Stud. Univ. Babeş-Bolyai Math. 69(2024), No. 4, 789–800
DOI: 10.24193/subbmath.2024.4.06

Bounds of third and fourth Hankel determinants for a generalized subclass of bounded turning functions subordinated to sine function

Gagandeep Singh and Gurcharanjit Singh

Abstract. The objective of this paper is to investigate the bounds of third and fourth Hankel determinants for a generalized subclass of bounded turning functions associated with sine function, in the open unit disc $E = \{z \in \mathbb{C} : |z| < 1\}$. The results are also extended to two-fold and three-fold symmetric functions. This investigation will generalize the resuls of some earlier works.

Mathematics Subject Classification (2010): 30C45, 30C50, 30C80.

Keywords: Analytic functions, Subordination, Coefficient inequalities, sine function, third Hankel determinant, fourth Hankel determinant.

1. Introduction

Let the complex plane is expressed by $\mathbb{C}$. By $\mathcal{A}$, let us denote the class of analytic functions of the form $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$, defined in the open unit disc $E = \{z \in \mathbb{C} : |z| < 1\}$ and normalized by the conditions $f(0) = f'(0) - 1 = 0$. By $\mathcal{S}$, we denote the subclass of $\mathcal{A}$ which consists of univalent functions in E.

Let f and g be two analytic functions in E. We say that f is subordinate to g (denoted as $f \prec g$) if there exists a function w with $w(0) = 0$ and $|w(z)| < 1$ for $z \in E$ such that $f(z) = g(w(z))$. Further, if g is univalent in E, then the subordination leads to $f(0) = g(0)$ and $f(E) \subset g(E)$.

In the theory of univalent functions, Bieberbach [5] stated a result that, for $f \in \mathcal{S}$, $|a_n| \leq n$, $n = 2, 3, \ldots$. This result is known as Bieberbach's conjecture and it remained as a challenge for the mathematicians for a long time. Finally, L. De-Branges [8],

Received 08 November 2022; Accepted 01 March 2023.

proved this conjecture in 1985. During the course of proving this conjecture, various results related to the coefficients were come into existence which gave rise to some new subclasses of analytic functions.

For better understanding of the main content, let's have a look on some fundamental subclasses of $\mathcal{A}$:

$\mathcal{S}^* = \left\{f : f \in \mathcal{A}, Re\left(\frac{zf'(z)}{f(z)}\right) > 0 \text{ or } \frac{zf'(z)}{f(z)} \prec \frac{1+z}{1-z}, z \in E\right\}$, the class of starlike functions.

$\mathcal{K} = \left\{f : f \in \mathcal{A}, Re\left(\frac{(zf'(z))'}{f'(z)}\right) > 0 \text{ or } \frac{(zf'(z))'}{f'(z)} \prec \frac{1+z}{1-z}, z \in E\right\}$, the class of convex functions.

Reade [24] introduced the class $\mathcal{CS}^*$ of close-to-star functions which is defined as $\mathcal{CS}^* = \left\{f : f \in \mathcal{A}, Re\left(\frac{f(z)}{g(z)}\right) > 0 \text{ or } \frac{f(z)}{g(z)} \prec \frac{1+z}{1-z}, g \in \mathcal{S}^*, z \in E\right\}$. Further for $g(z) = z$, MacGregor [17] studied the following subclass of close-to-star functions:

$$\mathcal{R}' = \left\{f : f \in \mathcal{A}, Re\left(\frac{f(z)}{z}\right) > 0 \text{ or } \frac{f(z)}{z} \prec \frac{1+z}{1-z}, z \in E\right\}.$$

MacGregor [16] established a very useful class $\mathcal{R}$ of bounded turning functions which is defined as

$$\mathcal{R} = \left\{f : f \in \mathcal{A}, Re(f'(z)) > 0 \text{ or } f'(z) \prec \frac{1+z}{1-z}, z \in E\right\}.$$

Later on, Murugusundramurthi and Magesh [19] studied the following class:

$$\mathcal{R}(\alpha) = \left\{f : f \in \mathcal{A}, Re\left((1-\alpha)\frac{f(z)}{z} + \alpha f'(z)\right) > 0, 0 \leq \alpha \leq 1, z \in E\right\}.$$

Particularly, $\mathcal{R}(1) \equiv \mathcal{R}$ and $\mathcal{R}(0) \equiv \mathcal{R}'$.

