

Conversely, assume that $(u_1, u_2) \in E$ such that for $j \in \{1,2\}$,

$$u_j(t,x) = \int_0^x \frac{1}{\theta_j(z)} \psi\left(\int_z^{+\infty} I_{0^+,t}^{\alpha,\sigma} \left(\frac{1}{\zeta_j(t)} \int_0^t h_j(\tau,s)\, d\tau \right)(t,s)\, ds \right) dz.$$

Then $u_j \in C^1(\mathbb{R}^+ \times \mathbb{R}^+)$ and verifies

$$u_j(x,0) = u_j(0,x) = 0.$$

Moreover, by derivating with respect to the variable x, we obtain

$$\frac{\partial u_j}{\partial x}(t,x) = \frac{1}{\theta_j(x)} \psi\left(\int_x^{+\infty} I_{0^+,t}^{\alpha,\sigma} \left(\frac{1}{\zeta_j(t)} \int_0^t h_j(\tau,s)\, d\tau \right)(t,s)\, ds \right), \tag{3.17}$$

and so

$$\frac{\partial}{\partial x} \phi\left(\theta_j(x) \frac{\partial u_j}{\partial x} \right) = -I_{0^+,t}^{\alpha,\sigma} \left(\frac{1}{\zeta_i(t)} \int_0^t h_j(\tau,x)\, d\tau \right)(t,x). \tag{3.18}$$

Applying $D_{0^+,t}^{\alpha,\omega,\sigma}$ on both sides of equation (3.18) and using Lemma (3.7) we have

$$\zeta_j(t).D_{0^+,t}^{\alpha,\omega,\sigma} \left[\frac{\partial}{\partial x} \left(\phi\left(\theta_j(x) \frac{\partial u_j}{\partial x} \right) \right) \right](t,x) = -\int_0^t h_j(\tau,x)\, d\tau,$$

so, u_j is solution of the equation

$$D_t^1 \left(\zeta_j(t).D_{0^+,t}^{\alpha,\omega,\sigma} \left[\frac{\partial}{\partial x} \left(\phi\left(\theta_j(x) \frac{\partial u_j}{\partial x} \right) \right) \right] \right)(t,x) = -h_j(t,x).$$

Now, we show that $\lim_{x \to +\infty} \frac{\partial u_j}{\partial x}(t,x) = 0$. Let $H_j = \sup\{h_j(t,x), t,x \geq 0\}$. We have

$$I_{0^+,t}^{\alpha,\sigma} \left(\frac{1}{\zeta_j(t)} \int_0^t h_j(\tau,s)\, d\tau \right)(t,s) \leq H_j.I_{0^+,t}^{\alpha,\sigma}(1)(t,s) = \frac{H_j}{\Gamma(\alpha+1)} \sigma^\alpha(t,s),$$

then

$$\int_x^{+\infty} I_{0^+,t}^{\alpha,\sigma} \left(\frac{1}{\zeta_j(t)} \int_0^t h_j(\tau,s)\, d\tau \right)(t,s)\, ds \leq \frac{H_j}{\Gamma(\alpha+1)} \int_x^{+\infty} \sigma^\alpha(t,s)\, ds$$

so, it follows from equation (3.17) that

$$\begin{aligned} \frac{\partial u_j}{\partial x}(t,x) &\leq \frac{1}{\theta_j(x)} \psi\left(\frac{H_j}{\Gamma(\alpha+1)} \int_x^{+\infty} \sigma^\alpha(t,s)\, ds \right) \\ &\leq \frac{1}{\theta_j(x)} \psi\left(\frac{H_j}{\Gamma(\alpha+1)} \int_0^{+\infty} (\sigma^+)^\alpha(s)\, ds \right) \end{aligned}$$

Since $\frac{1}{\theta_i(x).} \in L^1(\mathbb{R}^+, \mathbb{R}^+)$ then

$$\lim_{x \to +\infty} \frac{\partial u_j}{\partial x}(t,x) = 0.$$

Thus, (u_1, u_2) is solution of IVS (3.15). This completes the proof. □

Remark 3.14. We deduce from Lemma (3.13) that, $(u_1, u_2) \in C^1(\mathbb{R}^+ \times \mathbb{R}^+, \mathbb{R})$ is solution of IVS (1.1) if and only if (u_1, u_2) is a fixed point of T.

