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Existence results for Dirichlet double phase differential inclusions

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Dedicated to the memory of Professor Csaba Varga

Abstract. In this paper we consider a class of double phase differential inclusions of the type

$$\begin{cases} -\text{div }\left(|\nabla u|^{p-2}\nabla u+\mu(x)|\nabla u|^{q-2}\nabla u\right)\in\partial_C^2 f(x,u), & \text{in } \Omega,\\ u=0, & \text{on } \partial\Omega,\end{cases}$$

where $\Omega\subset\mathbb{R}^N$, with $N\geq 2$, is a bounded domain with Lipschitz boundary, $f(x,t)$ is measurable w.r.t. the first variable on Ω and locally Lipschitz w.r.t. the second variable and $\partial_C^2 f(x,\cdot)$ stands for the Clarke subdifferential of $t\mapsto f(x,t)$. The variational formulation of the problem gives rise to a so-called hemivariational inequality and the corresponding energy functional is not differentiable, but only locally Lipschitz. We use nonsmooth critical point theory to prove the existence of at least one weak solution, provided the $\partial_C^2 f(x,\cdot)$ satisfies an appropriate growth condition.

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1. Introduction and main results

In this paper we are interested in a class of boundary value problems of the following type:

$$(P):\quad \begin{cases} -\text{div }\left(|\nabla u|^{p-2}\nabla u+\mu(x)|\nabla u|^{q-2}\nabla u\right)\in\partial_C^2 f(x,u), & \text{in } \Omega,\\ u=0, & \text{on } \partial\Omega,\end{cases}$$

where $\Omega\subset\mathbb{R}^N$, with $N\geq 2$, is a bounded domain with Lipschitz boundary and $\partial_C^2 f(x,t)$ stands for the Clarke subdifferential of the locally Lipschitz mapping $t\mapsto f(x,t)$.

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The presence of the double phase operator in the left-hand side requires that weak solutions of problem (P) to be sought in the Musielak-Orlicz-Sobolev space $W_0^{1,\mathcal{H}}(\Omega)$ (see Section 2.2), while the presence of the Clarke subdifferential in the right-hand gives rise to a hemivariational inequality. More precisely, we say that $u \in W_0^{1,\mathcal{H}}(\Omega)$ is a *weak solution* of (P) if it satisfies the following *hemivariational inequality*

$$\int_\Omega \left[|\nabla u|^{p-2}\nabla u \cdot \nabla v + \mu(x)|\nabla u|^{q-2}\nabla u \cdot \nabla v\right] dx \leq \int_\Omega f^0(x,u;v)dx,$$

for all $v \in W_0^{1,\mathcal{H}}(\Omega)$. Here and hereafter, $f^0(x,\cdot;\cdot)$ denotes the *generalized directional derivative* of f (see Section 2.3).

The conditions, which guarantee the existence of weak solutions for problem (P), and the main results of the paper are listed as follows.

(H_1) $1 < p < q < +\infty$ and $0 \leq \mu(\cdot) \in L^1(\Omega)$.

(f_1) $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is such that:

(i). $x \mapsto f(x,t)$ is measurable on Ω for all $t \in \mathbb{R}$;

(ii). $t \mapsto f(x,t)$ is locally Lipschitz for a.a. $x \in \Omega$;

(iii). $f(x,0) = 0$ for a.a. $x \in \Omega$.

(f_2) There exist $r \in (1,p^*)$, $\alpha \in L^{\frac{r}{r-1}}(\Omega)$ and $k > 0$ such that

$$|\zeta| \leq \alpha(x) + k|t|^{r-1}, \text{ for a.a. } x \in \Omega \text{ all } t \in \mathbb{R} \text{ and all } \zeta \in \partial_C^2 f(x,t),$$

where p^* is the *critical exponent* corresponding to p, i.e.,

$$p^* := \begin{cases} \frac{Np}{N-p}, & \text{if } p < N, \\ +\infty, & \text{otherwise.} \end{cases}$$

The first existence result is devoted to the case when the exponent controlling the growth of $\partial_C f(x,\cdot)$ is sufficiently small. The proof relies on the fact that in this case the associated energy functional is coercive. More precisely, we have the following existence result.

