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Bernstein polynomials iterative method for weakly singular and fractional Fredholm integral equations

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Abstract. A novel iterative method based on Picard iterations and Berstein polynomials is proposed for solving weakly singular and fractional Fredholm integral equations. On a uniform mesh, at each iterative step a Bernstein type spline is constructed by using the values computed at the previous step. The error estimates are obtained in terms of the Lipschitz constants and the convergence of the method is proved. Some numerical examples are presented in order to illustrate the accuracy of this iterative method.

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Introduction

The interest for fractional order differential and integral equations is motivated by the multiple applications of fractional calculus in fluid dynamics, viscoelasticity (see [7] and [38] for the Bagley-Torvik fractional differential model), heat transfer, diffusive transport, signal processing and various areas of engineering, economy, plasma physics, hematopoiesis, epidemiology, and in modeling of memory and hereditary properties of materials (see [12], [14], [15], [20], [30], [37]). According to the Scot Blair model the fractional order of a derivative is an index of memory (see [14]). A significant development in the field of fractional calculus, including fractional differential and integral equations, was realized in recent years and the results are presented in the monographs of Baleanu et al. (see [8]), Diethelm (see [12]), Kilbas et al. (see [20]),

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Lakshmikantham et al. (see [21]), Miller and Ross (see [29]), Muskhelishvili and Radok (see [31]), and Podlubny (see [33]). The numerical integration of fractional type integrals is usually realized by product integration and adapted quadrature rules (see [6]). Fractional integral equations are suitable models for several phenomena from physics and electro-chemistry such as crystal growth and heat transfer (see [17] and [39]). The corresponding fractional integral equations equivalent with various types of boundary value problems associated to nonlinear fractional differential equations with Caputo fractional derivative and existence results can be found in [1]. Usually, the existence and uniqueness of the solution for fractional integral equations is investigated by using the Banach fixed point theorem (see [1], [12], and [27]). Regularity properties of the solution of weakly singular and fractional Fredholm integral equations were obtained in [19] and [35].

In order to solve Volterra fractional integral equations, various numerical methods were proposed based on the following techniques: product integration and quadrature rules (see [6], [5], [27], [28]), collocation (see [9], [10], [13], and [44]), Runge-Kutta techniques (see [23]), Adams-Bashforth procedures (see [12]), Bernstein's approximation (see [39]), Haar, Legendre and Riesz wavelets (see [30] and [43]), variational iteration (see [40]). In the case of weakly singular and fractional Fredholm integral equations, the numerical solution is obtained by applying sinc, spectral and Haar wavelet collocation (see [3], [24], [32] and [41]), B-spline wavelets Galerkin technique (see [25]), product integration (see [2] and [36]), Taylor-series expansion (see [34]), hybrid collocation (see [11]), Galerkin and iterated Galerkin methods (see [18] and [26]).

In this paper, we approximate the solution of the following type Fredholm integral equation with singularities

$$x(t) = g(t) + \lambda \int_0^T b(t)\, |t-s|^{\alpha-1} f(s, x(s))\, ds,\ t \in [0,T] \tag{0.1}$$

where $\lambda > 0$, $\alpha \in (0,1)$ and $g, b : [0,T] \to \mathbb{R}$, $f : [0,T] \times \mathbb{R} \to \mathbb{R}$ are continuous with $b(t) \geq 0$, $\forall t \in [0,T]$. The choice $\lambda = \frac{1}{\Gamma(\alpha)}$ corresponds to the case of fractional integral equations, while $\lambda = 1$ usually describes weakly singular integral equations.

In the case $\lambda = \frac{1}{\Gamma(\alpha)}$ of fractional integral equations, we use the left-sided and right-sided Riemann-Liouville fractional integrals which are defined as follows.

Definition 0.1. (see [39]) Let $f : [0,T] \to \mathbb{R}$.The left-sided fractional integral of f of order $\alpha \in (0,1)$ is defined as

$$I_+^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s) ds, \quad \text{for } t > 0$$

where $\Gamma(\alpha) = \int\limits_0^\infty e^{-x}x^{\alpha-1}dx$, for $x > 0$. The right-sided fractional integral of f of order $\alpha \in (0,1)$ is

$$I_-^\alpha f(t) = \frac{1}{\Gamma(\alpha)}\int\limits_t^T (s-t)^{\alpha-1} f(s)ds, \quad \text{for } t < T.$$

Our method comes from the product integration technique and an iterative procedure is obtained based on piecewise Bernstein polynomials involved at each iterative step. More precisely, at each iterative step we construct a Bernstein spline based on the values computed in the previous step and the integral is approximated by using the Bernstein type quadrature formula. This method differs by the technique developed in [39] where the solution was directly approximated by Bernstein polynomials inserted in the two sides of the integral equation and the convergence analysis was based on Voronovskaia's type theorem. The product integration method firstly appears in 1954, in the work of Young (see [42]), and as it is specified in [16] the most used procedures are rectangular and trapezoidal schemes with the order of convergence $O\left(h^{\min(1+\alpha,2)}\right)$. For integral equations such as (0.1), our Bernstein splines method has the order of convergence $O\left(h^\alpha\right)$ as it is specified in Theorem 2.1.