Lemma 3.15. *If equation (2.1) is L-Hyers-Ulam stable in E then IVS (1.1) has the Hyers-Ulam stability in E.*

Proof. Assume that equation (2.1) is L-Hyers-Ulam stable in E. Let $\epsilon > 0$ and $v = (v_1, v_2) \in E$ verifying inequalities (3.11). Let $w = (w_1, w_2) \in \bar{B}_E(0, \epsilon)$ such that

$$w_j(t) = -\left(\zeta_j(t) . D^{\alpha+1,\omega,\sigma}_{0^+,t} + D^{\alpha,\omega,\sigma}_{0^+,t}\right)\left(\frac{\partial}{\partial x}\left(\phi\left(\theta_j(x)\frac{\partial v_j}{\partial x}\right)\right)\right)(t,x) - f_j(t, v_1(t), v_2(t)),$$
$$j \in \{1, 2\}.$$

We have from Lemma (3.13) that

$$v_j(x) = T_j(v_1, v_2)(x) == L_j(F(v_1, v_2) + w),$$

then

$$v = L(F(v) + w).$$

If $w = (0, 0)$ then v is a fixed point of T, and so, $u = v$ is solution of IVS (1.1) and we have

$$\|u - v\| = 0 \leq N.\epsilon.$$

Now, if $w \in \bar{B}_E(0, \epsilon) \setminus \{0\}$, as (2.1) is L-Hyers-Ulam stable then there exists a fixed point u of T which is solution of IVS (1.1) such that

$$\|u - v\| \leq N.\epsilon.$$

Thus, IVS (1.1) has the Hyers-Ulam stability in E. □

Lemma 3.16. *Assume that*

$$p^+ \geq 2. \tag{3.19}$$

Then L verifies the condition (3.1), with $L^{(k)} = \left(L_1^{(k)}, L_2^{(k)}\right)$ *such that*

$$k = \frac{1}{p^+ - 1} \leq 1,$$

where for $j \in \{1, 2\}$

$$L_j^{(k)}(u_1, u_2)(t, x) = \int_0^x \frac{1}{\theta_j(z)}\psi_{p^+}\left(\int_z^{+\infty} I^{\alpha,\sigma}_{0^+,t}\left(\frac{1}{\zeta_j(t)}\int_0^t u_j(\tau, s)\, d\tau\right)(t, s)\, ds\right) dz.$$

Proof. Let $u = (u_1, u_2) \in E$. For $j \in \{1, 2\}$

$$\begin{aligned} |L_j(u_1, u_2)(t, x)| &= \left|\int_0^x \frac{1}{\theta_j(z)}\psi\left(\int_z^{+\infty} I^{\alpha,\sigma}_{0^+,t}\left(\frac{1}{\zeta_j(t)}\int_0^t u_j(\tau, s)\, d\tau\right)(t, s)\, ds\right) dz\right| \\ &\leq \int_0^x \frac{1}{\theta_j(z)}\psi\left(\int_z^{+\infty} I^{\alpha,\sigma}_{0^+,t}\left(\frac{1}{\zeta_j(t)}\int_0^t |u_j(\tau, s)|\, d\tau\right)(t, s)\, ds\right) dz. \end{aligned}$$

By using the inequality (3.13) we find that for all $t, x \geq 0$,

$$\begin{aligned} |L_j(u_1, u_2)(tx)| &\leq \int_0^x \frac{1}{\theta_j(z)}\psi_{p^+}\left(\int_z^{+\infty} I^{\alpha,\sigma}_{0^+,t}\left(\frac{1}{\zeta_j(t)}\int_0^t |u_j(\tau, s)|\, d\tau\right)(t, s)\, ds\right) dz \\ &= L_j^{(k)}(|u_1|, |u_2|)(x) \end{aligned}$$

and then $|L(u)| \leq L^{(k)}(|u|)$. Moreover, $L^{(k)}$ is bounded, increasing, $k-$positively homogeneous and verifies

$$L^{(k)}\left(E^{+}\backslash\{0\}\right) \subset E^{+}\backslash\{0\}.$$

And the condition (3.14) leads that $L^{(k)}$ is sub-additive. □

Lemma 3.17. *Assume that*

$$1 < p^{-} \leq 2. \tag{3.20}$$

Then For all $r > 0$ and for all $u, v \in \bar{B}(0, r)$,

$$|L(u) - L(v)| \leq \lambda L_{+} |u - v|.$$

where

$$L_{+} = (L_{+,1}, L_{+,2})$$

with

$$L_{+,j}(u_1, u_2) = \int_0^x \frac{1}{\theta_j(z)} \int_z^{+\infty} I_{0^+,t}^{\alpha,\sigma}\left(\frac{1}{\zeta_j(t)} \int_0^t u_j(\tau, s)\, d\tau\right)(t, s)\, ds dz,\ j \in \{1, 2\},$$

$$\lambda = \lambda(r) = \frac{1}{p^{-} - 1}\left(\frac{r.\left\|(\sigma^{+})^{\alpha}\right\|_{L^1}}{\Gamma(\alpha + 1)}\right)^{\dfrac{2 - p^{-}}{p^{-} - 1}} > 0,$$

and

$$\sigma^{+}(x) = \lim_{t \to \infty} \sigma(t, x).$$

Proof. Let $r > 0$ and $u, v \in \bar{B}(0, r)$, for each $j \in \{1, 2\}$,we have