Theorem 1.1. *Assume (H_1), (f_1) and (f_2) hold. Then for any $r \in (1,p)$ the problem (P) possesses at least one nontrivial weak solution.*

If the exponent controlling the growth of $t \mapsto \partial_C f(x,t)$ is "large", i.e., $r \in (q,p^*)$, then the energy functional is no longer coercive. In this case we use the Ekeland variational principle to prove the existence of at least one weak solution by replacing (f_2) with the slightly more restrictive condition (f_2') and assume in addition condition (f_3), listed below:

(f_2') There exist $r \in (1,p^*)$, and $k > 0$ such that

$$|\zeta| \leq k|t|^{r-1}, \text{ for a.a. } x \in \Omega \text{ all } t \in \mathbb{R} \text{ and all } \zeta \in \partial_C^2 f(x,t).$$

(f_3) There exist a nonempty open subset $\omega \subset \Omega$ and $\delta, K > 0, s \in (1,p)$ such that

$$f(x,t) \geq Kt^s, \quad \text{whenever } (x,t) \in \omega \times (0,\delta].$$

Theorem 1.2. *Assume (H_1), (f_1), (f_2') and (f_3) hold. Then for any $r \in (q,p^*)$ the problem (P) possesses at least one nontrivial weak solution.*

We point out the fact that Theorem 1.2 is new even in the "smooth case", i.e., $f(x, \cdot) \in C^1(\mathbb{R})$ on the one hand due to the fact that we do not impose the condition $p < N$ and we allow $\mu \in L^1(\Omega)$ and, on the other hand, due to the fact that we do not impose an Ambrosetti-Rabinowitz type condition.

2. Preliminaries

2.1. Generalized N-functions and Musielak-Orlicz spaces

In this subsection we recall some definitions and basic properties of generalized Orlicz spaces also referred to as Musielak-Orlicz spaces. For more details and connections see, e.g., [2, 7, 10].

Definition 2.1. A continuous and convex function $\varphi : \mathbb{R} \to [0, \infty)$ is called *N-function* if it satisfies the following conditions:

(i). $\varphi(t) = 0$ if and only if $t = 0$;

(ii). $\varphi(-t) = \varphi(t)$ for all $t \in \mathbb{R}$;

(iii). $\lim_{t\to 0} \frac{\varphi(t)}{t} = 0$ and $\lim_{t\to\infty} \frac{\varphi(t)}{t} = \infty$.

Definition 2.2. Assume $\Omega \subset \mathbb{R}^N$ is a bounded domain. An application $\Phi : \Omega \times \mathbb{R} \to [0, \infty)$ is called *generalized N-function* if $x \mapsto \Phi(x, t)$ is measurabe for all $t \in \mathbb{R}$ and $t \mapsto \Phi(x, t)$ is an N-function for a.a. $x \in \Omega$.

Note that if Φ is a generalized N-function, then the corresponding *Young conjugate function*, $\tilde{\Phi} : \Omega \times \mathbb{R} \to [0, \infty)$, defined by

$$\tilde{\Phi}(x, s) := \sup_{t\geq 0} \{st - \Phi(x, t)\},$$

is also a generalized N-function.

Definition 2.3. A generalized N-function Φ is said to satisfy the *Δ_2-condition* if there exist a constant $k > 0$ and a nonnegative function $h \in L^1(\Omega)$ such that

$$\Phi(x, 2t) \leq k\Phi(x, t) + h(x) \text{ for a.e. } x \in \Omega \text{ and all } t \in \mathbb{R}.$$

Let Φ_1, Φ_2 be two generalized N-functions. We say that Φ_1 *dominates* Φ_2, denoted $\Phi_1 \succeq \Phi_2$, if there exist two constants $K, L > 0$ and a nonnegative function $h \in L^1(\Omega)$ such that

$$\Phi_2(x, t) \leq K\Phi_1(x, Lt) + h(x), \text{ for a.a. } x \in \Omega \text{ and all } t \in \mathbb{R}.$$

The functions Φ_1, Φ_2 are called *equivalent*, denoted $\Phi_1 \simeq \Phi_2$, if $\Phi_1 \succeq \Phi_2$ and $\Phi_2 \succeq \Phi_1$.

