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An algorithm for solving a control problem for Kolmogorov systems

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Abstract. In this paper, a numerical algorithm is used for solving control problems related to Kolmogorov systems. It is proved the convergence of the algorithm and by this it is re-obtained, by a numerical approach, the controllability of the investigated problems.

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1. Introduction

Many real processes must be controlled to drive their evolution completion according to a desired plan. Mathematically, a control problem returns to determination of one or several parameters of the equation or system of equations so that the solution satisfies certain conditions, others than initial or boundary conditions.

The Kolmogorov system was introduced as a generalization of a model given by the mathematician Volterra from population dynamics. It operates at the general per capita rate of two species that interact with each other and has the following form:

$$\begin{cases} x' = xf(x,y) \\ y' = yg(x,y). \end{cases}$$

Here, the rates f and g are given in terms of parameters that cannot be changed, and others that can be modified in order to control the evolution. Kolmogorov systems arises in many areas, such as population dynamics, ecological balance and the spread of epidemics (for such models see [1], [3], [8]).

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The Lotka-Volterra system, also known as prey-predator system, consists in a pair of nonlinear differential equations dynamically describing the interaction between two species. Populations change over time according to the system of equations

$$\begin{cases} \dfrac{dx}{dt} = \alpha x - \beta xy \\ \dfrac{dy}{dt} = \delta xy - \gamma y, \end{cases}$$

where $x(t)$ represents the prey population, $y(t)$ predator population, $\frac{dx}{dt}, \frac{dy}{dt}$ represents the growth rates of the two populations, t time variable and $\alpha, \beta, \delta, \gamma$ are real positive parameters that describe the interaction of the two species.

In mathematical epidemiology, the SIR model (1.1) is well known. Here, $S(t)$ represented the number of susceptible population, $I(t)$ the number of population infected and $R(t)$ this number of recovered. Significant advances have been made by Kermack and McKendrick, where they studied those circumstances (represented by values of certain parameters) when behaviour of susceptible population falls below a threshold value.

The equations governing the SIR model are as follows:

$$\begin{cases} S'(t) = -aS(t)I(t) \\ I'(t) = aS(t)I(t) - bI(t) \\ R'(t) = bI(t). \end{cases} \tag{1.1}$$

The purpose of this paper is to present a numerical algorithm for solving control problems related to Kolmogorov systems. It is proved the convergence of the algorithm and by this it is reobtained, by a numerical approach, the controllability of the problems.

2. Main results

In what follows, we study the dynamics of the growth rates (not the per capita one) in order for certain conditions to be fulfilled. Consider the problem

$$\begin{cases} x'(t) = x(t)f(x(t), y(t)) - \lambda \\ y'(t) = y(t)g(x(t), y(t)) \\ x(0) = x_0, \ y(0) = y_0, \end{cases} \tag{2.1}$$

where λ is constant.
Here, the controllability condition is $\varphi(x, y) = 0$, where $\varphi : C([0, T], \mathbb{R}^2) \to \mathbb{R}$ represents a continuous function (for example, $\varphi(x, y) = \alpha x(T) + \beta y(T) - \gamma$ with $\alpha, \beta, \gamma \in \mathbb{R}$).

The initial value problem (2.1) has a unique solution $(S_1(\lambda), S_2(\lambda))$, for any fixed λ, which is continuous with respect to λ.

Assume the following conditions hold:

(i) $\varphi(x, y) < 0$ for $\lambda = 0$;
(ii) $\varphi(x, y) \geq 0$ for $\lambda = 1$.

The following iterative algorithm is aimed to bring us as close as possible to a value of λ that corresponds to a solution of the control problem.