The paper is organized as follows: in Section 1 we present some uniform boundedness and uniform Hölder type Lipschitz properties of the Picard iterations, including the description of the iterative algorithm for solving the integral equation (0.1). Section 2 is devoted to the convergence analysis of this iterative method. In order to confirm the obtained theoretical result and to illustrate the accuracy of the method, in Section 3 we present some numerical experimets. Finally, we point out some concluding remarks.

1. The properties of Picard's iterations and the iterative method

We see that in (0.1) the singularity appears inside the open interval $(0,T)$ which can be moved at extremeness by writing (0.1) as

$$x(t) = g(t) + \lambda\int\limits_0^t b\left(t\right)\left(t-s\right)^{\alpha-1} f\left(s, x(s)\right)ds + \lambda\int\limits_t^T b\left(t\right)\left(s-t\right)^{\alpha-1} f\left(s, x(s)\right)ds$$

and we consider the corresponding integral operator $A : C[0,T] \to C[0,T]$ that is well-defined according to [4],

$$\begin{aligned} A\left(x\right)\left(t\right) := g(t) + \lambda\int\limits_0^t b\left(t\right)\left(t-s\right)^{\alpha-1} f\left(s, x(s)\right)ds + \\ + \lambda\int\limits_t^T b\left(t\right)\left(s-t\right)^{\alpha-1} f\left(s, x(s)\right)ds. \end{aligned} \tag{1.1}$$

Concerning the existence and uniqueness of the solution and the properties of Picard's iterations we obtain the following result.

Theorem 1.1. *If* $g, b : [0, T] \to \mathbb{R}$, $f : [0, T] \times \mathbb{R} \to \mathbb{R}$ *are continuous,* $b(t) \geq 0$, $\forall t \in [0, T]$, $L \geq 0$ *is such that*

$$|f(s, u) - f(s, v)| \leq L|u - v|, \quad \forall s \in [0, T],\ u, v \in \mathbb{R} \tag{1.2}$$

and if $\theta = \frac{\lambda L M_b T^\alpha}{\alpha} < 1$, *then the integral equation (0.1) has unique solution* $x^* \in C[0, T]$ *where* $M_b \geq 0$, *with* $|b(t)| \leq M_b$, $\forall t \in [0, T]$, *and the sequence of Picard iterations given by* $x_0 = g$, $x_m = A(x_{m-1})$, $m \in \mathbb{N}^*$, *is uniformly bounded having* $\lim_{m\to\infty} x_m = x^*$ *in* $(C[0, T], \|\cdot\|_\infty)$ *and*

$$|x^*(t) - x_m(t)| \leq \frac{\theta^m \lambda M_b M_0 T^\alpha}{\alpha(1-\theta)}, \quad \forall t \in [0, T],\ m \in \mathbb{N}^*, \tag{1.3}$$

$$|x^*(t) - x_m(t)| \leq \frac{\theta}{1-\theta}|x_m(t) - x_{m-1}(t)|, \quad \forall t \in [0, T],\ m \in \mathbb{N}^*, \tag{1.4}$$

where $\|x\|_\infty = \max_{t\in[0,T]} |x(t)|$. *If in addition, there exist* $\beta, \gamma, \eta \geq 0$ *such that*

$$|g(t) - g(t')| \leq \eta|t - t'|, \quad |b(t) - b(t')| \leq \beta|t - t'|,\ \forall t, t' \in [0, T] \tag{1.5}$$

$$|f(s, u) - f(s', u)| \leq \gamma|s - s'|, \quad \forall s, s' \in [0, T],\ u \in \mathbb{R} \tag{1.6}$$

then the sequence $(x_m)_{m\in\mathbb{N}^*}$ *of Picard iterations is uniform Hölder type Lipschitz.*

Proof. Elementary calculus lead to

$$|A(x)(t) - A(y)(t)| \leq \frac{\lambda L M_b T^\alpha}{\alpha}\|x - y\|_\infty$$

for all $x, y \in C[0, T]$, $t \in [0, T]$ and according to Banach's fixed point principle the integral operator A has unique fixed point that is the unique solution $x^* \in C[0, T]$ of (0.1) with $\lim_{m\to\infty} x_m(t) = x^*(t)$ uniformly for $t \in [0, T]$ and the apriori and a posteriori error estimates (1.3) and (1.4) follows. For the Picard iterations

$$x_{m+1}(t) = g(t) + \lambda \int_0^t b(t)(t-s)^{\alpha-1} f(s, x_m(s))\, ds$$

$$+ \lambda \int_t^T b(t)(s-t)^{\alpha-1} f(s, x_m(s))\, ds \tag{1.7}$$

in inductive manner we get

$$|x_m(t) - x_{m-1}(t)| \leq \theta\|x_{m-1} - x_{m-2}\|_\infty \leq \dots \leq \theta^{m-1}\|x_1 - x_0\|_\infty$$

and thus,

$$|x_m(t)| \leq |x_m(t) - x_0(t)| + |x_0(t)| \leq \left(1 + \theta + \dots + \theta^{m-1}\right)\frac{\lambda M_b M_0 T^\alpha}{\alpha} + M_g$$

for all $t \in [0,T]$, $m \in \mathbb{N}^*$, where $M_0, M_g \geq 0$ are such that $|f(t,g(t))| \leq M_0$, $|g(t)| \leq M_g$, $\forall t \in [0,T]$. By denoting