Various subclasses of $\mathcal{S}$ were investigated by associating to different functions. Recently, Arif et al. [3], Cho et al. [7] and Khan et al. [11] studied the classes $\mathcal{S}^*_{sin}$, $\mathcal{K}_{sin}$ and $\mathcal{R}_{sin}$, which are the subclasses of starlike functions, convex functions and bounded turning functions associated with sine function, respectively. Getting motivated by these works, now we define the following class of analytic functions by subordinating to $1 + sinz$.

Definition 1.1. A function $f \in \mathcal{A}$ is said to be in the class $\mathcal{R}^\alpha_{sin}$ $(0 \leq \alpha \leq 1)$ if it satisfies the condition

$$(1-\alpha)\frac{f(z)}{z} + \alpha f'(z) \prec 1 + sinz.$$

We have the following observations:

(i) $\mathcal{R}^0_{sin} \equiv \mathcal{R}'_{sin}$.

(ii) $\mathcal{R}^1_{sin} \equiv \mathcal{R}_{sin}$.

For $q \geq 1$ and $n \geq 1$, Pommerenke [21] introduced the q^{th} Hankel determinant as

$$H_q(n) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ a_{n+q-1} & \cdots & \cdots & a_{n+2q-2} \end{vmatrix}.$$

For specific values of q and n, the Hankel determinant $H_q(n)$ reduces to the following functionals:
(i) For $q = 2$ and $n = 1$, it redues to $H_2(1) = a_3 - a_2^2$, which is the well known Fekete-Szegö functional.
(ii) For $q = 2$ and $n = 2$, the Hankel determinant takes the form of $H_2(2) = a_2a_4 - a_3^2$, which is known as Hankel determinant of second order.
(iii) For $q = 3$ and $n = 1$, the Hankel determinant reduces to $H_3(1)$, which is the Hankel determinant of third order.
(iv) For $q = 4$ and $n = 1$, $H_q(n)$ reduces to $H_4(1)$, which is the Hankel determinant of fourth order.

Ma [15] introduced the functional $J_{n,m}(f) = a_na_m - a_{m+n-1}$, $n, m \in \mathbb{N} - \{1\}$, which is known as generalized Zalcman functional. The functional $J_{2,3}(f) = a_2a_3 - a_4$ is a specific case of the generalized Zalcman functional. The upper bound for the functional $J_{2,3}(f)$ over different subclasses of analytic functions was computed by various authors. It is very useful in establishing the bounds for the third Hankel determinant.

On expanding, the third Hankel determinant can be expressed as

$$H_3(1) = a_3(a_2a_4 - a_3^2) - a_4(a_4 - a_2a_3) + a_5(a_3 - a_2^2),$$

and after applying the triangle inequality, it yields

$$|H_3(1)| \leq |a_3||a_2a_4 - a_3^2| + |a_4||a_2a_3 - a_4| + |a_5||a_3 - a_2^2|. \tag{1.1}$$

Also the expansion of fourth Hankel determinant can be expressed as

$$\begin{aligned} H_4(1) = a_7H_3(1) - 2a_4a_6(a_2a_4 - a_3^2) - 2a_5a_6(a_2a_3 - a_4) - a_6^2(a_3 - a_2^2) \\ + a_5^2(a_2a_4 - a_3^2) + a_5^2(a_2a_4 + 2a_3^2) - a_5^3 + a_4^4 - 3a_3a_4^2a_5. \end{aligned} \tag{1.2}$$

A lot of work has been done on the estimation of second Hankel determinant by various authors including Noor [20], Ehrenborg [9], Layman [12], Singh [26], Mehrok and Singh [18] and Janteng et al. [10]. The estimation of third Hankel determinant is little bit complicated. Babalola [4] was the first researcher who successfully obtained the upper bound of third Hankel determinant for the classes of starlike functions, convex functions and the class of functions with bounded boundary rotation. Further a few researchers including Shanmugam et al. [25], Bucur et al. [6], Altinkaya and Yalcin [1], Singh and Singh [27] have been actively engaged in the study of third Hankel determinant for various subclasses of analytic functions. Now a days, the study of fourth Hankel determinant for various subclasses of analytic functions, is an active topic of research. A few authors including Arif et al. [2], Singh et al. [28, 29] and Zhang and Tang [30] established the bounds of fourth Hankel determinant for certain subclasses of $\mathcal{A}$.

In this paper, we establish the upper bounds of the third and fourth Hankel determinants for the class $\mathcal{R}^{\alpha}_{sin}$. Also various known results follow as particular cases.