For a generalized N-function Φ, the *modular* $\varrho_\Phi : L^0(\Omega) \to \mathbb{R}$ is the functional given by

$$\varrho_\Phi(u) := \int_\Omega \Phi(x, |u|)\, dx,$$

where by $L^0(\Omega)$ we denote the set of measurable functions defined on Ω. We consider the following classes of functions:

(i). The *Musielak-Orlicz class* $K^\Phi(\Omega)$ defined by

$$K^\Phi(\Omega) := \left\{u \in L^0(\Omega) : \ \varrho_\Phi(u) < \infty\right\};$$

(ii). The *Musielak-Orlicz space* $L^\Phi(\Omega)$ is the linear space generated by $K^\Phi(\Omega)$. Note that $K^\Phi(\Omega) \subseteq L^\Phi(\Omega)$ and equality occurs if and only if $K^\Phi(\Omega)$ is a linear space, or equivalently Φ satisfies the Δ_2-condition.

The mapping $\|\cdot\|_\Phi : L^\Phi(\Omega \to [0,\infty)$ defined by

$$\|u\|_\Phi := \inf\left\{\beta > 0 : \ \varrho_\Phi\left(\frac{u}{\beta}\right) \leq 1\right\}$$

defines a norm (the so-called *Luxemburg norm*).

The following proposition highlights some useful properties of the Musielak-Orlicz spaces.

Proposition 2.4. *Let Φ, Ψ be two generalized N-functions. Then the following assertions hold:*

(i). *The Musielak-Orlicz space $(L^\Phi(\Omega), \|\cdot\|_\Phi)$ is a Banach space;*

(ii). *If $\Phi \succeq \Psi$, then $L^\Phi(\Omega) \hookrightarrow L^\Psi(\Omega)$;*

(iii). $\varrho_\Phi(u) < 1$*(resp.* $\varrho_\Phi(u) = 1; \varrho_\Phi(u) > 1)$ *if and only if* $\|u\|_\Phi < 1$ *(resp.* $\|u\|_\Phi = 1; \|u\|_\Phi > 1)$*;*

(iv). *The following Hölder-type inequality holds*

$$\int_\Omega |uv|\, dx \leq 2\|u\|_\Phi \|v\|_{\tilde{\Phi}}, \ \textit{for all } u \in L^\Phi(\Omega), v \in L^{\tilde{\Phi}}(\Omega).$$

For a generalized N-function Φ the corresponding *Musielak-Orlicz-Sobolev space* $W^{1,\Phi}(\Omega)$ is defined by

$$W^{1,\Phi}(\Omega) := \left\{u \in L^\Phi(\Omega) : \ |\nabla u| \in L^\Phi(\Omega)\right\}.$$

By a slight abuse, henceforth we denote $\|\nabla u\|_\Phi$ instead of $\||\nabla u|\|_\Phi$. Obviously the mapping $\|\cdot\|_{1,\Phi} : W^{1,\Phi}(\Omega) \to [0,\infty)$

$$\|u\|_{1,\Phi} := \|u\|_\Phi + \|\nabla u\|_\Phi$$

defines a norm.

The Musielak-Orlicz-Sobolev space $W_0^{1,\Phi}(\Omega)$ is defined as completion of $C_0^\infty(\Omega)$ in $W^{1,\Phi}(\Omega)$ w.r.t. the norm $\|\cdot\|_{1,\Phi}$.

Proposition 2.5 (Musielak [10]). *Assume Φ is a generalized N-function such that*

$$\inf_{x\in\Omega} \Phi(x,1) > 0. \tag{2.1}$$

Then $(W^{1,\Phi}(\Omega), \|\cdot\|_{1,\Phi})$ and $(W_0^{1,\Phi}(\Omega), \|\cdot\|_{1,\Phi})$ are Banach spaces. Furthermore, if $L^\Phi(\Omega)$ is reflexive, then $W^{1,\Phi}(\Omega)$ and $W_0^{1,\Phi}(\Omega)$ are also reflexive.