$$R = M_g + \frac{\lambda M_b M_0 T^\alpha}{\alpha(1-\theta)}$$

we have $|x_m(t)| \leq R$ for all $t \in [0,T]$, $m \in \mathbb{N}^*$, that is the uniform boundedness of the sequence $(x_m)_{m\in\mathbb{N}^*}$ of Picard iterations. If we denote $F_m(t) = f(t, x_m(t))$ for $t \in [0,T]$ and $m \in \mathbb{N}$, and use the Lipschitz property it obtains,

$$\begin{aligned} |F_m(t)| &\leq |f(t,x_m(t)) - f(t,x_0(t))| + |f(t,x_0(t))| \\ &\leq \frac{\lambda L M_b M_0 T^\alpha}{\alpha(1-\theta)} + M_0 = M \end{aligned} \tag{1.8}$$

for all $t \in [0,T]$ and $m \in \mathbb{N}^*$, and thus the sequence $(F_m)_{m\in\mathbb{N}}$ is uniformly bounded, too.

Now, by considering arbitrary $t,t' \in [0,T]$, if $t \leq t'$ (the case $t' \leq t$ being approached similarly) we have $(t'-s)^{\alpha-1} \leq (t-s)^{\alpha-1}$ and consequently,

$$\begin{aligned} &\int_t^{t'} \left| b(t')(t'-s)^{\alpha-1} - b(t)(s-t)^{\alpha-1} \right| ds \\ &\leq \int_t^{t'} |b(t') - b(t)|(t'-s)^{\alpha-1} ds + |b(t)| \int_t^{t'} \left(\left|(t'-s)^{\alpha-1}\right| + \left|(s-t)^{\alpha-1}\right| \right) ds \\ &\leq \beta|t-t'| \cdot \frac{|t-t'|^\alpha}{\alpha} + \frac{2M_b|t-t'|^\alpha}{\alpha} \end{aligned}$$

obtaining,

$$\begin{aligned} &|x_m(t') - x_m(t)| \leq \eta|t-t'| \\ &+\lambda \int_0^t \left| b(t')(t'-s)^{\alpha-1} - b(t)(t-s)^{\alpha-1} \right| \cdot |f(s, x_{m-1}(s))| ds \\ &+\lambda \int_t^{t'} (\left| b(t')|t'-s|^{\alpha-1} - b(t)(s-t)^{\alpha-1} \right| \cdot |f(s, x_{m-1}(s))| ds \\ &+\lambda \int_{t'}^T \left| b(t')(s-t')^{\alpha-1} - b(t)(s-t)^{\alpha-1} \right| \cdot |f(s, x_{m-1}(s))| ds \\ &\leq (\eta + \frac{2\lambda M\beta T^\alpha}{\alpha})|t-t'| + \frac{\lambda M\beta}{\alpha}|t-t'|^{\alpha+1} + \frac{4\lambda M M_b}{\alpha}|t-t'|^\alpha, \quad \forall m \in \mathbb{N}^*, \end{aligned}$$

that is the uniform Hölder type Lipschitz property of the sequence $(x_m)_{m\in\mathbb{N}^*}$ of Picard iterations. Under the Lipschitz conditions (1.5) and (1.6) we have

$$|F_m(t) - F_m(t')| \leq \gamma|t-t'| + L|x_m(t') - x_m(t)| \leq \frac{\lambda L M\beta}{\alpha}|t-t'|^{\alpha+1}$$

$$+ [\gamma + L\left(\eta + \frac{2\lambda\beta M T^{\alpha}}{\alpha}\right)] \left|t - t'\right| + \frac{4\lambda L M M_b}{\alpha} \left|t - t'\right|^{\alpha}, \quad \forall m \in \mathbb{N}^* \tag{1.9}$$

for all $t, t' \in [0, T]$, that is the uniform Hölder type Lipschitz property of the sequence $(F_m)_{m\in\mathbb{N}}$. In that follows, we will denote

$$L_0 = \gamma + L\left(\eta + \frac{2\lambda\beta M T^{\alpha}}{\alpha}\right), \ L'' = \frac{\lambda L M \beta}{\alpha}, \text{ and } L' = \frac{4\lambda L M M_b}{\alpha}. \qquad \square$$

For the case of fractional integral equations with $\lambda = \frac{1}{\Gamma(\alpha)}$ the contraction condition becomes

$$\theta = \frac{L M_b T^{\alpha}}{\Gamma(\alpha+1)} < 1.$$

Our iterative method is based on approximating the Picard iterations (1.7) and for this purpose we consider a uniform mesh of $[0, T]$ with the knots $t_i = i \cdot h$, $i = \overline{0, n}$, where $h = \frac{T}{n}$ is the stepsize. On these knots the Picard iterations become

$$x_{m+1}(t_i) = g(t_i) + \lambda \int_0^{t_i} b(t_i)(t_i - s)^{\alpha-1} f(s, x_m(s))\, ds$$

$$+ \lambda \int_{t_i}^{T} b(t_i)(s - t_i)^{\alpha-1} f(s, x_m(s))\, ds, \quad i = \overline{0, n}, \ m \in \mathbb{N}^* \tag{1.10}$$

and on each subinterval $[t_{i-1}, t_i]$, $i = \overline{1, n}$, we approximate the continuous function F_m by the Bernstein polynomial of a given degree $q \geq 1$:

$$B_{m,i}(t) = \frac{1}{h^q} \sum_{k=0}^{q} C_q^k (t - t_{i-1})^k (t_i - t)^{q-k} \cdot F_m\left(t_{i-1} + \frac{kh}{q}\right), \ t \in [t_{i-1}, t_i] \tag{1.11}$$

where $C_q^k = \frac{q!}{k! \cdot (q-k)!}$, and in this way F_m will be approximated on $[0, T]$ by a Bernstein spline B_m for all $m \in \mathbb{N}^*$. For estimating the remainder in the Bernstein approximation formula $F_m(t) = B_{m,i}(t) + R_{m,i}(t)$, we use the inequality of Lorentz (see [22]) described in terms of the modulus of continuity,

$$|R_{m,i}(t)| \leq \frac{5}{4} \cdot \omega\left(F_m, \frac{h}{\sqrt{q}}\right), \ \forall t \in [t_{i-1}, t_i], \ \forall i = \overline{1, n}, \ m \in \mathbb{N}.$$

According to the uniform Hölder type Lipschitz property of the sequence $(F_m)_{m\in\mathbb{N}^*}$, this inequality becomes

$$|R_{m,i}(t)| \leq \frac{5}{4}\left(\frac{L_0 h}{\sqrt{q}} + \frac{L' h^{\alpha}}{(\sqrt{q})^{\alpha}} + \frac{L'' h^{\alpha+1}}{(\sqrt{q})^{\alpha+1}}\right), \ \forall t \in [t_{i-1}, t_i], \ \forall i = \overline{1, n} \tag{1.12}$$

for all $m \in \mathbb{N}$.

Based on (1.10) and (1.11) we obtain the following iterative algorithm:

Step 1: $x_0(t) = g(t)$, $\forall t \in [0, T]$ and for $k = \overline{0, n-1}$, $l = \overline{0, q-1}$ let

$$x_1\left(t_k + \frac{lh}{q}\right) = g\left(t_k + \frac{lh}{q}\right) + \lambda b\left(t_k + \frac{lh}{q}\right) \int_0^{T} \left|t_k + \frac{lh}{q} - s\right|^{\alpha-1} f(s, g(s))\, ds$$

$$= g\left(t_k + \frac{lh}{q}\right) + \lambda b\left(t_k + \frac{lh}{q}\right)\sum_{i=1}^{n}\int\limits_{t_{i-1}}^{t_i}\left|t_k + \frac{lh}{q} - s\right|^{\alpha-1}\left(B_{0,i}\left(s\right) + R_{0,i}\left(s\right)\right)ds$$

$$= g\left(t_k + \frac{lh}{q}\right) + \frac{\lambda}{h^q}b\left(t_k + \frac{lh}{q}\right)\sum_{i=1}^{n}\sum_{j=0}^{q}C_q^j\varphi_{k,l,j}\cdot$$

$$\cdot f\left(t_{i-1} + \frac{jh}{q}, g\left(t_{i-1} + \frac{jh}{q}\right)\right) + \overline{R_{1,(k,l)}} = \overline{x_1}\left(t_k + \frac{lh}{q}\right) + \overline{R_{1,(k,l)}} \tag{1.13}$$

and

$$x_1\left(t_n\right) = g\left(t_n\right) + \lambda b\left(t_n\right)\int\limits_0^T\left(t_n - s\right)^{\alpha-1}f\left(s, g(s)\right)ds = g\left(t_n\right) + \lambda b\left(t_n\right)\cdot$$

$$\cdot\sum_{i=1}^{n}\int\limits_{t_{i-1}}^{t_i}\left(t_n - s\right)^{\alpha-1}\left(B_{0,i}\left(s\right) + R_{0,i}\left(s\right)\right)ds = g\left(t_n\right) + \frac{\lambda}{h^q}b\left(t_n\right)\sum_{i=1}^{n}\sum_{j=0}^{q}C_q^j\varphi_{n,j}\cdot$$

$$\cdot f\left(t_{i-1} + \frac{jh}{q}, g\left(t_{i-1} + \frac{jh}{q}\right)\right) + \overline{R_{1,n}} = \overline{x_1}\left(t_n\right) + \overline{R_{1,n}} \tag{1.14}$$

where

$$\varphi_{k,l,j} = \int\limits_{t_{i-1}}^{t_i}\left|t_k + \frac{lh}{q} - s\right|^{\alpha-1}\left(s - t_{i-1}\right)^j\left(t_i - s\right)^{q-j}ds,$$

$$k = \overline{0, n-1},\ l = \overline{0, q-1} \tag{1.15}$$

$$\varphi_{n,j} = \int\limits_{t_{i-1}}^{t_i}\left(t_n - s\right)^{\alpha-1}\left(s - t_{i-1}\right)^j\left(t_i - s\right)^{q-j}ds, \quad j = \overline{0, q}. \tag{1.16}$$

In the computation of the integrals (1.15)-(1.16) we use the change of variable

$$s = t_{i-1} + uh$$

obtaining $\varphi_{k,l,j} = h^{q+\alpha}\varphi_{k,l,j}\left(i\right)$ and $\varphi_{n,j} = h^{q+\alpha}\varphi_{n,j}\left(i\right)$ with

$$\varphi_{k,l,j}\left(i\right) = \int\limits_0^1 u^j\left(1 - u\right)^{q-j}\left|k + \frac{l}{q} - \left(i - 1\right) - u\right|^{\alpha-1}du$$

and $\varphi_{n,j}\left(i\right) = \int\limits_0^1 u^j\left(1 - u\right)^{q-j}\left(n - i - u + 1\right)^{\alpha-1}du.$