Let $\mathcal{P}$ denote the class of analytic functions p of the form

$$p(z) = 1 + \sum_{k=1}^{\infty} p_k z^k,$$

whose real parts are positive in E.

In order to prove our main results, the following lemmas have been used:

Lemma 1.2. [3] *If $p \in \mathcal{P}$, then*

$$|p_k| \leq 2, k \in \mathbb{N},$$

$$\left| p_2 - \frac{p_1^2}{2} \right| \leq 2 - \frac{|p_1|^2}{2},$$

$$|p_{i+j} - \mu p_i p_j| \leq 2, 0 \leq \mu \leq 1,$$

$$|p_{n+2k} - \lambda p_n p_k^2| \leq 2(1 + 2\lambda), (\lambda \in \mathbb{R}),$$

$$|p_m p_n - p_k p_l| \leq 4, (m + n = k + l; m, n \in \mathbb{N}),$$

and for complex number ρ, we have

$$|p_2 - \rho p_1^2| \leq 2 \max\{1, |2\rho - 1|\}.$$

Lemma 1.3. [3] *Let $p \in \mathcal{P}$, then*

$$|J p_1^3 - K p_1 p_2 + L p_3| \leq 2|J| + 2|K - 2J| + 2|J - K + L|.$$

In particular, it is proved in [22] *that*

$$|p_1^3 - 2p_1 p_2 + p_3| \leq 2.$$

Lemma 1.4. [13, 14] *If $p \in \mathcal{P}$, then*

$$2p_2 = p_1^2 + (4 - p_1^2)x,$$

$$4p_3 = p_1^3 + 2p_1(4 - p_1^2)x - p_1(4 - p_1^2)x^2 + 2(4 - p_1^2)(1 - |x|^2)z,$$

for $|x| \leq 1$ and $|z| \leq 1$.

Lemma 1.5. [23] *Let m, n, l and r satisfy the inequalities $0 < m < 1$, $0 < r < 1$ and $8r(1-r)\left[(mn - 2l)^2 + (m(r + m) - n)^2\right] + m(1-m)(n-2rm)^2 \leq 4m^2(1-m)^2 r(1-r)$. If $p \in \mathcal{P}$, then*

$$\left| l p_1^4 + r p_2^2 + 2m p_1 p_3 - \frac{3}{2} n p_1^2 p_2 - p_4 \right| \leq 2.$$

2. Coefficient bounds for the class $\mathcal{R}^{\alpha}_{sin}$

Theorem 2.1. *If $f \in \mathcal{R}^{\alpha}_{sin}$, then*

$$|a_2| \leq \frac{1}{1+\alpha}, \tag{2.1}$$

$$|a_3| \leq \frac{1}{1+2\alpha}, \tag{2.2}$$

$$|a_4| \leq \frac{1}{1+3\alpha}, \tag{2.3}$$

and

$$|a_5| \leq \frac{1}{1+4\alpha}. \tag{2.4}$$

Proof. Since $f \in \mathcal{R}^{\alpha}_{sin}$, by the principle of subordination, we have

$$(1-\alpha)\frac{f(z)}{z} + \alpha f'(z) = 1 + sin(w(z)). \tag{2.5}$$

Define $p(z) = \dfrac{1+w(z)}{1-w(z)} = 1 + p_1 z + p_2 z^2 + p_3 z^3 + ...$, which implies $w(z) = \dfrac{p(z)-1}{p(z)+1}$.
On expanding, we have

$$(1-\alpha)\frac{f(z)}{z} + \alpha f'(z) = 1 + (1+\alpha)a_2 z + (1+2\alpha)a_3 z^2 + (1+3\alpha)a_4 z^3 + (1+4\alpha)a_5 z^4 + ... \tag{2.6}$$

Also

$$1 + sin(w(z)) = 1 + \frac{1}{2}p_1 z + \left(\frac{p_2}{2} - \frac{p_1^2}{4}\right) z^2$$
$$+ \left(\frac{5p_1^3}{48} - \frac{p_1 p_2}{2} + \frac{p_3}{2}\right) z^3 + \left(-\frac{p_1^4}{32} + \frac{5p_1^2 p_2}{16} - \frac{p_3 p_1}{2} - \frac{p_2^2}{4} + \frac{p_4}{2}\right) z^4 + ... \tag{2.7}$$

