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On a singular elliptic problem with variable exponent

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Dedicated to the memory of Professor Csaba Varga

Abstract. In the present note we study a semilinear elliptic Dirichlet problem involving a singular term with variable exponent of the following type

$$\begin{cases} -\Delta u = \frac{f(x)}{u^{\gamma(x)}}, & \text{in } \Omega \\ u > 0, & \text{in } \Omega \\ u = 0, & \text{on } \partial\Omega \end{cases} \tag{$\mathcal{P}$}$$

Existence and uniqueness results are proved when $f \geq 0$.

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1. Introduction

In the present note we consider the following semilinear singular elliptic problem

$$\begin{cases} -\Delta u = \frac{f(x)}{u^{\gamma(x)}}, & \text{in } \Omega \\ u > 0, & \text{in } \Omega \\ u = 0, & \text{on } \partial\Omega \end{cases} \tag{$\mathcal{P}$}$$

where Ω is a bounded domain of $\mathbb{R}^N$ $(N > 2)$ with smooth boundary, $f \in L^p(\Omega)$ $(p > \frac{N}{2})$ is a nonnegative function and $\gamma \in C^1(\overline{\Omega})$ is positive. Singular nonlinear problems were introduced by Fulks and Maybee [10] as a mathematical model for describing the heat conduction in an electric medium and received a considerable attention after the seminal paper of Crandall, Rabinowitz and Tartar [8]. There is a wide literature dealing with singular term of the type $u^{-\gamma}$ (i.e. $\gamma(x) = const.$) when $0 < \gamma < 1$. In such a case one can associate to the problem an energy functional which, although not continuously Gâteaux differentiable, is strictly convex. Its global minimum turns out to be the unique (weak) solution of $(\mathcal{P})$ and variational methods

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apply (see for example [9, 11, 17] where the singular term is perturbed by suitable nonlinearities). When $\gamma \geq 1$ such kind of problems are less investigated. Notice in fact that the energy functional (when $\gamma > 1$) in general is not defined on the whole space $H_0^1(\Omega)$. However, one may still prove existence results in the framework of variational setting by constructing suitable approximation sequences or employing techniques from non smooth analysis (see for instance [3, 4, 5, 6, 13, 15, 16]).

As far as we know, the variable exponent case has been treated recently in [7]. Using Schauder's fixed point theorem, the authors prove the existence of an increasing sequence of solutions of non-singular approximating problems which converges to a weak solution of $(\mathcal{P})$ in the natural energy space $H_0^1(\Omega)$ or to a function of $H^1_{loc}(\Omega)$ according to the behaviour of γ on the boundary of Ω.

In the present note we will complete the result of [7] showing the uniqueness of the solution of $(\mathcal{P})$. For general variable exponent we don't expect to have solutions in $H_0^1(\Omega)$ (notice that in [2], where the uniqueness issue is addressed, the authors assume the solutions to be in $H_0^1(\Omega)$). As in [4], a weak solution is meant in the following sense:

Definition 1.1. A weak solution of $(\mathcal{P})$ is a function $u \in H^1_{loc}(\Omega)$ such that $u > 0$ in Ω, $(u-\varepsilon)^+ \in H_0^1(\Omega)$ for every $\varepsilon > 0$,

$$\frac{f(x)}{u^{\gamma(x)}} \in L^1_{loc}(\Omega),$$

and

$$\int_\Omega \nabla u \nabla \varphi \, dx = \int_\Omega \frac{f(x)}{u^{\gamma(x)}} \varphi \, dx \quad \text{for all } \varphi \in C_c^1(\Omega).$$

Our result reads as follows:

Theorem 1.2. *Assume that $f \in L^p(\Omega)$ $(p > \frac{N}{2})$ is a nonnegative function and $\gamma \in C^1(\overline{\Omega})$ is a positive function. Then, problem $(\mathcal{P})$ has a unique weak solution.*

2. Proof of Theorem 1.2

Existence of solution of $(\mathcal{P})$. The existence of a solution has been already proved in [7]. We propose here a slightly different approach which is purely variational and does not make use of the Schauder fixed point theorem. Denote by $g : \Omega \times (0, +\infty) \to \mathbb{R}$ and $g_n : \Omega \times \mathbb{R} \to \mathbb{R}$ the functions

$$g(x,t) = \frac{f(x)}{t^{\gamma(x)}}, \qquad \text{and}$$

$$g_n(x,t) = g(x, t^+ + \frac{1}{n}) \qquad \text{for every } n \in \mathbb{N}^+.$$