The algorithm:

Step 1. Initialize $\underline{\lambda}_0 := 0$, $\overline{\lambda}_0 := 1$

Step 2. At any iteration $k \geq 1$, define $\lambda_k := \dfrac{\underline{\lambda}_{k-1} + \overline{\lambda}_{k-1}}{2}$ and solve system (2.1) for $\lambda := \lambda_k$. Obtain the numerical solution

$$(x_k, y_k) = (S_1(\lambda_k), S_2(\lambda_k)).$$

If $\varphi(x_k, y_k) < 0$, then put $\underline{\lambda}_k = \lambda$, $\overline{\lambda}_k = \overline{\lambda}_{k-1}$, otherwise, take $\underline{\lambda}_k = \underline{\lambda}_{k-1}$, $\overline{\lambda}_k = \lambda$, we make $k = k+1$ and we repeat Step 2.

Step 3. The algorithm stops if

$$|\varphi(x_k, y_k)| < \delta,$$

where $0 < \delta < 1$ is the admitted error.

To demonstrate convergence we need the following two lemmas of continuous dependence on parameter.

Lemma 2.1. *Assume that $f, g : \mathbb{R}^2 \to \mathbb{R}$ are Lipschitz continuous on $\mathbb{R}^2$ and $|f| \leq C_f$, $|g| \leq C_g$. Then for any $\lambda \in \mathbb{R}$, the Cauchy problem (2.1) has a unique solution that depends continuously on the parameter λ.*

Proof. Problem (2.1) is equivalent to the Volterra integral system

$$\begin{cases} x(t) = x_0 + \int_0^t x(s)f(x(s), y(s))ds - \lambda t \\ y(t) = y_0 + \int_0^t y(s)g(x(s), y(s))ds, \end{cases} \tag{2.2}$$

which is a fixed point equation in (x, y), on the space $C([0,T]; \mathbb{R}^2)$.

For the proof we first show the boundedness of solutions.

I. Boundedness of solutions

We have to prove that there exist two constants $C_1, C_2 > 0$ such that

$$|S_1(\lambda)(t)| \leq C_1 \text{ and } |S_2(\lambda)(t)| \leq C_2,$$

for every $\lambda \in [0,1]$ and $t \in [0,T]$. Since $|f| \leq C_f$, $|g| \leq C_g$, the first equation in (2.2) yields

$$\begin{aligned} |x(t)| &\leq |x_0| + \int_0^t |x(s)||f(x(s), y(s))|ds + T \\ &\leq |x_0| + T + C_f \int_0^t |x(s)|ds. \end{aligned}$$

From Gronwall's inequality(see [2]), we obtain

$$|x(t)| \leq (|x_0| + T)e^{C_f T} =: C_1, \qquad t \in [0,T].$$

Under a similar reasoning we find $C_2 := (|y_0| + T)e^{C_g T}$ such that $|y(t)| \leq C_2$ for $t \in [0,T]$.

II. Existence and uniqueness

Let $\alpha_{ij}, i,j = 1,2$ be the Lipschitz constants for $f(x,y)$ and $g(x,y)$ with respect x and y.
Denote

$$\begin{aligned} A(x,y)(t) &= x_0 + \int_0^t x(s)f(x(s),y(s))ds - \lambda t, \\ B(x,y)(t) &= y_0 + \int_0^t y(s)g(x(s),y(s))ds. \end{aligned}$$

We prove that the operator $N := (A,B)$ is a contraction on $C\left([0,T];\mathbb{R}^2\right)$ with respect to the Bielecki norm $\|(x,y)\|_\theta := \|x\|_\theta + \|y\|_\theta$, where

$$\|x\|_\theta := \max_{t\in[0,T]} \left(|x(t)|\, e^{-\theta t}\right), \qquad \|y\|_\theta := \max_{t\in[0,T]} \left(|y(t)|\, e^{-\theta t}\right).$$

We have

$$\begin{aligned} |A(x,y)(t) - A(\bar{x},\bar{y})(t)| &\le \int_0^t |x(s)f(x(s),y(s)) - \bar{x}(s)f(\bar{x}(s),\bar{y}(s))|ds \\ &\le \int_0^t |x(s)f(x(s),y(s)) - x(s)f(\bar{x}(s),\bar{y}(s))|ds \\ &\quad + \int_0^t |x(s)f(\bar{x}(s),\bar{y}(s)) - \bar{x}(s)f(\bar{x}(s),\bar{y}(s))|ds \\ &\le C_1 \int_0^t (\alpha_{11}|x(s)-\bar{x}(s)| + \alpha_{12}|y(s)-\bar{y}(s)|)ds \\ &\quad + C_f \int_0^t |x(s)-\bar{x}(s)|ds. \end{aligned}$$