Step 2: Construct the Bernstein splines B_1 and $\overline{B_1}$ given for $i = \overline{1, n}$ by

$$B_{1,i}\left(t\right) = \frac{1}{h^q}\sum_{k=0}^{q}C_q^k(t - t_{i-1})^k(t_i - t)^{q-k}\cdot F_1\left(t_{i-1} + \frac{kh}{q}\right),\ t \in [t_{i-1}, t_i] \tag{1.17}$$

and

$$\overline{B_{1,i}}\,(t) = \frac{1}{h^q}\sum_{j=0}^{q} C_q^j (t-t_{i-1})^j (t_i - t)^{q-j} \cdot f\left(t_{i-1} + \frac{jh}{q}, \overline{x_1}\left(t_{i-1} + \frac{jh}{q}\right)\right), \quad (1.18)$$

$t \in [t_{i-1}, t_i]$.

Step 3: In inductive way, for $m \geq 2$, with $k = \overline{0, n-1}$, $l = \overline{0, q-1}$ let

$$\begin{aligned} x_m\left(t_k + \frac{lh}{q}\right) &= g\left(t_k + \frac{lh}{q}\right) \\ &+ \lambda b\left(t_k + \frac{lh}{q}\right)\int_0^T \left|t_k + \frac{lh}{q} - s\right|^{\alpha-1} f\left(s, x_{m-1}\left(s\right)\right) ds = g\left(t_k + \frac{lh}{q}\right) \\ &+ \lambda b\left(t_k + \frac{lh}{q}\right)\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i} \left|t_k + \frac{lh}{q} - s\right|^{\alpha-1} \left(B_{m-1,i}\left(s\right) + R_{m-1,i}\left(s\right)\right) ds \\ &= g\left(t_k + \frac{lh}{q}\right) + \frac{\lambda}{h^q} b\left(t_k + \frac{lh}{q}\right)\sum_{i=1}^{n}\sum_{j=0}^{q} C_q^j \varphi_{k,l,j} \\ &\cdot f\left(t_{i-1} + \frac{jh}{q}, \overline{x_{m-1}}\left(t_{i-1} + \frac{jh}{q}\right)\right) + \overline{R_{m,(k,l)}} \\ &= \overline{x_m}\left(t_k + \frac{lh}{q}\right) + \overline{R_{m,(k,l)}} \end{aligned} \quad (1.19)$$

and

$$x_m\left(t_n\right) = g\left(t_n\right) + \lambda b\left(t_n\right)\int_0^T \left(t_n - s\right)^{\alpha-1} f\left(s, x_{m-1}\left(s\right)\right) ds = g\left(t_n\right)$$

$$+\lambda b\left(t_n\right)\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i} \left(t_n - s\right)^{\alpha-1}\left(B_{m-1,i}\left(s\right) + R_{m-1,i}\left(s\right)\right) ds = g\left(t_n\right) + \frac{\lambda}{h^q} b\left(t_n\right)$$

$$\cdot \sum_{i=1}^{n}\sum_{j=0}^{q} C_q^j \varphi_{n,j} \cdot f\left(t_{i-1} + \frac{jh}{q}, \overline{x_{m-1}}\left(t_{i-1} + \frac{jh}{q}\right)\right) + \overline{R_{m,n}} = \overline{x_m}\left(t_n\right) + \overline{R_{m,n}} \quad (1.20)$$

where

$$B_{m,i}\left(t\right) = \frac{1}{h^q}\sum_{k=0}^{q} C_q^k (t-t_{i-1})^k (t_i - t)^{q-k} \cdot F_m\left(t_{i-1} + \frac{kh}{q}\right), t \in [t_{i-1}, t_i] \quad (1.21)$$

and

$$\overline{B_{m,i}}\left(t\right) = \frac{1}{h^q}\sum_{k=0}^{q} C_q^k (t-t_{i-1})^k (t_i - t)^{q-k} \cdot f\left(t_{i-1} + \frac{kh}{q}, \overline{x_m}\left(t_{i-1} + \frac{kh}{q}\right)\right) \quad (1.22)$$

for $t \in [t_{i-1}, t_i]$ and $i = \overline{1, n}$ are the Bernstein splines B_{m-1} and $\overline{B_{m-1}}$. The algorithm is stopped to a previously chosen iteration m and at this iterative step we construct the Bernstein spline $\widetilde{B_m}$ given on the subintervals $[t_{i-1}, t_i]$, $i = \overline{1, n}$, by

$$\widetilde{B_{m,i}}\,(t) = \frac{1}{h^q}\sum_{k=0}^{q} C_q^k (t-t_{i-1})^k (t_i - t)^{q-k}\cdot \overline{x_m}\left(t_{i-1} + \frac{kh}{q}\right), t \in [t_{i-1}, t_i]. \quad (1.23)$$

This spline $\widetilde{B_m}$ will be the continuous approximation of the solution.

2. Convergence analysis

Concerning the convergence of the above presented iterative method we obtain the following main result.