For every $n \in \mathbb{N}^+$, g_n is a Carathéodory function and if $G_n : \Omega \times \mathbb{R} \to \mathbb{R}$ is its primitive, i.e.

$$G_n(x,t) = \int_0^t g_n(x,s) ds,$$

the following inequalities hold:

$$0 < g_n(x,t) \le f(x) n^{\|\gamma\|_\infty}$$
$$|G_n(x,t)| \le f(x) n^{\|\gamma\|_\infty} |t|$$

Denote by $\mathscr{E}_n : H_0^1(\Omega) \to \mathbb{R}$ the functional

$$\mathscr{E}_n(u) = \frac{1}{2}\|u\|^2 - \int_\Omega G_n(x, u(x))$$

which is well defined, coercive, sequentially weakly lower semicontinuous. Let u_n be its global minimum.

Since the functional $\mathscr{E}_n$ is of class $C^1(H_0^1(\Omega))$ with derivative at u given by

$$\mathscr{E}_n'(u)(\varphi) = \int_\Omega \nabla u \nabla \varphi - \int_\Omega g_n(x,u)\varphi \qquad \text{for every } \varphi \in H_0^1(\Omega)$$

u_n turns out to be a weak solution of

$$\begin{cases} -\Delta u = g_n(x,u), & \text{in } \Omega \\ u = 0, & \text{on } \partial\Omega. \end{cases} \tag{$\mathcal{P}_n$}$$

Thus, in particular,

$$\int_\Omega \nabla u_n \nabla \varphi = \int_\Omega g_n(x,u_n)\varphi \qquad \text{for all } \varphi \in H_0^1(\Omega). \tag{2.1}$$

Testing the above equality with $\varphi = u_n^-$ we obtain at once that $u_n \ge 0$. By classical regularity results, $u_n \in C^{1,\alpha}(\overline{\Omega})$ for some $\alpha \in (0,1)$ and by the strong maximum principle, $u_n > 0$ in Ω. Moreover, since the function $g_n(x,\cdot)$ is decreasing, in a standard way one can prove that u_n is the unique solution to $(\mathcal{P}_n)$.

As in [8], let $n > m$ and denote by $w = u_n - u_m$. Then $w \in C_0^1(\overline{\Omega})$ and

$$-\Delta w = \frac{f(x)}{\left(u_n + \frac{1}{n}\right)^{\gamma(x)}} - \frac{f(x)}{\left(u_m + \frac{1}{m}\right)^{\gamma(x)}}.$$

Using $w^- \in H_0^1(\Omega)$ as test function in the above equality, we deduce that

$$-\|w^-\|^2 = \int_{\{x\in\Omega \,:\, u_n < u_m\}} \left(\frac{f(x)}{\left(u_n + \frac{1}{n}\right)^{\gamma(x)}} - \frac{f(x)}{\left(u_m + \frac{1}{m}\right)^{\gamma(x)}} \right) w^- \ge 0,$$

which implies $w^- = 0$, i.e. $u_n(x) \ge u_m(x)$ for every $x \in \overline{\Omega}$.

Put now $z = u_m + \frac{1}{m} - \left(u_n + \frac{1}{n}\right)$. Then, $z \in C^1(\overline{\Omega})$ and $z^- \in H_0^1(\Omega)$ (recall that $n > m$) so, using z^- as test function in

$$-\Delta z = \frac{f(x)}{\left(u_m + \frac{1}{m}\right)^{\gamma(x)}} - \frac{f(x)}{\left(u_n + \frac{1}{n}\right)^{\gamma(x)}},$$

we obtain

$$-\|z^-\|^2 = \int_{\{x\in\Omega \,:\, u_m + \frac{1}{m} < u_n + \frac{1}{n}\}} \left(\frac{f(x)}{\left(u_m + \frac{1}{m}\right)^{\gamma(x)}} - \frac{f(x)}{\left(u_n + \frac{1}{n}\right)^{\gamma(x)}} \right) z^- \ge 0,$$

which implies $z^- = 0$, i.e. $u_n(x) + \frac{1}{n} \le u_m(x) + \frac{1}{m}$ for every $x \in \overline{\Omega}$. In conclusion, if $n > m$ then

$$0 \le u_n(x) - u_m(x) \le \frac{1}{m} - \frac{1}{n} \text{ for all } x \in \overline{\Omega}.$$