Using (2.6) and (2.7), (2.5) yields

$$1 + (1+\alpha)a_2 z + (1+2\alpha)a_3 z^2 + (1+3\alpha)a_4 z^3 + (1+4\alpha)a_5 z^4 + ...$$
$$= 1 + \frac{1}{2}p_1 z + \left(\frac{p_2}{2} - \frac{p_1^2}{4}\right) z^2 + \left(\frac{5p_1^3}{48} - \frac{p_1 p_2}{2} + \frac{p_3}{2}\right) z^3$$
$$+ \left(-\frac{p_1^4}{32} + \frac{5p_1^2 p_2}{16} - \frac{p_3 p_1}{2} - \frac{p_2^2}{4} + \frac{p_4}{2}\right) z^4 + ... \tag{2.8}$$

Equating the coefficients of z, z^2, z^3 and z^4 in (2.8) and on simplification, we obtain

$$a_2 = \frac{1}{2(1+\alpha)} p_1, \tag{2.9}$$

$$a_3 = \frac{1}{1+2\alpha}\left[\frac{p_2}{2} - \frac{p_1^2}{4}\right], \tag{2.10}$$

$$a_4 = \frac{1}{48(1+3\alpha)}\left[5p_1^3 - 24p_1 p_2 + 24p_3\right], \tag{2.11}$$

and

$$a_5 = \frac{1}{2(1+4\alpha)}\left[\frac{p_1^4}{16} + \frac{p_2^2}{2} + p_3 p_1 - \frac{5p_1^2 p_2}{8} - p_4\right]. \tag{2.12}$$

Using first inequality of Lemma 1.2 in (2.9), the result (2.1) is obvious.
From (2.10), we have

$$|a_3| = \frac{1}{2(1+2\alpha)}\left|p_2 - \frac{1}{2}p_1^2\right|. \tag{2.13}$$

Using sixth inequality of Lemma 1.2 in (2.13), the result (2.2) can be easily obtained.
(2.11) can be expressed as

$$|a_4| = \frac{1}{48(1+3\alpha)}\left|5p_1^3 - 24p_1p_2 + 24p_3\right|. \tag{2.14}$$

On applying Lemma 1.3 in (2.14), the result (2.3) is obvious.
Further, on using Lemma 1.5 in (2.12), the result (2.4) is obvious. □

On putting $\alpha = 0$, Theorem 2.1 yields the following result:

Remark 2.2. If $f \in \mathcal{R}'_{sin}$, then

$$|a_2| \leq 1, |a_3| \leq 1, |a_4| \leq 1, |a_5| \leq 1.$$

For $\alpha = 1$, Theorem 2.1 gives the following result due to Khan et al. [11]:

Remark 2.3. If $f \in \mathcal{R}_{sin}$, then

$$|a_2| \leq \frac{1}{2}, |a_3| \leq \frac{1}{3}, |a_4| \leq \frac{1}{4}, |a_5| \leq \frac{1}{5}.$$

Conjecture. If $f \in \mathcal{R}^{\alpha}_{sin}$, then

$$|a_n| \leq \frac{1}{1+(n-1)\alpha}, n \geq 2.$$

Theorem 2.4. *If $f \in \mathcal{R}^{\alpha}_{sin}$ and μ is any complex number, then*

$$|a_3 - \mu a_2^2| \leq \frac{1}{1+2\alpha} \max\left\{1, \frac{(1+2\alpha)}{(1+\alpha)^2}|\mu|\right\}. \tag{2.15}$$

Proof. From (2.9) and (2.10), we obtain

$$|a_3 - \mu a_2^2| = \frac{1}{2(1+2\alpha)}\left|p_2 - \frac{(1+\alpha)^2 + \mu(1+2\alpha)}{2(1+\alpha)^2}p_1^2\right|. \tag{2.16}$$

Using sixth inequality of Lemma 1.2, (2.16) takes the form

$$|a_3 - \mu a_2^2| \leq \frac{1}{1+2\alpha} \max\left\{1, \frac{(1+2\alpha)}{(1+\alpha)^2}|\mu|\right\}. \tag{2.17}$$

□

Substituting for $\alpha = 0$, Theorem 2.4 yields the following result:

Remark 2.5. If $f \in \mathcal{R}'_{sin}$, then

$$|a_3 - \mu a_2^2| \leq \max\{1, |\mu|\}.$$

Putting $\alpha = 1$, Theorem 2.4 yields the following result due to Khan et al. [11]:

Remark 2.6. If $f \in \mathcal{R}_{sin}$, then

$$|a_3 - \mu a_2^2| \leq \max\left\{\frac{1}{3}, \frac{1}{4}|\mu|\right\}.$$

Putting $\mu = 1$, Theorem 2.4 yields the following result:

Remark 2.7. If $f \in \mathcal{R}^{\alpha}_{sin}$, then

$$|a_3 - a_2^2| \leq \frac{1}{1+2\alpha}.$$

Theorem 2.8. *If $f \in \mathcal{R}^{\alpha}_{sin}$, then*

$$|a_2a_3 - a_4| \leq \frac{1}{1+3\alpha}. \tag{2.18}$$

Proof. Using (2.9), (2.10), (2.11) and after simplification, we have

$$|a_2a_3 - a_4| = \frac{1}{48(1+\alpha)(1+2\alpha)(1+3\alpha)}$$

$$\cdot\left|(11+33\alpha+10\alpha^2)p_1^3 - (36+108\alpha+48\alpha^2)p_1p_2 + 24(1+\alpha)(1+2\alpha)p_3\right|. \tag{2.19}$$

On applying Lemma 1.3 in (2.19), it yields (2.18). □

For $\alpha = 0$, the following result is a consequence of Theorem 2.8:

Remark 2.9. If $f \in \mathcal{R}^{'}_{sin}$, then

$$|a_2a_3 - a_4| \leq 1.$$

On putting $\alpha = 1$ in Theorem 2.8, we can obtain the following result due to Khan et al. [11]:

Remark 2.10. If $f \in \mathcal{R}_{sin}$, then

$$|a_2a_3 - a_4| \leq \frac{1}{4}.$$

Theorem 2.11. *If $f \in \mathcal{R}^{\alpha}_{sin}$, then*

$$|a_2a_4 - a_3^2| \leq \frac{1}{(1+2\alpha)^2}. \tag{2.20}$$

Proof. Using (2.9), (2.10) and (2.11), we have

$$|a_2a_4 - a_3^2| = \frac{1}{96(1+\alpha)(1+2\alpha)^2(1+3\alpha)}$$

$$\cdot\left|24(1+2\alpha)^2p_1p_3 - 24\alpha^2p_1^2p_2 + (-1-4\alpha+2\alpha^2)p_1^4 - 24(1+\alpha)(1+3\alpha)p_2^2\right|.$$

Substituting for p_2 and p_3 from Lemma 1.4 and letting $p_1 = p$, we get

$$|a_2a_4 - a_3^2| = \frac{1}{96(1+\alpha)(1+2\alpha)^2(1+3\alpha)}\Bigg| - (4\alpha^2+4\alpha+1)p^4$$

$$-6(1+2\alpha)^2p^2(4-p^2)x^2 - 6(1+\alpha)(1+3\alpha)(4-p^2)^2x^2 + 12(1+2\alpha)^2p(4-p^2)(1-|x|^2)z\Bigg|.$$

Since $|p| = |p_1| \leq 2$, we may assume that $p \in [0, 2]$. By using triangle inequality and $|z| \leq 1$ with $|x| = t \in [0, 1]$, we obtain

$$|a_2a_4 - a_3^2| \leq \frac{1}{96(1+\alpha)(1+2\alpha)^2(1+3\alpha)}\Big[(4\alpha^2+4\alpha+1)p^4 + 6(1+2\alpha)^2p^2(4-p^2)t^2$$

$$+6(1+\alpha)(1+3\alpha)(4-p^2)^2t^2 + 12(1+2\alpha)^2p(4-p^2) - 12(1+2\alpha)^2p(4-p^2)t^2\Big] = F(p,t).$$

$$\frac{\partial F}{\partial t} = \frac{(4-p^2)t}{8(1+\alpha)(1+2\alpha)^2(1+3\alpha)}\Big[\alpha^2p^2 - 2(1+2\alpha)^2p + 4(1+\alpha)(1+3\alpha)\Big] \geq 0,$$

and so $F(p,t)$ is an increasing function of t for $p \leq \frac{3}{2}$.
Therefore,

$$\max\{F(p,t)\} = F(p,1) = \frac{1}{192(1+\alpha)(1+2\alpha)^2(1+3\alpha)}\Big[(\alpha^2+4\alpha+1)p^4$$

$$+12\alpha^2p^2(4-p^2) + 12(1+2\alpha)^2p^2(4-p^2) + 12(1+\alpha)(1+3\alpha)(4-p^2)^2\Big] = H(p).$$

$H'(p) = 0$ gives $p = 0$. Also $H''(p) < 0$ for $p = 0$.
This implies $\max\{H(p)\} = H(0) = \dfrac{1}{(1+2\alpha)^2}$, which proves (2.20). □