2.2. The double phase space

Throughout this section we consider the particular case of the double-phase space, required to study problem (P). Note that if (H_1) holds, then the *double phase* function $\mathcal{H} : \Omega \times \mathbb{R} \to [0,\infty)$ given by

$$\mathcal{H}(x,t) := |t|^p + \mu(x)|t|^q$$

is a generalized N-function satisfying (2.1). Simple computations yield

$$\mathcal{H}(x,2t) \leq 2^q \mathcal{H}(x,t), \ \text{for a.a. } x \in \Omega \text{ and all } t \in \mathbb{R},$$

i.e., $\mathcal{H}$ satisfies the Δ_2-condition. Moreover, according to Colasuonno & Squassina [4, Proposition 2.14] the space $(L^{\mathcal{H}}(\Omega), \|\cdot\|_{\mathcal{H}})$ is uniformly convex. Consequently, Proposition 2.5 ensures that $W^{1,\mathcal{H}}(\Omega)$ and $W_0^{1,\mathcal{H}}(\Omega)$ are reflexive. Moreover, if (H_1) holds, then the following Poincaré-type inequality holds

$$\|u\|_{\mathcal{H}} \leq C\|\nabla u\|_{\mathcal{H}}, \text{ for all } u \in W_0^{1,\mathcal{H}}(\Omega),$$

for some positive constant C independent of u. Thus, on the space $W_0^{1,\mathcal{H}}(\Omega)$ we can use the equivalent norm

$$\|u\| := \|\nabla u\|_{\mathcal{H}}.$$

We introduce next the space

$$L^q_\mu(\Omega) := \left\{ u \in L^0(\Omega) : \int_\Omega \mu(x)|u|^q \, dx < \infty \right\},$$

endowed with the seminorm

$$|u|_{q,\mu} := \left(\int_\Omega \mu(x)|u|^q \, dx \right)^{1/q}.$$

The definition of the Luxemburg norm together with the fact that $\mathcal{H}(x,t)$ is a generalized N-function which satisfies the Δ_2-condition. So, the following estimates hold:

$$\|u\|^q_{\mathcal{H}} \leq \int_\Omega [|u|^p + \mu(x)|u|^q]dx \leq \|u\|^p_{\mathcal{H}}, \ \forall u \in L^{\mathcal{H}}(\Omega), \text{ with } \|u\|_{\mathcal{H}} < 1, \tag{2.2}$$

and

$$\|u\|^p_{\mathcal{H}} \leq \int_\Omega [|u|^p + \mu(x)|u|^q]dx \leq \|u\|^q_{\mathcal{H}}, \ \forall u \in L^{\mathcal{H}}(\Omega), \text{ with } \|u\|_{\mathcal{H}} > 1. \tag{2.3}$$

The following proposition highlights some embedding results that will play a crucial role throughout the subsequent sections.

Proposition 2.6 (Colasuonno & Squassina [4]). *Assume (H_1) holds. The following statements are true:*

- (i). *The embedding $L^{\mathcal{H}}(\Omega) \hookrightarrow L^p(\Omega) \cap L^q_\mu(\Omega)$ is continuous;*
- (ii). *If $\mu \in L^\infty(\Omega)$, then the embedding $L^q(\Omega) \hookrightarrow L^{\mathcal{H}}(\Omega)$ is continuous;*
- (iii). *If $p \leq N$, then the embedding $W_0^{1,\mathcal{H}}(\Omega) \hookrightarrow L^r(\Omega)$ is compact for all $r \in (1, p^*)$;*
- (iv). *If $p > N$, then the embedding $W_0^{1,\mathcal{H}}(\Omega) \hookrightarrow L^r(\Omega)$ is compact for all $r \in (1, +\infty)$.*

2.3. Locally Lipschitz functionals

We recall that a functional $\phi : X \to \mathbb{R}$, with X being a Banach space, is said to be *locally Lipschitz* if, for every $u \in X$ there exists a neighborhood V of u and a positive constant L, which depends on the neighborhood V, such that

$$|\phi(w) - \phi(v)| \leq L\|w - v\|, \ \forall v, w \in V.$$

Definition 2.7. Let $\phi : X \to \mathbb{R}$ be a locally Lipschitz function. The *generalized directional derivative* of ϕ at $u \in X$ in the direction $v \in X$, denoted $\phi^0(u;v)$, is defined by

$$\phi^0(u;v) := \limsup_{\substack{w\to u\\ t\downarrow 0}} \frac{\phi(w+tv)-\phi(w)}{t}. \tag{2.4}$$

The following result points out some important properties of generalized directional derivatives that will be used in the sequel. For the proof one can consult Clarke [3].