Theorem 2.1. *Under the conditions of Theorem 1.1, including (1.5) and (1.6), the sequence* $\left(\overline{x_m}\left(t_k + \frac{lh}{q}\right)\right)_{m\in\mathbb{N}^*}$ *with* $k = \overline{0, n-1}$, $l = \overline{0, q}$, *approximates the solution of the integral equation (0.1) on the knots of a uniform mesh and the sequence of Bernstein splines* $\left(\widetilde{B_m}\right)_{m\in\mathbb{N}^*}$ *approximates the same solution on* $[0, T]$. *The error estimates in the discrete and continuous approximation is*

$$\left|x^*\left(t_k + \frac{lh}{q}\right) - \overline{x_m}\left(t_k + \frac{lh}{q}\right)\right| \le \frac{\theta^m \lambda M_b M_0 T^\alpha}{\alpha\,(1-\theta)}$$

$$+\frac{5\lambda M_b T^\alpha}{4\alpha\left(1 - \frac{\lambda L M_b T^\alpha}{\alpha}\right)}\left(\frac{L_0 h}{\sqrt{q}} + \frac{L' h^\alpha}{\left(\sqrt{q}\right)^\alpha} + \frac{L'' h^{\alpha+1}}{\left(\sqrt{q}\right)^{\alpha+1}}\right), \quad \forall m \in \mathbb{N}^* \quad (2.1)$$

for $k = \overline{0, n-1}$, $l = \overline{0, q}$, *and*

$$\left|x^*\,(t) - \widetilde{B_m}\,(t)\right|$$

$$\le \frac{\theta^m \lambda M_b M_0 T^\alpha}{\alpha\,(1-\theta)} + \frac{5\lambda M_b T^\alpha}{4\alpha\left(1 - \frac{\lambda L M_b T^\alpha}{\alpha}\right)}\left(\frac{L_0 h}{\sqrt{q}} + \frac{L' h^\alpha}{\left(\sqrt{q}\right)^\alpha} + \frac{L'' h^{\alpha+1}}{\left(\sqrt{q}\right)^{\alpha+1}}\right)$$

$$+\frac{5}{4}\left(\left(\eta + \frac{2\lambda\beta M T^\alpha}{\alpha}\right)\frac{h}{\sqrt{q}} + \frac{L' h^\alpha}{\left(\sqrt{q}\right)^\alpha} + \frac{L'' h^{\alpha+1}}{\left(\sqrt{q}\right)^{\alpha+1}}\right), \quad \forall t \in [0, T] \quad (2.2)$$

where $\theta = \frac{\lambda L M_b T^\alpha}{\alpha}$.

Proof. Since

$$\left|x^*\left(t_k + \frac{lh}{q}\right) - \overline{x_m}\left(t_k + \frac{lh}{q}\right)\right| \le \left|x^*\left(t_k + \frac{lh}{q}\right) - x_m\left(t_k + \frac{lh}{q}\right)\right|$$

$$+\left|x_m\left(t_k + \frac{lh}{q}\right) - \overline{x_m}\left(t_k + \frac{lh}{q}\right)\right|$$

by (1.3) we have to estimate $\left|\overline{R_{m,(k,l)}}\right| = \left|x_m\left(t_k+\frac{lh}{q}\right) - \overline{x_m}\left(t_k+\frac{lh}{q}\right)\right|$ for $m \in \mathbb{N}^*$, $k=\overline{0,n-1}$, $l=\overline{0,q}$. Based on (1.12) and (1.13) we have

$$\begin{aligned}\left|\overline{R_{1,(k,l)}}\right| &= \left|x_1\left(t_k+\frac{lh}{q}\right) - \overline{x_1}\left(t_k+\frac{lh}{q}\right)\right| \\ &\leq \lambda M_b \sum_{i=1}^{n}\int_{t_{i-1}}^{t_i} \left|R_{0,i}(s)\right| \left|t_k+\frac{lh}{q}-s\right|^{\alpha-1} ds \\ &\leq \frac{5\lambda M_b}{4}\left(\frac{L_0 h}{\sqrt{q}} + \frac{L' h^{\alpha}}{\left(\sqrt{q}\right)^{\alpha}} + \frac{L'' h^{\alpha+1}}{\left(\sqrt{q}\right)^{\alpha+1}}\right)\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}\left|t_k+\frac{lh}{q}-s\right|^{\alpha-1} ds \\ &\leq \frac{5\lambda M_b T^{\alpha}}{4\alpha}\left(\frac{L_0 h}{\sqrt{q}} + \frac{L' h^{\alpha}}{\left(\sqrt{q}\right)^{\alpha}} + \frac{L'' h^{\alpha+1}}{\left(\sqrt{q}\right)^{\alpha+1}}\right),\ k=\overline{0,n-1},\ l=\overline{0,q-1}\end{aligned}$$

and by (1.14) we get

$$\left|\overline{R_{1,n}}\right| \leq \frac{5\lambda M_b T^{\alpha}}{4\alpha}\left(\frac{L_0 h}{\sqrt{q}} + \frac{L' h^{\alpha}}{\left(\sqrt{q}\right)^{\alpha}} + \frac{L'' h^{\alpha+1}}{\left(\sqrt{q}\right)^{\alpha+1}}\right).$$