Putting $\alpha = 0$, Theorem 2.11 gives the following result:

Remark 2.12. If $f \in \mathcal{R}'_{sin}$, then

$$|a_2a_4 - a_3^2| \leq 1.$$

Substituting for $\alpha = 1$ in Theorem 2.11, the following result due to Khan et al. [11], is obvious:

Remark 2.13. If $f \in \mathcal{R}_{sin}$, then

$$|a_2a_4 - a_3^2| \leq \frac{1}{9}.$$

Theorem 2.14. *If $f \in \mathcal{R}^{\alpha}_{sin}$, then*

$$|H_3(1)| \leq \frac{(2+8\alpha+4\alpha^2)(1+3\alpha)^2 + (1+4\alpha)(1+2\alpha)^3}{(1+2\alpha)^3(1+3\alpha)^2(1+4\alpha)}. \tag{2.21}$$

Proof. By using (2.2), (2.3), (2.4), (2.18), (2.20) and Remark 2.7 in (1.1), the result (2.21) can be easily obtained. □

For $\alpha = 0$, Theorem 2.14 yields the following result:

Remark 2.15. If $f \in \mathcal{R}'_{sin}$, then

$$|H_3(1)| \leq 3.$$

For $\alpha = 1$, Theorem 2.14 yields the following result due to Khan et al. [11]:

Remark 2.16. If $f \in \mathcal{R}_{sin}$, then

$$|H_3(1)| \leq \frac{359}{2160}.$$

Theorem 2.17. *If $f \in \mathcal{R}^{\alpha}_{sin}$, then*

$$|H_4(1)| \leq \frac{2}{(1+2\alpha)^2(1+4\alpha)}\left[\frac{1+4\alpha+2\alpha^2}{(1+2\alpha)(1+6\alpha)}+\frac{3+12\alpha+2\alpha^2}{(1+4\alpha)^2}+\frac{2+8\alpha+4\alpha^2}{(1+3\alpha)(1+5\alpha)}\right]$$
$$+\frac{1}{(1+3\alpha)^2}\left[\frac{2+12\alpha+9\alpha^2}{(1+6\alpha)(1+3\alpha)^2}+\frac{3}{(1+2\alpha)(1+4\alpha)}\right].$$

Proof. We have

$$|a_2a_4+2a_3^2| \leq |a_2a_4-a_3^2|+3|a_3|^2.$$

Applying the triangle inequality in (1.2) and using the above inequality along with Theorem 2.1, Theorem 2.4, Theorem 2.8, Theorem 2.11 and Theorem 2.14, the proof of the Theorem 2.17 is obvious. □

For $\alpha = 0$, Theorem 2.17 yields the following result:

Remark 2.18. If $f \in \mathcal{R}^{'}_{sin}$, then

$$|H_4(1)| \leq 17.$$

For $\alpha = 1$, Theorem 2.17 yields the following result due to Khan et al. [11]:

Remark 2.19. If $f \in \mathcal{R}_{sin}$, then

$$|H_4(1)| \leq 0.10556.$$

3. Bounds of $|H_3(1)|$ for two-fold and three-fold symmetric functions

A function f is said to be n-fold symmetric if is satisfy the following condition:

$$f(\xi z) = \xi f(z)$$

where $\xi = e^{\frac{2\pi i}{n}}$ and $z \in E$.
By $S^{(n)}$, we denote the set of all n-fold symmetric functions which belong to the class S.
The n-fold univalent function have the following Taylor-Maclaurin series:

$$f(z) = z + \sum_{k=1}^{\infty} a_{nk+1} z^{nk+1}. \tag{3.1}$$

An analytic function f of the form (3.1) belongs to the family $\mathcal{R}^{\alpha(n)}_{sin}$ if and only if

$$(1-\alpha)\frac{f(z)}{z}+\alpha f'(z) = 1 + sin\left(\frac{p(z)-1}{p(z)+1}\right), p \in \mathcal{P}^{(n)},$$

where

$$\mathcal{P}^{n} = \left\{p \in \mathcal{P} : p(z) = 1 + \sum_{k=1}^{\infty} p_{nk} z^{nk}, z \in E\right\}. \tag{3.2}$$

Theorem 3.1. *If $f \in \mathcal{R}^{\alpha(2)}_{sin}$, then*

$$|H_3(1)| \leq \frac{1}{(1+2\alpha)(1+4\alpha)}. \tag{3.3}$$

Proof. If $f \in \mathcal{R}_{sin}^{\alpha(2)}$, so there exists a function $p \in \mathcal{P}^{(2)}$ such that

$$(1-\alpha)\frac{f(z)}{z} + \alpha f'(z) = 1 + sin\left(\frac{p(z)-1}{p(z)+1}\right). \tag{3.4}$$