Proposition 2.8. *Let $\phi, \rho : X \to \mathbb{R}$ be two locally Lipschitz functions. Then we have*

(*i*). *for each fixed $u \in X$, the function $v \mapsto \phi^0(u;v)$ is finite, subadditive and satisfies*
$$|\phi^0(u;v)| \le L\|v\|,$$
where $L > 0$ is the Lipschitz constant near the point u;

(*ii*). *the function $(u,v) \mapsto \phi^0(u;v)$ is upper semicontinuous;*

(*iii*). *$\phi^0(u;-v) = (-\phi)^0(u;v)$, for all $u, v \in X$;*

(*iv*). *$\phi^0(u;\mu v) = \mu\phi^0(u;v)$, for all $u, v \in X$ and all $\mu > 0$;*

(*v*). *$(\phi+\rho)^0(u;v) \le \phi^0(u;v) + \rho^0(u;v)$, for all $u, v \in X$.*

Definition 2.9. The *Clarke subdifferential* of a locally Lipschitz function $\phi : X \to \mathbb{R}$ at a point $u \in X$, denoted $\partial_C\phi(u)$, is the subset of X^* defined by

$$\partial_C\phi(u) := \left\{\xi \in X^* :\ \phi^0(u;v) \ge \langle \xi, v\rangle,\ \forall v \in X\right\}. \tag{2.5}$$

We point out the fact that if ϕ is convex, then the Clarke subdifferential $\partial_C\phi$ coincides with the subdifferential of ϕ in the sense of Convex Analysis. Although is no longer monotone, for each $u \in X$ the generalized gradient $\partial_C\phi(u)$ is a nonempty, convex and weak*-compact subset of X^* (see, e.g., Clarke [3, Proposition 2.1.2]). Furthermore, if $\phi \in C^1(X,\mathbb{R})$, then $\partial_C\phi(u) = \{\phi'(u)\}$.

Theorem 2.10 (Lebourg's Mean Value Theorem [8]**).** *Let U be an open subset of a Banach space X and u, v be two points of U such that the line segment*

$$[u,v] := \{(1-t)u + tv : 0 \le t \le 1\} \subset U.$$

If $\phi : U \to \mathbb{R}$ is a locally Lipschitz function, then there exist $t \in (0,1)$ and $\zeta \in \partial_C\phi(u + t(v-u))$ such that

$$\phi(v) - \phi(u) = \langle \zeta, v-u\rangle.$$

Definition 2.11. We say that $u \in X$ is a *critical point* for the locally Lipschitz functional $\phi : X \to \mathbb{R}$ if $0 \in \partial_C\phi(u)$.

Remark 2.12. The point $u \in X$ is critical for ϕ if and only if $\phi^0(u;v) \ge 0,\ \forall v \in X$. Furthermore, any local extremum of ϕ is in fact a critical point.

We close this subsection by recalling the well-known Ekeland variational principle (see, e.g., [6]) which will play a key role in the proof of Theorem 1.2.

Theorem 2.13. *Let (Y,d) be a complete metric space and let $\varphi : Y \to (-\infty,\infty]$ be a proper, lower semicontinuous and bounded from below functional. Then for any $\varepsilon, \lambda > 0$ and any $v \in Y$ satisfying $\varphi(v) \le \inf_Y \varphi + \varepsilon$ there exists $u \in Y$ such that:*

(i). $\varphi(u) \leq \varphi(v)$;
(ii). $d(v,u) \leq \frac{1}{\lambda}$;
(iii). $-\varepsilon\lambda d(u,w) \leq \varphi(w)-\varphi(u)$, *for all* $w \in Y$.

3. Proof of the main results

Define the functionals $I : W_0^{1,\mathcal{H}}(\Omega) \to \mathbb{R}$ and $F : L^r(\Omega) \to \mathbb{R}$ by

$$I(u) := \int_\Omega \left[\frac{1}{p}|\nabla u|^p + \frac{\mu(x)}{q}|\nabla u|^q\right] dx \text{ for all } u \in W_0^{1,\mathcal{H}}(\Omega)$$

and

$$F(w) := \int_\Omega f(x,w)dx \text{ for all } w \in L^r(\Omega)$$

respectively. Then $I \in C^1(W_0^{1,\mathcal{H}}(\Omega),\mathbb{R})$ with its derivative given by

$$\langle I'(u),v\rangle = \int_\Omega \left[|\nabla u|^{p-2}\nabla u\cdot\nabla v + \mu(x)|\nabla u|^{q-2}\nabla u\cdot\nabla v\right] dx, \tag{3.1}$$

for all $v \in W_0^{1,\mathcal{H}}(\Omega)$.