$$\begin{aligned} &|L_j(u) - L_j(v)| \\ &= \left|\int_0^x \frac{1}{\theta_j(z)}\left[\psi\left(\int_z^{+\infty} I_{0^+,t}^{\alpha,\sigma}(B_j u_j(t, s))\, ds\right) - \psi\left(\int_z^{+\infty} I_{0^+,t}^{\alpha,\sigma}(B_j v_j(t, s))\, ds\right)\right] dz\right| \\ &\leq \int_0^x \frac{1}{\theta_j(z)}\left|\psi\left(\int_z^{+\infty} I_{0^+,t}^{\alpha,\sigma}(B_j u_j(t, s))\, ds\right) - \psi\left(\int_z^{+\infty} I_{0^+,t}^{\alpha,\sigma}(B_j v_j(t, s))\, ds\right)\right| dz, \end{aligned}$$

where

$$B_j u_j(t, s) = \frac{1}{\zeta_j(t)} \int_0^t u_j(\tau, s)\, d\tau \leq \|u\|,\ \text{for all } u \in E.$$

Let $t, x > 0$ such that $u_j \neq v_j$ on $[0, t] \times [x, +\infty[$, and let $\chi_{t,x} \in [b_{t,x}, c_{t,x}] \setminus \{0\}$ where

$$\begin{aligned} b_{t,x} &= \min\left(\int_z^{+\infty} I_{0^+,t}^{\alpha,\sigma}(B_j u_j(t, s))\, ds, \int_z^{+\infty} I_{0^+,t}^{\alpha,\sigma}(B_j v_j(t, s))\, ds\right) \text{ and} \\ c_{t,x} &= \max\left(\int_z^{+\infty} I_{0^+,t}^{\alpha,\sigma}(B_j u_j(t, s))\, ds, \int_z^{+\infty} I_{0^+,t}^{\alpha,\sigma}(B_j v_j(t, s))\, ds\right), \end{aligned}$$

such that

$$\begin{aligned} &\psi\left(\int_z^{+\infty} I_{0^+,t}^{\alpha,\sigma}(B_j u_j(t, s))\, ds\right) - \psi\left(\int_z^{+\infty} I_{0^+,t}^{\alpha,\sigma}(B_j v_j(t, s))\, ds\right) \\ &\qquad = A(\chi_{t,x}) \int_z^{+\infty} I_{0^+,t}^{\alpha,\sigma}(B_j(u_j - v_j))(t, s)\, ds \end{aligned}$$

where

$$A(\chi_t) = \frac{1}{(p^+ - 1)\,|\psi(\chi_{t,x})|^{p^+-2} + (p^- - 1)\,|\psi(\chi_{t,x})|^{p^--2}}.$$

We have

$$\begin{aligned} A(\chi_t) &= \frac{1}{(p^+ - 1)\,(\psi(|\chi_{t,x}|))^{p^+-2} + (p^- - 1)\,(\psi(|\chi_{t,x}|))^{p^--2}} \\ &\leq \frac{(\psi(|\chi_{t,x}|))^{2-p^-}}{p^- - 1} \\ &\leq \frac{\left(\psi_{p^-}(|\chi_{t,x}|)\right)^{2-p^-}}{p^- - 1}. \end{aligned}$$

Moreover,

$$\begin{aligned} |\chi_{t,x}| &\leq |c_{t,x}| \\ &\leq \max\left(\int_z^{+\infty} I_{0^+,t}^{\alpha,\sigma}(B_j(|u_j|)(t,s))\,ds, \int_z^{+\infty} I_{0^+,t}^{\alpha,\sigma}(B_j(|v_j|)(t,s))\,ds\right) \\ &\leq r.\int_z^{+\infty} I_{0^+,t}^{\alpha,\sigma}(B_j(1))\,ds \\ &\leq r.\int_0^{+\infty} I_{0^+,t}^{\alpha,\sigma}(1)\,ds = r.\int_0^{+\infty} \frac{\sigma^\alpha(s,t)}{\Gamma(\alpha+1)}ds \\ &\leq r.\frac{\left\|(\sigma^+)^\alpha\right\|_{L^1}}{\Gamma(\alpha+1)}, \end{aligned}$$

this leads

$$\left|\psi\left(\int_z^{+\infty} I_{0^+,t}^{\alpha,\sigma}(B_j u_j(t,s))\,ds\right) - \psi\left(\int_z^{+\infty} I_{0^+,t}^{\alpha,\sigma}(B_j v_j(t,s))\,ds\right)\right|$$

$$\leq \lambda \int_z^{+\infty} I_{0^+,t}^{\alpha,\sigma}(B_j(|u_j - v_j|))(t,s)\,ds$$

and so,

$$|L_j(u) - L_j(v)| \leq \lambda \int_0^x \frac{1}{\theta_j(z)} \int_z^{+\infty} I_{0^+,t}^{\alpha,\sigma}(B_j(|u_j - v_j|))(t,s)\,ds.$$

Thus

$$|L(u) - L(v)| \leq \lambda L_+ |u - v|.$$

□

Remark 3.18. Since L_+ is linear, bounded and strictly positive on E, then Lemma (3.17) implies that the condition (3.4) holds for all $r_* > 0$. Moreover, the operator

$$L_0 = \lambda L_+ = (\lambda L_{+,1}, \lambda L_{+,2})$$

is linear, compact and strictly positive operator, so, the condition (3.7) is also satisfied.