Using (3.1) and (3.2) for $n = 2$, (3.4) yields

$$a_3 = \frac{1}{2(1+2\alpha)}p_2, \tag{3.5}$$

$$a_5 = \frac{1}{2(1+4\alpha)}\left(p_4 - \frac{1}{2}p_2^2\right). \tag{3.6}$$

Also

$$H_3(1) = a_3a_5 - a_3^3. \tag{3.7}$$

Using (3.5) and (3.6) in (3.7), it yields

$$H_3(1) = \frac{1}{4(1+2\alpha)(1+4\alpha)}p_2\left[p_4 - \frac{(1+2\alpha)^2 + (1+4\alpha)}{2(1+2\alpha)^2}p_2^2\right]. \tag{3.8}$$

On applying triangle inequality in (3.8) and using fourth inequality of Lemma 1.2, we can easily get the result (3.3). □

Putting $\alpha = 0$, the following result can be easily obtained from Theorem 3.1:

Remark 3.2. If $f \in \mathcal{R}_{sin}^{'(2)}$, then

$$|H_3(1)| \leq 1.$$

For $\alpha = 1$, Theorem 3.1 agrees with the following result:

Remark 3.3. If $f \in \mathcal{R}_{sin}^{(2)}$, then

$$|H_3(1)| \leq \frac{1}{15}.$$

Theorem 3.4. *If $f \in \mathcal{R}_{sin}^{\alpha(3)}$, then*

$$|H_3(1)| \leq \frac{1}{(1+3\alpha)^2}. \tag{3.9}$$

Proof. If $f \in \mathcal{R}_{sin}^{\alpha(3)}$, so there exists a function $p \in \mathcal{P}^{(3)}$ such that

$$(1-\alpha)\frac{f(z)}{z} + \alpha f'(z) = 1 + sin\left(\frac{p(z)-1}{p(z)+1}\right). \tag{3.10}$$

Using (3.1) and (3.2) for $n = 3$, (3.10) gives

$$a_4 = \frac{1}{2(1+3\alpha)}p_3. \tag{3.11}$$

Also

$$H_3(1) = -a_4^2. \tag{3.12}$$

Using (3.11) in (3.12), it yields

$$H_3(1) = -\frac{1}{4(1+3\alpha)^2}p_3^2. \tag{3.13}$$

On applying triangle inequality and using first inequality of Lemma 1.2, (3.9) can be easily obtained. □

For $\alpha = 0$, Theorem 3.4 yields the following result:

Remark 3.5. If $f \in \mathcal{R}_{sin}^{'(3)}$, then

$$|H_3(1)| \leq 1.$$

For $\alpha = 1$, Theorem 3.4 yields the following result:

Remark 3.6. If $f \in \mathcal{R}_{sin}^{(3)}$, then

$$|H_3(1)| \leq \frac{1}{16}.$$

References

[1] Altinkaya, Ş., Yalçin, S., *Third Hankel determinant for Bazilevic functions*, Adv. Math., Sci. J., **5**(2016), no. 2, 91-96.

[2] Arif, M., Rani, L., Raza, M., Zaprawa, P., *Fourth Hankel determinant for the family of functions with bounded turning*, Bull. Korean Math. Soc., **55**(2018), no. 6, 1703-1711.

[3] Arif, M., Raza, M., Tang, H., Hussain, S., Khan, H., *Hankel determinant of order three for familiar subsets of ananlytic functions related with sine function*, Open Math., **17**(2019), 1615-1630.

[4] Babalola, K. O., *On $H_3(1)$ Hankel determinant for some classes of univalent functions*, Inequal. Theory Appl., **6**(2010), 1-7.

[5] Bieberbach, L., *Über die koeffizienten derjenigen Potenzreihen, welche eine schlichte Abbildung des Einheitskreises vermitteln*, Sitzungsberichte Preussische Akademie der Wissenschaften, **138**(1916), 940-955.

[6] Bucur, R., Breaz, D., Georgescu, L., *Third Hankel determinant for a class of analytic functions with respect to symmetric points*, Acta Univ. Apulensis, **42**(2015), 79-86.

[7] Cho, N.E., Kumar, V., Kumar, S.S., Ravichandran, V., *Radius problems for starlike functions associated with sine function*, Bull. Iranian Math. Soc., **45**(2019), 213-232.