Now, let us consider $\left|\overline{R_{m-1}}\right| = \max\{\left|\overline{R_{m-1,n}}\right|, \max_{k=\overline{0,n-1},l=\overline{0,q-1}}\left|\overline{R_{m-1,(k,l)}}\right|\}$, and since

$$\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}\left(\sum_{j=0}^{q} C_q^j \left(s-t_{i-1}\right)^j \left(t_i-s\right)^{q-j}\right)\left|t_k+\frac{lh}{q}-s\right|^{\alpha-1} ds \leq \frac{h^q T^{\alpha}}{\alpha}$$

$$\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}\left[\sum_{j=0}^{q} C_q^j \left(s-t_{i-1}\right)^j \left(t_i-s\right)^{q-j}\right]\left(t_n-s\right)^{\alpha-1} ds \leq \frac{h^q T^{\alpha}}{\alpha}$$

by induction for $m \geq 2$, and by (1.2) and (1.17)-(1.22) it obtains

$$\begin{aligned}\left|\overline{R_{m,(k,l)}}\right| \leq \lambda M_b &\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}\left|R_{m-1,i}(s)\right|\left|t_k+\frac{lh}{q}-s\right|^{\alpha-1} ds \\ &+\frac{\lambda}{h^q} M_b \sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}\left[\sum_{j=0}^{q} C_q^j \left(s-t_{i-1}\right)^j \left(t_i-s\right)^{q-j}\right. \\ \cdot L &\left.\left|x_{m-1}\left(t_{i-1}+\frac{jh}{q}\right) - \overline{x_{m-1}}\left(t_{i-1}+\frac{jh}{q}\right)\right|\right]\left(t_k+\frac{lh}{q}-s\right)^{\alpha-1} ds \\ \leq &\frac{5\lambda M_b T^{\alpha}}{4\alpha}\left(\frac{L_0 h}{\sqrt{q}} + \frac{L' h^{\alpha}}{\left(\sqrt{q}\right)^{\alpha}} + \frac{L'' h^{\alpha+1}}{\left(\sqrt{q}\right)^{\alpha+1}}\right) + \frac{\lambda L M_b T^{\alpha}}{\alpha}\left|\overline{R_{m-1}}\right|\end{aligned}$$

with $k=\overline{0,n-1}$, $l=\overline{0,q-1}$. Similarly, we get

$$\left|\overline{R_{m,n}}\right| \leq \frac{5\lambda M_b T^\alpha}{4\alpha}\left(\frac{L_0 h}{\sqrt{q}}+\frac{L' h^\alpha}{\left(\sqrt{q}\right)^\alpha}+\frac{L'' h^{\alpha+1}}{\left(\sqrt{q}\right)^{\alpha+1}}\right)+\frac{\lambda L M_b T^\alpha}{\alpha}\left|\overline{R_{m-1}}\right|.$$

For estimating $\left|\overline{R_{m-1}}\right|$ we have

$$\left|\overline{R_{2,(k,l)}}\right| \leq \left[1+\frac{\lambda L M_b T^\alpha}{\alpha}\right]\frac{5\lambda M_b T^\alpha}{4\alpha}\left(\frac{L_0 h}{\sqrt{q}}+\frac{L' h^\alpha}{\left(\sqrt{q}\right)^\alpha}+\frac{L'' h^{\alpha+1}}{\left(\sqrt{q}\right)^{\alpha+1}}\right),$$
$$k=\overline{0,n-1}, l=\overline{0,q-1}$$

and

$$\left|\overline{R_{2,n}}\right| \leq \frac{5\lambda M_b T^\alpha}{4\alpha}\left(\frac{L_0 h}{\sqrt{q}}+\frac{L' h^\alpha}{\left(\sqrt{q}\right)^\alpha}+\frac{L'' h^{\alpha+1}}{\left(\sqrt{q}\right)^{\alpha+1}}\right)\left(1+\frac{\lambda L M_b T^\alpha}{\alpha}\right),$$

obtaining in inductive manner for $m \geq 3$, the estimate

$$\begin{aligned}\left|\overline{R_{m,(k,l)}}\right| &\leq \left[1+\frac{\lambda L M_b T^\alpha}{\alpha}+\ldots+\left(\frac{\lambda L M_b T^\alpha}{\alpha}\right)^{m-1}\right]\frac{5\lambda M_b T^\alpha}{4\alpha}\\ &\quad\cdot\left(\frac{L_0 h}{\sqrt{q}}+\frac{L' h^\alpha}{\left(\sqrt{q}\right)^\alpha}+\frac{L'' h^{\alpha+1}}{\left(\sqrt{q}\right)^{\alpha+1}}\right)\\ &\leq \frac{5\lambda M_b T^\alpha}{4\alpha\left(1-\frac{\lambda L M_b T^\alpha}{\alpha}\right)}\left(\frac{L_0 h}{\sqrt{q}}+\frac{L' h^\alpha}{\left(\sqrt{q}\right)^\alpha}+\frac{L'' h^{\alpha+1}}{\left(\sqrt{q}\right)^{\alpha+1}}\right)\end{aligned}$$

with $k=\overline{0,n-1}$, $l=\overline{0,q}$. Now, the inequality (2.1) follows.
The estimate (2.2) will be obtained by using the scheme

$$x^* \to x_m \to \widehat{B_m} \to \widetilde{B_m}$$

where $\widehat{B_{m,i}}(t) = \frac{1}{h^q}\sum\limits_{k=0}^{q} C_q^k (t-t_{i-1})^k (t_i-t)^{q-k}\cdot x_m\left(t_{i-1}+\frac{kh}{q}\right)$ for $t\in[t_{i-1},t_i]$, $i=\overline{1,n}$. According to the proof of Theorem 1.1 we have