Lemma 3.19. *Let* $\theta_0 = \min\{\theta_1, \theta_2\}$. *Then*

$$r(L_0) \leq \beta = \frac{\lambda \left\|(\sigma^+)^\alpha\right\|_{L^1}}{\Gamma(\alpha+1)} \int_0^\infty \frac{dt}{\theta_0(t)}, \tag{3.21}$$

where $r(L_0)$ *is the spectral raidus of* L_0.

Proof. Assume that (3.21) holds. Let $u = (u_1, u_2) \in \partial B_E(0,1)$. For $j \in \{1,2\}$

$$\begin{aligned} L_{0,j}(u) &= \lambda \int_0^x \frac{1}{\theta_j(z)} \int_z^{+\infty} I_{0^+,t}^{\alpha,\sigma} \left(\frac{1}{\zeta_j(t)} \int_0^t u_j(\tau, s)\, d\tau \right)(t,s)\, ds dz \\ &\leq \lambda \int_0^x \frac{1}{\theta_0(z)} \int_z^{+\infty} I_{0^+,t}^{\alpha,\sigma}(1)(t,s)\, ds dz \\ &\leq \frac{\lambda \left\|(\sigma^+)^\alpha\right\|_{L^1}}{\Gamma(\alpha+1)} \int_0^\infty \frac{dz}{\theta_0(z)}, \end{aligned}$$

then for all $n \in \mathbb{N}^*$,

$$L_0^n(\mu) \leq (\beta^n, \beta^n).$$

Thus,

$$r(L_0) = \lim_{n\to+\infty} \sqrt[n]{\|L_0^n\|} \leq \beta.$$

□

We consider the following hypothesis:

$$\left\{ \begin{array}{c} \text{There exist } (g_1, g_2) \in E^+ \backslash \{0\} \text{ and } (h_1, h_2) \in E^+ \text{ such that} \\ \left\|L^{(k)}(g_1, g_2)\right\| < 1, \text{ and for all } (t, x, y_1, y_2) \in \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^2 \\ |f_j(t, x, y_1, y_2)| \leq g_j(t,x) . (\max(|y_1|, |y_2|))^{\frac{1}{k}} + h_j(t,x), \ \forall j \in \{1,2\}. \end{array} \right. \tag{3.22}$$

Let $r_* = \max\left\{r_0, (r_0)^{\frac{1}{k}} \|(g_1, g_2)\| + \|(h_1, h_2)\|\right\}$ and

$$r_0 = \frac{\left\|L^{(k)}(h_1, h_2)\right\|}{1 - \left\|L^{(k)}(g_1, g_2)\right\|}.$$

Theorem 3.20. *Assume that the condition (3.22) holds and*

$$1 < p^- \leq 2 \leq p^+.$$

If there exist $r > r_*$, $\rho^* > 0$ *and* $\rho_0 \in G\backslash\{0\}$ *such that for all* $j \in \{1,2\}$, f_j *verifies one of the following conditions for all* $t, x \in \mathbb{R}^+$ *and all* $(x_1, x_2), (y_1, y_2) \in [-r, r]^2$;

$$\left\{ \begin{array}{c} |f_j(t, x, x_1, x_2) - f_j(t, x, y_1, y_2)| \leq \rho_0(t) . \max(|x_1 - y_1|, |x_2 - y_2|) \\ \text{and} \\ \lambda < \|L_+(\rho_0, \rho_0)\|^{-1} \end{array} \right. \tag{3.23}$$

or

$$\left\{ \begin{array}{c} |f_j(t, x, x_1, x_2) - f_j(t, x, y_1, y_2)| \leq \rho^* . |x_j - y_j|, \\ \text{and} \\ \frac{\lambda . \left\|(\sigma^+)^\alpha\right\|_{L^1}}{\Gamma(\alpha+1)} \int_0^\infty \frac{dt}{\theta_j(t)} < (\rho^*)^{-1}, \end{array} \right. \tag{3.24}$$

then IVS (1.1) is Hyers-Ulam stable in E.