[8] De-Branges, L., *A proof of the Bieberbach conjecture*, Acta Math., **154**(1985), 137-152.

[9] Ehrenborg, R., *The Hankel determinant of exponential polynomials*, Amer. Math. Monthly, **107**(2000), 557-560.

[10] Janteng, A., Halim, S.A., Darus, M., *Hankel determinant for starlike and convex functions*, Int. J. Math. Anal., **1**(2007), no. 13, 619-625.

[11] Khan, M.G., Ahmad, B., Sokol, J., Muhammad, Z., Mashwani, W.K., Chinram, R., Petchkaew, P., *Coefficient problems in a class of functions with bounded turning associated with sine function*, Eur. J. Pure Appl. Math., **14**(2021), no. 1, 53-64.

[12] Layman, J.W., *The Hankel transform and some of its properties*, J. Integer Seq., **4**(2001), 1-11.

[13] Libera, R.J., Zlotkiewiez, E.J., *Early coefficients of the inverse of a regular convex function*, Proc. Amer. Math. Soc., **85**(1982), 225-230.

[14] Libera, R.J., Zlotkiewiez, E.J., *Coefficient bounds for the inverse of a function with derivative in $\mathcal{P}$*, Proc. Amer. Math. Soc., **87**(1983), 251-257.

[15] Ma, W., *Generalized Zalcman conjecture for starlike and typically real functions*, J. Math. Anal. Appl. **234**(1999), 328-329.

[16] MacGregor, T.H., *Functions whose derivative has a positive real part*, Trans. Amer. Math. Soc., **104**(1962), 532-537.

[17] MacGregor, T.H., *The radius of univalence of certain analytic functions*, Proc. Amer. Math. Soc. **14**(1963), 514-520.

[18] Mehrok, B.S., Singh, G., *Estimate of second Hankel determinant for certain classes of analytic functions*, Sci. Magna, **8**(2012), no. 3, 85-94.

[19] Murugusundramurthi, G., Magesh, N., *Coefficient inequalities for certain classes of analytic functions associated with Hankel determinant*, Bull. Math. Anal. Appl., **1**(2009), no. 3, 85-89.

[20] Noor, K.I., *Hankel determinant problem for the class of functions with bounded boundary rotation*, Rev. Roumaine Math. Pures Appl., **28**(1983), no. 8, 731-739.

[21] Pommerenke, C., *On the coefficients and Hankel determinants of univalent functions*, J. Lond. Math. Soc., **41**(1966), 111-122.

[22] Pommerenke, C., *Univalent Functions*, Math. Lehrbucher, Vandenhoeck and Ruprecht, Gottingen, 1975.

[23] Ravichandran, V., Verma, S., *Bound for the fifth coefficient of certain starlike functions*, Comptes Rendus Math., **353**(2015), 505-510.

[24] Reade, M.O., *On close-to-convex univalent functions*, Michigan Math. J., **3**(1955-56), 59-62.

[25] Shanmugam, G., Stephen, B.A., Babalola, K.O., *Third Hankel determinant for α starlike functions*, Gulf J. Math., **2**(2014), no. 2, 107-113.

[26] Singh, G., *Hankel determinant for a new subclass of analytic functions*, Sci. Magna, **8**(2012), no. 4, 61-65.

[27] Singh, G., Singh, G., *Hankel determinant problems for certain subclasses of Sakaguchi-type functions defined with subordination*, Korean J. Math., **30**(2022), no. 1, 81-90.

[28] Singh, G., Singh, G., Singh, G., *Upper bound on fourth Hankel determinant for certain subclass of multivalent functions*, Jnanabha, **50**(2020), no. 2, 122-127.

[29] Singh, G., Singh, G., Singh, G., *Fourth Hankel determinant for a subclass of analytic functions defined by generalized Salagean operator*, Creat. Math. Inform., **31**(2022), no. 2, 229-240.

[30] Zhang, H. Y., Tang, H., *A study of fourth order Hankel determinant for starlike functions connected with the sine function*, J. Funct. Spaces, 2021, Article Id. 9991460, 8 pages.

Gagandeep Singh
Department of Mathematics, Khalsa College,
Amritsar-143001, Punjab, India
e-mail: `kamboj.gagandeep@yahoo.in`

Gurcharanjit Singh
Department of Mathematics, G.N.D.U. College, Chungh,
Tarn-Taran-143304, Punjab, India
e-mail: `dhillongs82@yahoo.com`