Hence, there exists $u \in C^0(\overline{\Omega})$ such that $u_n \rightrightarrows u$ in $\overline{\Omega}$ and

$$u_n \le u \le u_n + \frac{1}{n} \text{ for every } n \in \mathbb{N}. \tag{2.2}$$

Let us prove that u is a solution of $(\mathcal{P})$. It is clear that $u > 0$ in Ω. Moreover if $K \subset \Omega$ is a compact set, then, for suitable constants $c_0, c_1, c_2 > 0$,

$$u(x) \ge c_0 \text{ for all } x \in K,$$

$$0 \le \frac{f(x)}{u(x)^{\gamma(x)}} \le c_1 f(x) \text{ and } 0 \le \frac{f(x)}{u_n(x)^{\gamma(x)}} \le c_2 f(x) \text{ for all } x \in K,$$

thus in particular, $\frac{f(x)}{u^{\gamma(x)}}$ is in $L^1_{\text{loc}}(\Omega)$.

Let δ be a positive number and denote

$$\Omega_\delta = \{x \in \Omega : d(x, \partial\Omega) < \delta\}.$$

We distinguish two cases. Assume that $\|\gamma\|_{L^\infty(\Omega_\delta)} \le 1$. Following [7], the sequence $\{u_n\}$ is bounded in $H^1_0(\Omega)$ (for completeness we give the details). For a suitable constant c we obtain

$$\begin{aligned}
\|u_n\|^2 &= \int_{\Omega_\delta} \frac{f(x)}{(u_n + \frac{1}{n})^{\gamma(x)}} u_n + \int_{\Omega\setminus\Omega_\delta} \frac{f(x)}{(u_n + \frac{1}{n})^{\gamma(x)}} u_n \\
&\le \int_{\Omega_\delta} f(x) u_n^{1-\gamma(x)} + c \int_{\Omega\setminus\Omega_\delta} f(x) u_n \\
&\le \int_\Omega f(x)(1 + (1+c)u_n) = \|f\|_1 + \mathcal{S}(1+c)\|f\|_{\frac{2N}{N+2}} \|u_n\|,
\end{aligned}$$

being $\mathcal{S}$ the embedding constant of $H^1_0(\Omega) \hookrightarrow L^{2^\star}(\Omega)$.

Thus, u turns out to be also the limit in the weak topology of $H^1_0(\Omega)$ of $\{u_n\}$. Being $u \in H^1_0(\Omega)$, for every $\varepsilon > 0$, $(u - \varepsilon)^+ \in H^1_0(\Omega)$. Let $\varphi \in C^1_c(\Omega)$ and denote by c_1 the positive constant such that $u_n \ge c_1$ on suppφ. Since

$$g_n(x, u_n(x))\varphi(x) \to \frac{f(x)}{u(x)^{\gamma(x)}}\varphi(x) \text{ for all } \ x \in \Omega$$

and

$$0 \le g_n(x, u_n(x))\varphi(x) \le \frac{f(x)}{c_1^{\gamma(x)}}\varphi(x) \in L^1(\Omega),$$

passing to the limit in

$$\int_\Omega \nabla u_n \nabla\varphi = \int_\Omega g_n(x, u_n)\varphi \text{ for all } n \in \mathbb{N}$$

we obtain

$$\int_\Omega \nabla u \nabla\varphi = \int_\Omega \frac{f(x)}{u(x)^{\gamma(x)}}\varphi,$$

as we claimed.