Next

$$\begin{aligned} & |A(x,y)(t) - A(\bar{x},\bar{y})(t)| \\ \le\; & C_1 \int_0^t (\alpha_{11}|x(s)-\bar{x}(s)|e^{-\theta s}e^{\theta s} + \alpha_{12}|y(s)-\bar{y}(s)|e^{-\theta s}e^{\theta s})ds \\ & + C_f \int_0^t |x(s)-\bar{x}(s)|e^{-\theta s}e^{\theta s}ds \\ \le\; & (C_1\alpha_{11} + C_f)\,\|x-\overline{x}\|_\theta \int_0^t e^{\theta s}ds + C_1\alpha_{12}\,\|y-\overline{y}\|_\theta \int_0^t e^{\theta s}ds \\ \le\; & \frac{C_1\alpha_{11}+C_f}{\theta}\,\|x-\overline{x}\|_\theta\, e^{\theta t} + \frac{C_1\alpha_{12}}{\theta}\,\|y-\overline{y}\|_\theta\, e^{\theta t}. \end{aligned}$$

Now, multipling the above relation with $e^{-\theta t}$ and taking the supremum over t, we obtain

$$||A(x,y) - A(\bar{x},\bar{y})||_\theta \le \frac{C_1\alpha_{11}+C_f}{\theta}\,\|x-\overline{x}\|_\theta + \frac{C_1\alpha_{12}}{\theta}\,\|y-\overline{y}\|_\theta\,. \tag{2.3}$$

Similarly

$$||B(x,y) - B(\bar{x},\bar{y})||_\theta \leq \frac{C_2\alpha_{21}}{\theta}||x-\bar{x}||_\theta + \frac{C_2\alpha_{22}+C_g}{\theta}||y-\bar{y}||_\theta. \tag{2.4}$$

Further, adding relations (2.3) and (2.4), we deduce

$$\|(A(x,y),B(x,y)) - (A(\overline{x},\overline{y}),B(\overline{x},\overline{y}))\|_\theta \leq \overline{C}_1||x-\bar{x}||_\theta + \overline{C}_2||y-\bar{y}||_\theta,$$

where

$$\overline{C}_1 = \frac{C_1\alpha_{11}+C_f}{\theta} + \frac{C_2\alpha_{21}}{\theta},$$

$$\overline{C}_2 = \frac{C_1\alpha_{12}}{\theta} + \frac{C_2\alpha_{22}+C_g}{\theta}.$$

Therefore,

$$||N(x,y) - N(\bar{x},\bar{y})||_\theta \leq L\left(||x-\bar{x}||_\theta + ||y-\bar{y}||_\theta\right) = L||(x,y)-(\bar{x},\bar{y})||_\theta,$$

where $L := \max\{\overline{C}_1, \overline{C}_2\}$.

If we now take a sufficiently large number θ, then $L < 1$, and thus the operator $N = (A,B)$ is a contraction on the space $C([0,T];\mathbb{R}^2)$ endowed with the Bielecki norm $||\cdot||_\theta$. Therefore, Banach contraction principle applies and gives the result.