Proof. We have from hypothesis (3.22) and remark 3.18 that the conditions (3.1), (3.2), (3.4) and (3.7) hold.
1. Assume that the condition (3.23), this means that the hypothesis (3.3) and (3.5) hold with

$$\rho = (\rho_1, \rho_2) = (\rho_0, \rho_0),$$

so, it follows from theorem 3.3) that equation (2.1) is L-Hyers-Ulam stable, and from Lemma (3.15) that IVS (1.1) is Hyers-Ulam stable in E.
2. Now, assume that f verifies (3.24). It follows from Lemma (3.19) and (3.24) that

$$r(L_0) \leq \beta = \frac{\lambda . \left\|(\sigma^+)^\alpha\right\|_{L^1}}{\Gamma(\alpha+1)} \int_0^\infty \frac{dt}{\theta_0(t)} < (\rho^*)^{-1}$$

and so, the conditions (3.6) and (3.8) of theorem (3.4) hold with

$$\lambda_0 = \rho^*.$$

Consequently, IVS (1.1) is Hyers-Ulam stable in E. □

3.3. Existence and controllability

In this section, we assume that for all $(t, x, u_1, u_2) \in (\mathbb{R}^+)^2 \times \mathbb{R}^2$:

$$f(t, x, u_1, u_2) = G(t, x, u_1, u_2) + h(t, x),$$

where $h \in E$ is the control function of IVS (1.1) and $G \in E^+$ such that, for each $j \in \{1, 2\}$,

$$G_j(u_1, u_2) \leq \bar{\lambda} \max\left(|u_1|^{p^+-1}, |u_2|^{p^+-1}\right), \tag{3.25}$$

with

$$\bar{\lambda} \left\|\frac{1}{\theta_j}\right\|_{L^1}^{p^+-1} \left(\frac{\left\|(\sigma^+)^\alpha\right\|_{L^1}}{\Gamma(\alpha+1)}\right) < 1. \tag{3.26}$$

We denote by $C^1_{0,\phi}(\mathbb{R}^+)$ the set

$$C^1_{0,\phi}(\mathbb{R}^+) = \left\{u \in C^1(\mathbb{R}^+) : \phi(u) \in AC(\mathbb{R}^+),\ u(0) = \lim_{x\to+\infty} u'(x) = 0\right\}.$$

Definition 3.21. IVS (1.1) is said to be controllable in E at ∞, if given any $x^\infty \in C^1_{0,\phi}(\mathbb{R}^+) \times C^1_{0,\phi}(\mathbb{R}^+)$, there exists a control function $h \in E$, such that the solution u of IVS (1.1) satisfies $\lim\limits_{x\to+\infty} u(t, x) = x^\infty$.

Lemma 3.22. *We have* $\lim\limits_{t\to\infty} I^{\alpha,\sigma}_{0^+,t}\left(\frac{t}{\zeta_j(t)}\right)(t, x) > 0$, $\forall x \geq 0$.

Proof. Let $x \geq 0$. Since $\dfrac{\partial \sigma}{\partial t}(t,x) > 0$;

$$\begin{aligned}
&\lim_{t\to\infty} I^{\alpha,\sigma}_{0^+,t}\left(\frac{t}{\zeta_j(t)}\right)\\
&= \frac{1}{\Gamma(\alpha)}\lim_{t\to\infty}\int_0^{\sigma(t,x)} \frac{T\sigma'_t(T,x)}{\zeta_j(T)}\left(\sigma(t,x)-\sigma(T,x)\right)^{\alpha-1} dT\\
&\geq \frac{1}{\Gamma(\alpha)}\lim_{t\to\infty}\int_{\sigma(1,x)}^{\sigma(t,x)} \frac{T\sigma'_t(T,x)}{a_j+T}\left(\sigma(t,x)-\sigma(T,x)\right)^{\alpha-1} dT\\
&\geq \lim_{t\to\infty}\frac{\sigma(1,x)}{\Gamma(\alpha)(a_j+\sigma(t,x))}\int_{\sigma(1,x)}^{\sigma(t,x)} \sigma'_t(T,x)\left(\sigma(t,x)-\sigma(T,x)\right)^{\alpha-1} dT\\
&\geq \lim_{t\to\infty}\frac{\sigma(1,x)}{\Gamma(\alpha)(a_j+\sigma(t,x))}\int_{\sigma(1,x)}^{\sigma(t,x)} \left(\sigma(t,x)-\sigma\right)^{\alpha-1} d\sigma\\
&\geq \frac{\sigma(1,x)}{\Gamma(\alpha+1)(a_j+\sigma^+(x))}\left(\sigma^+(x)-\sigma(1,x)\right)^{\alpha} > 0.
\end{aligned}$$

□

Theorem 3.23. *Assume that (3.25) and (3.26) hold true. Then for all $h \in E$, IVS (1.1) admits a solution.*

Proof. Let $h \in E$. We show that there exists $R > 0$ such that $T\left(\bar{B}(0,R)\right) \subset \bar{B}(0,R)$ and then we deduce from Schauder's theorem that the compactness of T guarantees the existence of at least one fixed point of T which is, from Lemma (3.13), a solution of IVS (1.1).