Due to the Aubin-Clarke Theorem (see, e.g., [3, Theorem 2.7.5]) F is locally Lipschitz and

$$\partial_C F(w) \subseteq \int_\Omega \partial_C^2 f(x,w)dx, \forall w \in L^r(\Omega),$$

in the sense that for any $\xi \in \partial_C F(w)$ there exists $\zeta \in L^{\frac{r}{r-1}}(\Omega)$ such that

$$\begin{cases} \langle \xi, z\rangle = \displaystyle\int_\Omega \zeta(x)z(x)dx, & \forall z \in L^r(\Omega), \\ \zeta(x) \in \partial_C^2 f(x,w(x)), & \text{for a.a. } x \in \Omega. \end{cases} \tag{3.2}$$

On the other hand, Proposition 2.6 ensures that for any $r \in (1,p^*)$ the embedding operator $i : W_0^{1,\mathcal{H}}(\Omega) \to L^r(\Omega)$ is compact and its adjoint operator, $i^* : L^{\frac{r}{r-1}}(\Omega) \to (W_0^{1,\mathcal{H}}(\Omega))^*$, is also compact. Consequently, the *energy functional* associated to problem (P), $E : W_0^{1,\mathcal{H}}(\Omega) \to \mathbb{R}$

$$E(u) := I(u) - F(i(u)), \tag{3.3}$$

is well defined, weakly lower semicontinuous and locally Lipschitz. Moreover, basic subdifferential calculus (see, e.g., Carl, Le & Motreanu [1, Propositions 2.173, 2.174 & Corollary 2.180] ensures that

$$\partial_C E(u) \subseteq I'(u) - i^*\partial_C F(i(u)),\ \forall u \in W_0^{1,\mathcal{H}}(\Omega).$$

Henceforth, for any $u \in W_0^{1,\mathcal{H}}(\Omega)$ and any $\zeta \in L^{\frac{r}{r-1}}(\Omega)$ we simply write u and ζ instead of $i(u)$ and $i^*(\zeta)$, respectivelly.

Lemma 3.1. *If* $r \in (1,p^*)$, *then any critical point of* E (*in the sense of Definition* 2.11) *is a weak solution for problem* (P).

Proof. Let $u \in W_0^{1,\mathcal{H}}(\Omega)$ be a critical point of E. Then $0 \in \partial_C E(u)$, or equivalently

$$I'(u) \in \partial_C F(u).$$

Keeping in mind (3.2), there exists $\zeta \in L^{\frac{r}{r-1}}(\Omega)$ such that $\zeta \in \partial_C^2 f(x,u)$ a.a. in Ω and

$$\langle I'(u), v\rangle = \int_\Omega \zeta v dx, \ \forall v \in W_0^{1,\mathcal{H}}(\Omega).$$

Now, using the definition of the Clarke subdifferential and (3.1) we get that

$$\int_\Omega \left[|\nabla u|^{p-2}\nabla u \cdot \nabla v + \mu(x)|\nabla u|^{q-2}\nabla u \cdot \nabla v\right] dx \le \int_\Omega f^0(x,u;v)dx,$$

for all $v \in W_0^{1,\mathcal{H}}(\Omega)$, i.e., u is indeed a weak solution for problem (P). $\square$

Proof of Theorem 1.1. Let $u \in W_0^{1,\mathcal{H}}(\Omega)$ be fixed. Simple computations show that

$$\frac{1}{q}\varrho_{\mathcal{H}}(|\nabla u|) \le I(u) \le \frac{1}{p}\varrho_{\mathcal{H}}(|\nabla u|),$$

which combined with (2.2)-(2.3) leads to the following inequalities:

$$\frac{1}{q}\|u\|^q \le I(u) \le \frac{1}{p}\|u\|^p, \quad \text{if } \|u\| < 1, \tag{3.4}$$

and

$$\frac{1}{q}\|u\|^p \le I(u) \le \frac{1}{p}\|u\|^q, \quad \text{if } \|u\| > 1, \tag{3.5}$$