III. Continuous dependence of parameter λ

Using (2.2), where, first $x = S_1(\lambda)$ and $y = S_2(\lambda)$, and next $x = S_1(\mu)$ and $y = S_2(\mu)$, we have

$$\begin{aligned}
& |S_1(\lambda)(t) - S_1(\mu)(t)| \\
\leq & \int_0^t |S_1(\lambda)(s)f(S_1(\lambda)(s),S_2(\lambda)(s)) - S_1(\mu)(s)f(S_1(\mu)(s),S_2(\mu)(s))|ds \\
& +|\lambda-\mu|T \\
\leq & \int_0^t |S_1(\lambda)(s)f(S_1(\lambda)(s),S_2(\lambda)(s)) - S_1(\lambda)(s)f(S_1(\mu)(s),S_2(\mu)(s))|ds \\
& + \int_0^t |S_1(\lambda)(s)f(S_1(\mu)(s),S_2(\mu)(s)) - S_1(\mu)(s)f(S_1(\mu)(s),S_2(\mu)(s))|ds \\
& +|\lambda-\mu|T \\
\leq & \int_0^t |S_1(\lambda)(s)||f(S_1(\lambda)(s),S_2(\lambda)(s)) - f(S_1(\mu)(s),S_2(\mu)(s))|ds \\
& + \int_0^t |f(S_1(\mu)(s),S_2(\mu)(s))||S_1(\lambda)(s) - S_1(\mu)(s)|ds + |\lambda-\mu|T.
\end{aligned}$$

Furthermore, using the Lipschitz property of f, g, and their boundedness, we obtain

$$\begin{aligned}
&|S_1(\lambda)(t) - S_1(\mu)(t)| \\
\leq\ & C_1 \int_0^t (\alpha_{11}|S_1(\lambda)(s) - S_1(\mu)(s)| + \alpha_{12}|S_2(\lambda)(s) - S_2(\mu)(s)|)ds \\
&+C_f \int_0^t |S_1(\lambda)(s) - S_1(\mu)(s)|ds + |\lambda - \mu|T \\
\leq\ & C_1 \int_0^t (\alpha_{11}|S_1(\lambda)(s) - S_1(\mu)(s)|e^{-\theta s}e^{\theta s} + \alpha_{12}|S_2(\lambda)(s) - S_2(\mu)(s)|e^{-\theta s}e^{\theta s})ds \\
&+C_f \int_0^t |S_1(\lambda)(s) - S_1(\mu)(s)|e^{-\theta s}e^{\theta s}ds + |\lambda - \mu|T \\
\leq\ & C_1||S_1(\lambda) - S_1(\mu)||_\theta \frac{\alpha_{11}}{\theta}e^{\theta t} + C_1||S_2(\lambda) - S_2(\mu)||_\theta \frac{\alpha_{12}}{\theta}e^{\theta t} \\
&+\frac{C_f}{\theta}||S_1(\lambda) - S_1(\mu)||_\theta e^{\theta t} + |\lambda - \mu|T.
\end{aligned}$$

Thus

$$\begin{aligned}
&|S_1(\lambda)(t) - S_1(\mu)(t)| \\
&\leq \frac{C_1\alpha_{11} + C_f}{\theta}||S_1(\lambda) - S_1(\mu)||_\theta e^{\theta t} + \frac{C_1\alpha_{12}}{\theta}||S_2(\lambda) - S_2(\mu)||_\theta e^{\theta t} \\
&+ |\lambda - \mu|T.
\end{aligned}$$

Multiply by $e^{-\theta t}$, go to the maximum and introduce the Bielecki norm, to obtain

$$\begin{aligned}
||S_1(\lambda) - S_1(\mu)||_\theta \leq\ & \frac{C_1\alpha_{11} + C_f}{\theta}||S_1(\lambda) - S_1(\mu)||_\theta \\
&+ \frac{C_1\alpha_{12}}{\theta}||S_2(\lambda) - S_2(\mu)||_\theta + |\lambda - \mu|T.
\end{aligned}$$

Similarly

$$\begin{aligned}
||S_2(\lambda) - S_2(\mu)||_\theta \leq\ & \frac{C_2\alpha_{21}}{\theta}||S_1(\lambda) - S_1(\mu)||_\theta \\
&+ \frac{C_2\alpha_{22} + C_g}{\theta}||S_2(\lambda) - S_2(\mu)||_\theta.
\end{aligned}$$