Otherwise, $\|\gamma\|_{L^\infty(\Omega_\delta)} > 1$. Set $\gamma^* = \|\gamma\|_{L^\infty(\Omega_\delta)}$. In this case we prove that $\left\{u_n^{\frac{\gamma^*+1}{2}}\right\}$ is bounded in $H_0^1(\Omega)$. Since

$$\int_\Omega \nabla u_n \nabla u_n^{\gamma^*} = \gamma^* \int_\Omega u_n^{\gamma^*-1}|\nabla u_n|^2 = \gamma^* \left(\frac{2}{\gamma^*+1}\right)^2 \int_\Omega \left|\nabla u_n^{\frac{\gamma^*+1}{2}}\right|^2,$$

using $u_n^{\gamma^*}$ as test function in (2.1), we obtain

$$\begin{aligned}
\int_\Omega |\nabla u_n^{\frac{\gamma^*+1}{2}}|^2 &= \frac{4\gamma^*}{(\gamma^*+1)^2} \int_\Omega g_n(x,u_n)u_n^{\gamma^*} \\
&\le \frac{4\gamma^*}{(\gamma^*+1)^2} \left(\int_{\Omega_\delta} f(x)u_n^{\gamma^*-\gamma(x)} + c_0 \int_{\Omega\setminus\Omega_\delta} f(x)u_n^{\gamma^*}\right) \\
&\le \frac{4\gamma^*}{(\gamma^*+1)^2} \left(\int_\Omega f(x)(1+(1+c_0)u_n^{\gamma^*})\right) \\
&= \frac{4\gamma^*}{(\gamma^*+1)^2}\|f\|_1 + \frac{4\gamma^*}{(\gamma^*+1)^2}(1+c_0)\int_\Omega f(x)u_n^{\gamma^*}.
\end{aligned}$$

By the assumption, $f \in L^{\frac{N(\gamma^*+1)}{N+2\gamma^*}}$ and applying Hölder inequality we obtain

$$\begin{aligned}
\int_\Omega f(x)u_n(x)^{\gamma^*} &\le \left(\int_\Omega f(x)^{\frac{N(\gamma^*+1)}{N+2\gamma^*}}\right)^{\frac{N+2\gamma^*}{N(\gamma^*+1)}} \left(\int_\Omega u_n(x)^{\frac{N(\gamma^*+1)}{N-2}}\right)^{\frac{N(\gamma^*+1)}{(N-2)\gamma^*}} \\
&\le \|f\|_{\frac{N(\gamma^*+1)}{N+2\gamma^*}} \|u_n^{\frac{\gamma^*+1}{2}}\|_{2^*}^{\frac{\gamma^*+1}{2\gamma^*}} \le \|f\|_{\frac{N(\gamma^*+1)}{N+2\gamma^*}}(1+\mathcal{S}\|u_n^{\frac{\gamma^*+1}{2}}\|).
\end{aligned}$$

Thus, for suitable constants one has

$$\|u_n^{\frac{\gamma^*+1}{2}}\|^2 \le c_1 + c_2\|u_n^{\frac{\gamma^*+1}{2}}\|,$$

that is our claim. Thus, $u^{\frac{\gamma^*+1}{2}} \in H_0^1(\Omega)$ and from [5, Theorem 1.3], it follows that $(u-\varepsilon)^+ \in H_0^1(\Omega)$ for every $\varepsilon > 0$.

Moreover, if $K \subset \Omega$ is a compact set, there exists a constant $c > 0$ such that $u_n^{\gamma^*-1} \ge c$ uniformly on K. Since

$$c\int_K |\nabla u_n|^2 \le \int_K u_n^{\gamma^*-1}|\nabla u_n|^2 = \frac{4}{(\gamma^*+1)^2}\int_K |\nabla u_n^{\frac{\gamma^*+1}{2}}|^2 \le const,$$

we deduce at once that $\{u_n\}$ is bounded in $H^1_{\mathrm{loc}}(\Omega)$, thus $u \in H^1_{\mathrm{loc}}(\Omega)$. We conclude as above.

Uniqueness of solution of $(\mathcal{P})$**.**

In order to prove the uniqueness of the solution we follow [6] and prove that inequality (2.2) holds for every solution u of $(\mathcal{P})$.

Let $u \in H^1_{\mathrm{loc}}(\Omega)$ be a solution of $(\mathcal{P})$, $n \in \mathbb{N}^+$ and u_n be the solution of $(\mathcal{P}_n)$. Let us prove that $u_n \le u \le u_n + \frac{1}{n}$. We first prove that $u \le u_n + \frac{1}{n}$.