Assume on the contrary that for all $n \in \mathbb{N}^*$, there is $u^{(n)} = \left(u_1^{(n)}, u_2^{(n)}\right) \in \bar{B}(0,n)$, $(t,x) \in \mathbb{R}^+ \times \mathbb{R}^+$ and $j \in \{1,2\}$, such that

$$n \leq \left|T_j\left(u^{(n)}\right)(t,x)\right|$$

$$= \left|\int_0^x \frac{1}{\theta_j(z)}\psi\left(\int_z^{+\infty} I^{\alpha,\sigma}_{0^+,t}\left(\frac{1}{\zeta_j(t)}\int_0^t \left(G_j\left(u_1^{(n)},u_2^{(n)}\right)+h_j\right)(\tau,s)\,d\tau\right) ds\right) dz\right|.$$

By using the inequality (3.13) of Remark (3.11), it follows:

$$\begin{aligned}
1 &\leq \frac{1}{n}\int_0^x \frac{1}{\theta_j(z)}\psi_{p^+}\left(\int_z^{+\infty} I^{\alpha,\sigma}_{0^+,t}\left(\frac{1}{\zeta_j(t)}\int_0^t \left(G_j\left(u_1^{(n)},u_2^{(n)}\right)+|h_j|\right)(\tau,s)\,d\tau\right) ds\right) dz\\
&\leq \int_0^x \frac{1}{\theta_j(z)}\psi_{p^+}\left(\int_z^{+\infty} I^{\alpha,\sigma}_{0^+,t}\left(\frac{1}{\zeta_j(t)}\int_0^t \left(\frac{G_j\left(u_1^{(n)},u_2^{(n)}\right)+|h_j|}{n^{p^+-1}}\right)(\tau,s)\,d\tau\right) ds\right) dz\\
&\leq \psi_{p^+}\left(\bar{\lambda}+\frac{\|h_j\|_0}{n^{p^+-1}}\right)\int_0^x \frac{1}{\theta_j(z)}\psi_{p^+}\left(\int_z^{+\infty} I^{\alpha,\sigma}_{0^+,t}\left(\frac{t}{\zeta_j(t)}\right) ds\right) dz\\
&\leq \psi_{p^+}\left(\bar{\lambda}+\frac{\|h_j\|_0}{n^{p^+-1}}\right)\int_0^x \frac{1}{\theta_j(z)}\psi_{p^+}\left(\int_0^{+\infty} I^{\alpha,\sigma}_{0^+,t}(1)\,ds\right) dz
\end{aligned}$$

$$\leq \left(\bar{\lambda}+\frac{\|h_j\|_0}{n^{p^+-1}}\right)^{\frac{1}{p^+-1}} \left\|\frac{1}{\theta_j}\right\|_{L^1} \left(\frac{\left\|(\sigma^+)^{\alpha}\right\|_{L^1}}{\Gamma(\alpha+1)}\right)^{\frac{1}{p^+-1}}.$$

Letting $n \to \infty$, we have

$$\bar{\lambda}\left\|\frac{1}{\theta_j}\right\|_{L^1}^{p^+-1} \left(\frac{\left\|(\sigma^+)^{\alpha}\right\|_{L^1}}{\Gamma(\alpha+1)}\right) \geq 1.$$

This contradicts hypothesis (3.26) and the proof is finished. □

Theorem 3.24. *Assume that (3.25) and (3.26) hold true. Then IVS (1.1) is controllable.*

Proof. For each $u^{\infty}=(u_1^{\infty},u_2^{\infty}) \in C_0^2(\mathbb{R}^+) \times C_0^2(\mathbb{R}^+ \times \mathbb{R}^+)$, let

$$h(t,x) = -\frac{1}{\lim\limits_{t\to\infty} I_{0^+,t}^{\alpha,\sigma}\left(\frac{t}{\zeta_j(t)}\right)} \left(\frac{\partial}{\partial x}\phi\left(\theta_j.\frac{\partial u_j^{\infty}}{\partial x}\right)(x) + \lim_{t\to\infty} I_{0^+,t}^{\alpha,\sigma}\left(\frac{1}{\zeta_j(t)}\int_0^t G_j(u_1,u_2)(\tau,x)\,d\tau\right)\right). \tag{3.27}$$

Let $u=(u_1,u_2) \in C^1(\mathbb{R}^+ \times \mathbb{R}^+) \times C^2(\mathbb{R}^+ \times \mathbb{R}^+)$ be solution of IVS (1.1). We have from Lemma (3.13) that for each $j \in \{1,2\}$;

$$u_j(t,x) = \int_0^x \frac{1}{\theta_j(z)}\psi\left(\int_z^{+\infty} I_{0^+,t}^{\alpha,\sigma}\left(\frac{1}{\zeta_j(t)}\int_0^t (G_j(u_1,u_2)+h_j)(\tau,s)\,d\tau\right)ds\right)dz.$$