Summing up, gives

$$\begin{aligned}
&||S_1(\lambda) - S_1(\mu)||_\theta + ||S_2(\lambda) - S_2(\mu)||_\theta \\
&\leq \frac{C_1\alpha_{11} + C_f + C_2\alpha_{21}}{\theta}||S_1(\lambda) - S_1(\mu)||_\theta \\
&+ \frac{C_1\alpha_{12} + C_2\alpha_{22} + C_g}{\theta}||S_2(\lambda) - S_2(\mu)||_\theta + |\lambda - \mu|T,
\end{aligned}$$

whence

$$\begin{aligned}
||(S_1(\lambda), S_2(\lambda)) - (S_1(\mu), S_2(\mu))||_\theta \leq\ & M_\theta||(S_1(\lambda), S_2(\lambda)) - (S_1(\mu), S_2(\mu))||_\theta \\
&+ |\lambda - \mu|T,
\end{aligned}$$

where

$$M_\theta = \max\left\{\frac{C_1\alpha_{11} + C_f + C_2\alpha_{21}}{\theta},\ \frac{C_1\alpha_{12} + C_2\alpha_{22} + C_g}{\theta}\right\}.$$

Notice that $M_\theta \to 0$, as $\theta \to +\infty$, so if θ is large enough, one has $M_\theta < 1$. Then

$$(1 - M_\theta)||(S_1(\lambda), S_2(\lambda)) - (S_1(\mu), S_2(\mu))||_\theta \le |\lambda - \mu|T,$$

where since $1 - M_\theta > 0$ we get that

$$||(S_1(\lambda), S_2(\lambda)) - (S_1(\mu), S_2(\mu))||_\theta \le \frac{1}{1 - M_\theta}|\lambda - \mu|T.$$

So, if $\mu \to \lambda$, then $(S_1(\mu), S_2(\mu)) \to (S_1(\lambda), S_2(\lambda))$, which means that the solution depends continuously on the parameter λ. □

Alternatively, we have

Lemma 2.2. *Let $f, g : \mathbb{R}^2 \to \mathbb{R}$ be such that the functions $xf(x, y)$ and $yg(x, y)$ are Lipschitz continuous on the entire $\mathbb{R}^2$. Then for any $\lambda \in \mathbb{R}$, the Cauchy problem (2.2) has a unique solution that depends continuously on the parameter λ.*

Proof. **I. Existence and uniqueness.** Let $\alpha_{ij}, i, j = 1, 2$ be the Lipschitz constants of the functions $xf(x, y)$ and $yg(x, y)$. Hence

$$\begin{aligned} |xf(x, y) - \bar{x}f(\bar{x}, \bar{y})| &\le \alpha_{11}|x - \bar{x}| + \alpha_{12}|y - \bar{y}|, \\ |yg(x, y) - \bar{y}g(\bar{x}, \bar{y})| &\le \alpha_{21}|x - \bar{x}| + \alpha_{22}|y - \bar{y}|. \end{aligned}$$

Using the notations from the proof of Lemma 2.1, we have that the operator $N = (A, B)$ is a contraction on the space $C([0, T]; \mathbb{R}^2)$ with respect to a suitable Bielecki norm. Indeed, one has

$$\begin{aligned} &|A(x, y)(t) - A(\bar{x}, \bar{y})(t)| \\ &\le \int_0^t |x(s)f(x(s), y(s)) - \bar{x}(s)f(\bar{x}(s), \bar{y}(s))|ds \\ &\le \int_0^t (\alpha_{11}|x(s) - \bar{x}(s)| + \alpha_{12}|y(s) - \bar{y}(s)|)ds \\ &\le \int_0^t (\alpha_{11}|x(s) - \bar{x}(s)|e^{-\theta s}e^{\theta s} + \alpha_{12}|y(s) - \bar{y}(s)|e^{-\theta s}e^{\theta s})ds \\ &\le \frac{\alpha_{11}}{\theta}||x - \bar{x}||_\theta e^{\theta t} + \frac{\alpha_{12}}{\theta}||y - \bar{y}||_\theta e^{\theta t}. \end{aligned}$$