Fix a sequence $\{\varphi_k\} \subset C_c^1(\Omega)$ converging in $H_0^1(\Omega)$ to $\left(u-u_n-\frac{1}{n}\right)^+$ and let $\tilde{\varphi}_k = \min\{\varphi_k, \left(u-u_n-\frac{1}{n}\right)^+\}$. Thus, $\{\tilde{\varphi}_k\} \subset C_c^1(\Omega)$ still converges in $H_0^1(\Omega)$ to $\left(u-u_n-\frac{1}{n}\right)^+$ and $\mathrm{supp}\tilde{\varphi}_k \subseteq \mathrm{supp}\left(u-u_n-\frac{1}{n}\right)^+ \subseteq \mathrm{supp}(u-\frac{1}{n})^+$. Then,

$$\int_\Omega \nabla u \nabla \tilde{\varphi}_k = \int_\Omega \frac{f(x)}{u^{\gamma(x)}}\tilde{\varphi}_k.$$

Since u is $H^1(\mathrm{supp}\left(u-\frac{1}{n}\right)^+)$, passing to the limit one has also that

$$\int_\Omega \nabla u \nabla \tilde{\varphi}_k \to \int_\Omega \nabla u \nabla \left(u-u_n-\frac{1}{n}\right)^+.$$

From the definition of $\tilde{\varphi}_k$ and Fatou lemma, one also has

$$\int_\Omega \frac{f(x)}{u^{\gamma(x)}}\tilde{\varphi}_k \to \int_\Omega \frac{f(x)}{u^{\gamma(x)}}\left(u-u_n-\frac{1}{n}\right)^+.$$

Combining the above outcomes,

$$\int_\Omega \nabla u \nabla \left(u-u_n-\frac{1}{n}\right)^+ = \int_\Omega \frac{f(x)}{u^{\gamma(x)}}\left(u-u_n-\frac{1}{n}\right)^+.$$

Since u_n is a solution of $(\mathcal{P}_n)$,

$$\int_\Omega \nabla u_n \nabla \left(u-u_n-\frac{1}{n}\right)^+ = \int_\Omega \frac{f(x)}{\left(u_n+\frac{1}{n}\right)^{\gamma(x)}}\left(u-u_n-\frac{1}{n}\right)^+,$$

and subtracting one has

$$\left\|\left(u-u_n-\frac{1}{n}\right)^+\right\|^2 = \int_\Omega f(x)\left(\frac{1}{u^{\gamma(x)}}-\frac{1}{\left(u_n+\frac{1}{n}\right)^{\gamma(x)}}\right)\left(u-u_n-\frac{1}{n}\right)^+ \le 0,$$

which implies the claim.

Let us prove now that $u \ge u_n$. Let $\varepsilon \le \frac{1}{n}$. Put $\psi_\varepsilon = (u_n-u-\varepsilon)^+$ for every $n \in \mathbb{N}$. Notice that ψ_ε has compact support since $u_n \le \varepsilon$ in a neighborhood of the boundary. Thus,

$$\int_\Omega \nabla u \nabla (u_n-u-\varepsilon)^+ = \int_\Omega \frac{f(x)}{u^{\gamma(x)}}(u_n-u-\varepsilon)^+,$$

and

$$\int_\Omega \nabla u_n \nabla (u_n-u-\varepsilon)^+ = \int_\Omega \frac{f(x)}{\left(u_n+\frac{1}{n}\right)^{\gamma(x)}}(u_n-u-\varepsilon)^+.$$

Subtracting,

$$\left\|(u_n-u-\varepsilon)^+\right\|^2 = \int_\Omega f(x)\left(\frac{1}{\left(u_n+\frac{1}{n}\right)^{\gamma(x)}}-\frac{1}{u^{\gamma(x)}}\right)(u_n-u-\varepsilon)^+ \le 0,$$

which implies $u_n \le u+\varepsilon$. Letting $\varepsilon \to 0$ we obtain the desired inequality.

The proof of uniqueness follows at once: let u, v be solutons of $(\mathcal{P})$. Then, for every $n \in \mathbb{N}^+$ one has

$$u \leq v + \frac{1}{n},$$

which implies, passing to the limit that $u \leq v$. Analogously we get the converse inequality. $\square$

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