This means that for every $x \geq 0$,

$$y_j(x) = \lim_{t\to\infty} u_j(t,x)$$

$$= \int_0^x \frac{1}{\theta_j(z)}\psi\left(\int_z^{+\infty} \lim_{t\to\infty} I_{0^+,t}^{\alpha,\sigma}\left(\frac{1}{\zeta_j(t)}\int_0^t (G_j(u_1,u_2)+h_j)(\tau,s)\,d\tau\right)ds\right)dz$$

$$\Rightarrow -\frac{\partial}{\partial x}\phi\left(\theta_j.\frac{\partial y_j}{\partial x}\right)(x) = \lim_{t\to\infty} I_{0^+,t}^{\alpha,\sigma}\left(\frac{1}{\zeta_j(t)}\int_0^t (G_j(u_1,u_2)+h_j)(\tau,x)\,d\tau\right)$$

$$\Rightarrow -\frac{\partial}{\partial x}\phi\left(\theta_j.\frac{\partial y_j}{\partial x}\right)(x) - \lim_{t\to\infty} I_{0^+,t}^{\alpha,\sigma}\left(\frac{1}{\zeta_j(t)}\int_0^t G_j(u_1,u_2)(\tau,x)\,d\tau\right) = \lim_{t\to\infty} I_{0^+,t}^{\alpha,\sigma}\left(\frac{1}{\zeta_j(t)}\int_0^t h_j(\tau,x)\,d\tau\right).$$

then

$$-\frac{\partial}{\partial x}\phi\left(\theta_j.\frac{\partial y_j}{\partial x}\right)(x) - \lim_{t\to\infty} I_{0^+,t}^{\alpha,\sigma}\left(\frac{1}{\zeta_j(t)}\int_0^t G_j(u_1,u_2)(\tau,x)\,d\tau\right) = \lim_{t\to\infty} I_{0^+,t}^{\alpha,\sigma}\left(\frac{1}{\zeta_j(t)}\int_0^t h_j(\tau,x)\,d\tau\right). \tag{3.28}$$

Substituting (3.27) into (3.28), we find that

$$\frac{\partial}{\partial x}\phi\left(\theta_j.\frac{\partial u_j^{\infty}}{\partial x}\right)(x) = \frac{\partial}{\partial x}\phi\left(\theta_j.\frac{\partial y_j}{\partial x}\right)(x),$$

and using $\lim\limits_{x\to\infty} \dfrac{\partial u_j^{\infty}}{\partial x}(x) = \lim\limits_{x\to\infty} \dfrac{\partial y_j}{\partial x}(x) = 0$ and the fact that ϕ is invertible, we can get

$$\frac{\partial u_j^{\infty}}{\partial x}(x) = \frac{\partial y_j}{\partial x}(x),$$

and also, from $u_j^{\infty}(0) = y_j(0)$, it follows that

$$\lim_{t\to\infty} u_j(t,x) = y_j(x) = u_j^{\infty}(x).$$

Thus, at the stat ∞, $u(\infty,.) = u_j^{\infty}$. So, IVS (1.1) is controllable. □

Example 3.25. Let $\alpha = \frac{1}{2}$, $\sigma(t,x) = \frac{\pi}{4}\left(1-e^{-t}\right)^2 e^{-2x}$ and $\phi(x) = |x|^{-\frac{1}{2}}.x + |x|.x$. For $j \in \{0,1\}$, we have

$$\begin{aligned} f_j(t,x,x_1,x_2) &= G_j(t,x,x_1,x_2) + h_j(t,x), \\ \theta_j(x) &= 1+x^2, \end{aligned}$$

where $h_j(t,x) \in E$ is a control function.
1. If $G_j(t,x,x_1,x_2) = g_j(t,x).x_j$, with

$$g_j(t,x) = \frac{1}{\pi^2} = \bar{\lambda}.$$

Then $p^- = \frac{3}{2} < 2 < p^+ = 3$, $\left\|\left(\sigma^+\right)^{\alpha}\right\|_{L^1} = \left\|\sqrt{\sigma^+}\right\|_{L^1} = \dfrac{\sqrt{\pi}}{2}$

$$\sigma^+(x) = \frac{\pi}{4}e^{-2x}.$$

We have $\bar{\lambda} = \dfrac{1}{\pi^2}$ and

$$\bar{\lambda}\left\|\frac{1}{\theta_j}\right\|_{L^1}^{p^+-1}\left(\frac{\left\|\left(\sigma^+\right)^{\alpha}\right\|_{L^1}}{\Gamma(\alpha+1)}\right) = \bar{\lambda}\left(\frac{\pi}{2}\right)^2\left(\frac{1}{\Gamma\left(\frac{3}{2}\right)}\sqrt{\frac{\pi}{4}}\right) = \frac{1}{4} < 1.$$