Multiplying by $e^{-\theta t}$, and taking the maximum, we obtain

$$\begin{aligned} ||A(x, y) - A(\bar{x}, \bar{y})||_\theta &\le \frac{\alpha_{11}}{\theta}||x - \bar{x}||_\theta + \frac{\alpha_{12}}{\theta}||y - \bar{y}||_\theta, \\ ||B(x, y) - B(\bar{x}, \bar{y})||_\theta &\le \frac{\alpha_{21}}{\theta}||x - \bar{x}||_\theta + \frac{\alpha_{22}}{\theta}||y - \bar{y}||_\theta. \end{aligned}$$

Summing up gives

$$\begin{aligned} &||A(x, y) - A(\bar{x}, \bar{y})||_\theta + ||B(x, y) - B(\bar{x}, \bar{y})||_\theta \\ \le\ &\frac{\alpha_{11} + \alpha_{21}}{\theta}||x - \bar{x}||_\theta + \frac{\alpha_{12} + \alpha_{22}}{\theta}||y - \bar{y}||_\theta \end{aligned}$$

So

$$||N(x,y) - N(\bar{x},\bar{y})||_\theta \leq L(||x-\bar{x}||_\theta + ||y-\bar{y}||_\theta),$$

where

$$L = \max\left\{\frac{\alpha_{11}+\alpha_{21}}{\theta}, \frac{\alpha_{12}+\alpha_{22}}{\theta}\right\}.$$

It turns out that

$$||N(x,y) - N(\bar{x},\bar{y})||_\theta \leq L||(x,y)-(\bar{x},\bar{y})||_\theta.$$

Here again if θ is chosen large enough, then $L < 1$, and so $N = (A, B)$ is a contraction on $C([0,T];\mathbb{R}^2)$ endowed with the Bielecki norm $\|.\|_\theta$. It follows that the Cauchy problem has a unique solution.

II. Continuous dependence of parameter λ. Using (2.2) where $x = S_1(\lambda)$ and $y = S_2(\lambda)$, we have

$$\begin{aligned}
& |S_1(\lambda)(t) - S_1(\mu)(t)| \\
\leq\ & \int_0^t |S_1(\lambda)(s)f(S_1(\lambda)(s), S_2(\lambda)(s)) - S_1(\mu)(s)f(S_1(\mu)(s), S_2(\mu)(s))|ds \\
& +|\lambda-\mu|T \\
\leq\ & \int_0^t (\alpha_{11}|S_1(\lambda)(s) - S_1(\mu)(s)| + \alpha_{12}|S_2(\lambda)(s) - S_2(\mu)(s)|)ds + |\lambda-\mu|T \\
\leq\ & \int_0^t (\alpha_{11}|S_1(\lambda)(s) - S_1(\mu)(s)|e^{-\theta s}e^{\theta s} + \alpha_{12}|S_2(\lambda)(s) - S_2(\mu)(s)|e^{-\theta s}e^{\theta s})ds \\
& +|\lambda-\mu|T.
\end{aligned}$$

Then

$$\begin{aligned}
|S_1(\lambda)(t) - S_1(\mu)(t)| \leq\ & \frac{\alpha_{11}}{\theta}||S_1(\lambda) - S_1(\mu)||_\theta e^{\theta t} + \frac{\alpha_{12}}{\theta}||S_2(\lambda) - S_2(\mu)||_\theta e^{\theta t} \\
& + |\lambda-\mu|T.
\end{aligned}$$

It follows that

$$\begin{aligned}
||S_1(\lambda) - S_1(\mu)||_\theta \leq\ & \frac{\alpha_{11}}{\theta}||S_1(\lambda) - S_1(\mu)||_\theta + \frac{\alpha_{12}}{\theta}||S_2(\lambda) - S_2(\mu)||_\theta \\
& + |\lambda-\mu|T.
\end{aligned}$$