respectivelly. On the other hand, using Lebourg's Mean Value Theorem and the compact embedding $W_0^{1,\mathcal{H}}(\Omega) \hookrightarrow L^r(\Omega)$ we get

$$\begin{aligned} |F(u)| &= \left|\int_\Omega f(x,u)dx\right| \le \int_\Omega |f(x,u) - f(x,0)|dx \le \int_\Omega |\zeta||u|dx \\ &\le \int_\Omega (\alpha(x) + k|u|^{r-1})|u|dx \le \|\alpha\|_{\frac{r}{r-1}}\|u\|_r - k\|u\|_r^r \\ &\le C_0\|\alpha\|_{\frac{r}{r-1}}\|u\| + C_1\|u\|^r, \end{aligned}$$

for some suitable constants $C_0, C_1 > 0$. Thus, for any $u \in W_0^{1,\mathcal{H}}(\Omega)$ with $\|u\| > 1$ one has

$$E(u) = I(u) - F(u) \ge \frac{1}{q}\|u\|^p - C_0\|\alpha\|_{\frac{r}{r-1}}\|u\| - C_1\|u\|^r \to \infty, \text{ as } \|u\| \to \infty,$$

i.e., the energy functional E is coercive. Since E is also weakly lower semicontinuous, The Direct Method in the Calculus of Variations (see, e.g., [5, Theorem 1.7]) ensures the existence of a global minimizer u_0 of E, i.e.,

$$E(u_0) = \inf_{u \in W_0^{1,\mathcal{H}}(\Omega)} E(u).$$

The conclusion follows now from Remark 2.12 and Lemma 3.1. $\square$

Lemma 3.2. *Assume* (H_1), (f_1) *and* (f_2') *hold. If* $r \in (q, p^*)$, *then there exist* $\rho \in (0,1)$ *and* $\gamma > 0$ *such that*

$$\inf_{u \in \partial B_\rho(0)} E(u) \geq \gamma,$$

where $\partial B_\rho(0) := \{u \in W_0^{1,\mathcal{H}}(\Omega) : \ \|u\| = \rho\}$.

Proof. Since (f_2') is in fact condition (f_2) with $\alpha \equiv 0$, it follows that there exists $C_1 > 0$ such that

$$|F(u)| \leq C_1 \|u\|^r, \ \forall u \in W_0^{1,\mathcal{H}}(\Omega).$$

Thus, for fixed $\rho \in \left(0, \min\left\{1, (qC_1)^{\frac{1}{q-r}}\right\}\right)$ and any $u \in \partial B_\rho(0)$ one has

$$E(u) \geq \frac{1}{q}\|u\|^q - C_1\|u\|^r = \frac{1}{q}\rho^q\left(1 - qC_1\rho^{r-q}\right).$$

The choice of ρ implies that $\gamma := \frac{1}{q}\rho^q\left(1 - qC_1\rho^{r-q}\right) > 0$, thus completing the proof. □

Lemma 3.3. *Assume* (H_1), (f_1) *and* (f_3) *hold. If* $r \in (q, p^*)$, *then there exist* $w_0 \in W_0^{1,\mathcal{H}}(\Omega) \setminus \{0\}$ *and* $t_0 \in (0,1)$ *such that*

$$E(tw_0) < 0, \ \forall t \in (0, t_0).$$

Proof. Let $x_0 \in \omega$ be fixed and choose $R > 0$ such that $\bar{B}_R(x_0) \subset \omega$. Then there exists $w_0 \in C_0^\infty(\omega)$ such that

$$\begin{cases} w_0(x) = 1, & \text{in } B_R(x_0), \\ 0 \leq w_0(x) \leq 1, & \text{on } \omega \setminus \bar{B}_R(x_0). \end{cases}$$