So, the conditions (3.25) and (3.26) of theorems (3.23) and (3.24) hold true. Then IVS (1.1) is controllable.
2. Now, we assume that $G_j(t,x,x_1,x_2) = g_j(t,x).x_j^2$ and $h_j(t,x) = \eta \in \mathbb{R}^+$ with

$$g_j(t,x) = \frac{1}{\pi^2} = g^+$$

and η verifies

$$\eta < \min\left\{\frac{\sqrt{\pi}}{4\pi}, \frac{\sqrt{\pi\sqrt{\pi}}}{2(2\pi+1)}\right\}. \tag{3.29}$$

We have

$$\begin{aligned} L_j^{(k)}(g_1,g_2)(t,x) &= \int_0^x \frac{1}{\theta_j(z)}\psi_{p^+}\left(\int_z^{+\infty} I_{0^+,t}^{\alpha,\sigma}\left(\frac{1}{\zeta_j(t)}\int_0^t g_j(\tau,s)\,d\tau\right)(t,s)\,ds\right)dz \\ &= \int_0^x \frac{1}{1+z^2}\sqrt{g^+\left(\int_z^{+\infty} I_{0^+,t}^{\alpha,\sigma}(1)(t,s)\,ds\right)}dz \\ &\leq \int_0^x \frac{dz}{1+z^2}\sqrt{\frac{g^+\left\|(\sigma^+)^\alpha\right\|_{L^1}}{\Gamma\left(\frac{3}{2}\right)}} \\ &\leq \int_0^x \frac{dz}{1+z^2}\sqrt{\frac{g^+\frac{\sqrt{\pi}}{2}}{\Gamma\left(\frac{3}{2}\right)}} = \sqrt{g^+}.\arctan(x), \end{aligned}$$

then

$$\left\|L^{(k)}(g_1,g_2)\right\| \leq \frac{1}{2} < 1.$$

This means that 3.22 holds.
Moreover,

$$\begin{aligned} L_j^{(k)}(h_1,h_2) &\leq \int_0^x \frac{1}{\theta_j(z)}\sqrt{\left(\int_z^{+\infty} I_{0^+,t}^{\alpha,\sigma}\left(\frac{1}{\zeta_j(t)}\int_0^t \eta.d\tau\right)(t,s)\,ds\right)}dz \\ &< \frac{\pi}{2}\sqrt{\eta} \end{aligned}$$

then

$$r_0 = \frac{\left\|L^{(k)}(h_1,h_2)\right\|}{1-\left\|L^{(k)}(g_1,g_2)\right\|} \leq 2\left\|L^{(k)}(h_1,h_2)\right\| < \pi\sqrt{\eta}.$$

Then, from (3.29), we have

$$\begin{aligned} r_* &= \max\left\{r_0, \frac{2}{\pi}(r_0)^2 + \|(h_1,h_2)\|\right\} \leq \max\left\{\pi\sqrt{\eta}, (2\pi+1).\eta\right\} \\ &< \frac{\sqrt{\pi\sqrt{\pi}}}{2}. \end{aligned}$$

Now, let $r > 0$ such that

$$r_* < r < \frac{\sqrt{\pi\sqrt{\pi}}}{2}.$$

For all $t, x \geq 0$, $(x_1,x_2)\,[-r,r]^2, (y_1,y_2) \in [-r,r]^2$ we have

$$\begin{aligned} |f_j(t,x,x_1,x_2) - f_j(t,x,y_1,y_2)| &= g_j(t,x).\left|x_j^2 - y_j^2\right| \\ &\leq 2.r.g^+.|x_j - y_j| = \rho^*.|x_j - y_j|, \end{aligned}$$

where

$$\rho^* = \frac{2.r}{\pi^2},$$

and

$$\begin{aligned}\lambda &= \frac{1}{p^{-}-1}\left(\frac{r.\left\|(\sigma^{+})^{\alpha}\right\|_{L^1}}{\Gamma(\alpha+1)}\right)^{\frac{2-p^{-}}{p^{-}-1}} \\ &= \frac{4}{\sqrt{\pi}}r.\end{aligned}$$

As $r < \dfrac{\sqrt{\pi\sqrt{\pi}}}{2}$, we have

$$\begin{aligned}\frac{\rho^*}{\Gamma(\alpha+1)}\int_0^{\infty}\frac{\left\|(\sigma^{+})^{\alpha}\right\|_{L^1}}{\theta_j(t)}dt &\leq \frac{2.r}{\pi^2}\int_0^{\infty}\frac{1}{1+t^2}dt \\ &\leq \frac{r}{\pi} < \frac{\sqrt{\pi}}{4r} = \lambda^{-1}.\end{aligned}$$

Then, hypothesis (3.24) is also satisfied. Thus, we deduce from theorem (3.20) that IVS (1.1) is Hyers-Ulam stable in E.

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