Similarly

$$||S_2(\lambda) - S_2(\mu)||_\theta \leq \frac{\alpha_{21}}{\theta}||S_1(\lambda) - S_1(\mu)||_\theta + \frac{\alpha_{22}}{\theta}||S_2(\lambda) - S_2(\mu)||_\theta.$$

Summing up gives

$$\begin{aligned}
& ||S_1(\lambda) - S_1(\mu)||_\theta + ||S_2(\lambda) - S_2(\mu)||_\theta \\
& \leq \frac{\alpha_{11}+\alpha_{21}}{\theta}||S_1(\lambda) - S_1(\mu)||_\theta + \frac{\alpha_{12}+\alpha_{22}}{\theta}||S_2(\lambda) - S_2(\mu)||_\theta \\
& + |\lambda-\mu|T.
\end{aligned}$$

Thus

$$\begin{aligned}
& ||(S_1(\lambda), S_2(\lambda)) - (S_1(\mu), S_2(\mu))||_\theta \\
\leq\ & m_\theta||(S_1(\lambda), S_2(\lambda)) - (S_1(\mu), S_2(\mu))||_\theta + |\lambda-\mu|T,
\end{aligned}$$

where

$$m_\theta := \max\left\{\frac{\alpha_{11}+\alpha_{21}}{\theta},\ \frac{\alpha_{12}+\alpha_{22}}{\theta}\right\}.$$

Then we have

$$(1-m_\theta)||(S_1(\lambda),S_2(\lambda))-(S_1(\mu),S_2(\mu))||_\theta \le |\lambda-\mu|T.$$

Here again, if $\theta\to+\infty$, then $m_\theta\to 0$ and thus one can choose $\theta>0$ sufficiently large that $m_\theta<1$. Then

$$||(S_1(\lambda),S_2(\lambda))-(S_1(\mu),S_2(\mu))||_\theta \le \frac{1}{1-m_\theta}|\lambda-\mu|T.$$

So, if $\mu\to\lambda$, then $(S_1(\mu),S_2(\mu))\to(S_1(\lambda),S_2(\lambda))$, which proves the continuous dependence of the solution of λ. □

Using Lemmas 2.1 and 2.2 we obtain two convergence results regarding the above algorithm.

Theorem 2.3. *Assume that $f,g:\mathbb{R}^2\to\mathbb{R}$ are Lipschitz continuous on $\mathbb{R}^2$ and $|f|\le C_f$, $|g|\le C_g$. Then the algorithm is convergent to a solution of the control problem.*

Proof. For $k\ge 1$ we have the solution (x_k,y_k) corresponding to $\lambda=\lambda_k$. In addition, the algorithm gives an increasing sequence $(\underline{\lambda}_k)$ and a decreasing sequence $(\overline{\lambda}_k)$ with the following properties

$$\varphi(S_1(\underline{\lambda}_k),S_2(\underline{\lambda}_k))<0,\qquad \varphi(S_1(\overline{\lambda}_k),S_2(\overline{\lambda}_k))\ge 0, \tag{2.5}$$

and

$$\overline{\lambda}_k-\underline{\lambda}_k=\frac{1}{2^k}. \tag{2.6}$$

The two sequences being monotone and bounded are convergent. Moreover, from (2.6) they have the same limit λ^*. Using the continuity of φ and of S_1, S_2 with respect to λ, and (2.5) we deduce that

$$\varphi(S_1(\lambda^*),S_2(\lambda^*))=0. \tag{2.7}$$

Finally, denote $x^*:=S_1(\lambda^*)$ and $y^*:=S_2(\lambda^*)$. The (2.7) shows that (x^*,y^*,λ^*) is a solution the control problem. □

Similarly, using Lemma 2.2, one can prove the follwoing result.

Theorem 2.4. *Let $f,g:\mathbb{R}^2\to\mathbb{R}$ be such that the functions $xf(x,y)$ and $yg(x,y)$ are Lipschitz continuous on the entire $\mathbb{R}^2$. Then the algorithm is convergent to a solution of the control problem.*

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