Obviously $w_0 \in W_0^{1,\mathcal{H}}(\Omega)$ and $\|w_0\| > 0$. Then, for any $0 < t < \min\{1, \delta, \|w_0\|^{-1}\}$ the following estimates hold

$$F(tw_0) = \int_\Omega f(x, tw_0(x))dx = \int_\omega f(x, tw_0(x))dx \geq \int_\omega K t^s dx = K\text{meas}(\omega)t^s,$$

and

$$\begin{aligned} E(tw_0) &= I(tw_0) - F(tw_0) \leq \frac{1}{p}\|tw_0\|^p - K\text{meas}(\omega)t^s \\ &= K\text{meas}(\omega)t^s\left[\frac{\|w_0\|^p t^{p-s}}{pK\text{meas}(\omega)} - 1\right], \end{aligned}$$

which shows $E(tw_0) < 0$ for all $t \in (0, t_0)$ with

$$t_0 := \min\left\{1, \delta, \|w_0\|^{-1}, \left(\frac{pK\text{meas}(\omega)}{\|w_0\|^p}\right)^{\frac{1}{p-s}}\right\}.$$

□

Proof of Theorem 1.2. Lemmas 3.2 and 3.3 ensure that there exists $\rho \in (0,1)$ such that

$$\inf_{\bar{B}_\rho(0)} E < 0 < \inf_{\partial B_\rho(0)} E.$$

Let $\{w_n\} \subset \bar{B}_\rho(u)$ be a minimizing sequence for $E|_{\bar{B}_\rho(0)}$, i.e., $E(w_n) \to \inf_{\bar{B}_\rho(0)} E,$ as $n \to \infty$. Passing, if necessary, to a subsequence we may assume that

$$E(w_n) < \inf_{\bar{B}_\rho(0)} E + \frac{1}{n}, \ \forall n \geq 1. \tag{3.6}$$

Applying Ekeland's variational principle with $\varepsilon := \frac{1}{n}$ and $\lambda := \sqrt{n}$ we get that there exists $\{u_n\} \subset \bar{B}_\rho(0)$ such that

$$E(u_n) \leq E(w_n), \ \forall n \geq 1, \tag{3.7}$$

and

$$-\frac{1}{\sqrt{n}}\|v - u_n\| \leq E(v) - E(u_n), \ \forall v \in \bar{B}_\rho(0). \tag{3.8}$$

The sequence $\{u_n\}$ is clearly bounded, hence there exists $u \in \bar{B}_\rho(0)$ and a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ such that

$$u_{n_k} \rightharpoonup u \text{ in } W_0^{1,\mathcal{H}}(\Omega) \quad \text{and} \quad u_{n_k} \to u \text{ in } L^r(\Omega).$$

For any $t \in (0,1)$ the element $v_t := u_{n_k} + t(u - u_{n_k})$ lies in $\bar{B}_\rho(0)$ and using (3.8) we have

$$-\frac{t}{\sqrt{n}}\|u - u_{n_k}\| \leq E(u_{n_k} + t(u - u_{n_k})) - E(u_{n_k}).$$

Dividing the last relation by $t > 0$ then taking the lim sup as $t \searrow 0$ we obtain

$$\begin{aligned} -\frac{1}{\sqrt{n}} \leq & \limsup_{t \searrow 0} \left[\frac{I(u_{n_k} + t(u - u_{n_k})) - I(u_{n_k})}{t} \right. \\ & \left. + \frac{(-F)(u_{n_k} + t(u - u_{n_k})) - (-F)(u_{n_k})}{t} \right] \\ \leq & \langle I'(u_{n_k}), u - u_{n_k} \rangle + (-F)^0(u_{n_k}; u - u_{n_k}), \end{aligned}$$

which can be rewritten as

$$\langle I'(u_{n_k}), u_{n_k} - u \rangle \leq \frac{1}{\sqrt{n}} + F^0(u_{n_k}; u_{n_k} - u), \ \forall n \geq 1.$$

Taking the lim sup as $n \to \infty$ and using Proposition 2.8 we have

$$\limsup_{n \to \infty} \langle I'(u_{n_k}, u_{n_k} - u) \rangle \leq F^0(u; 0) = 0.$$

Keeping in mind that I' of type $(S)_+$ (see, e.g., [9, Proposition 3.1]) we infer that

$$u_{n_k} \to u \text{ in } W_0^{1,\mathcal{H}}(\Omega).$$

But, due to (3.6) and (3.7), we conclude

$$E(u) = \lim_{n \to \infty} E(u_{n_k}) = \inf_{\bar{B}_\rho(0)} E < 0,$$

which shows that u is a nonzero local minimizer of E, and, according to Remark 2.12 a nontrivial critical point. □

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