$$\left|x_m(t')-x_m(t)\right| \leq \left(\eta+\frac{2\lambda\beta M T^\alpha}{\alpha}\right)\left|t-t'\right|+L'\left|t-t'\right|^\alpha+L''\left|t-t'\right|^{\alpha+1}$$

for $t\in[0,T]$, $m\in\mathbb{N}^*$, and with the inequality of Lorentz we get

$$\left|x_m(t)-\widehat{B_m}(t)\right| \leq \frac{5}{4}\left(\left(\eta+\frac{2\lambda\beta M T^\alpha}{\alpha}\right)\frac{h}{\sqrt{q}}+\frac{L' h^\alpha}{\left(\sqrt{q}\right)^\alpha}+\frac{L'' h^{\alpha+1}}{\left(\sqrt{q}\right)^{\alpha+1}}\right)$$

for all $t\in[0,T]$. By (1.23) and (2.1) it follows,

$$\left|\widehat{B_m}(t)-\widetilde{B_m}(t)\right|$$

$$\leq \frac{1}{h^q}\sum_{k=0}^{q} C_q^k (t-t_{i-1})^k (t_i-t)^{q-k}\cdot\left|x_m\left(t_{i-1}+\frac{kh}{q}\right)-\overline{x_m}\left(t_{i-1}+\frac{kh}{q}\right)\right|$$

$$\leq \frac{5\lambda M_b T^\alpha}{4\alpha\left(1-\frac{\lambda L M_b T^\alpha}{\alpha}\right)}\left(\frac{L_0 h}{\sqrt{q}}+\frac{L' h^\alpha}{\left(\sqrt{q}\right)^\alpha}+\frac{L'' h^{\alpha+1}}{\left(\sqrt{q}\right)^{\alpha+1}}\right)$$

for $t\in[t_{i-1},t_i]$, $i=\overline{1,n}$, and with (1.3) we obtain the error estimate (2.2). □

By the error estimate (2.2) we observe that the order of convergence of this iterative method is $\left\| x_m-\widetilde{B_m}\right\|_\infty = O\left(h^\alpha\right).$

3. Numerical experiments

In order to test the obtained theoretical result and to illustrate the accuracy of the Bernstein spline iterative method, in that follows we present some numerical examples.

Example 3.1. The weakly singular linear integral equation (Example 6.1. in [32])

$$x\left(t\right)=g\left(t\right)+\frac{1}{4}\int_0^1\sqrt{ts}\left|t-s\right|^{-\frac{1}{2}}x\left(s\right)ds,\quad t\in[0,1] \tag{3.1}$$

with $\lambda=\frac{1}{4}$, $b\left(t\right)=\sqrt{t}$, $\alpha=\frac{1}{2}$, and

$$g\left(t\right)=\frac{1}{5}\sqrt{t}\left(1-t\right)\left[15-\sqrt{1-t}\left(1+4t\right)\right]+\frac{1}{5}t^2\left(4t-5\right)$$

has the exact solution $x^*\left(t\right)=3\sqrt{t}\left(1-t\right).$ By considering separately the degree of Bernstein polynomials $q=1$ and $q=4$, we apply the algorithm (1.13)-(1.22) with $m=30$ iterations, and take $n=10$, $n=50$, and $n=100$ for the test of convergence. The results are presented in Tables 1 and 2, where $e_i=\left|\overline{x_m}(t_i)-x^*(t_i)\right|$, $i=\overline{0,n}$, are the pointwise errors. Investigating Tables 1 and 2, the convergence is confirmed and improved results can be observed when the degree of Bernstein polynomials increases by $q=1$ to $q=4$. It is interesting to see that the case $q=1$ corresponds to the trapezoidal product integration and as was expected, the Bernstein splines iterative method provides better results.

Table 1. Numerical results for (3.1) with $q=1$

	$m=30$ $q=1$		
t_i/e_i	$n=10$	$n=50$	$n=100$
$0,0$	$0,00$	$0,00$	$0,00$
$0,2$	$1,86E-03$	$7,88E-05$	$2,00E-05$
$0,4$	$2,98E-03$	$1,26E-04$	$3,20E-05$
$0,6$	$3,96E-03$	$1,68E-04$	$4,25E-05$
$0,8$	$4,63E-03$	$1,97E-04$	$5,00E-05$
$1,0$	$3,88E-03$	$1,64E-04$	$4,15E-05$

Table 2. Numerical results for (3.1) with $q = 4$

	$m = 30$ $q = 4$		
t_i/e_i	$n = 10$	$n = 50$	$n = 100$
$0,0$	$0,00$	$0,00$	$0,00$
$0,2$	$4,75E-04$	$1,99E-05$	$5,02E-06$
$0,4$	$7,66E-04$	$3,20E-05$	$8,06E-06$
$0,6$	$1,02E-03$	$4,25E-05$	$1,07E-05$
$0,8$	$1,20E-03$	$5,00E-05$	$1,26E-05$
$1,0$	$1,01E-03$	$4,16E-05$	$1,05E-05$

Example 3.2. We test the Bernstein spline iterative method (1.17)-(1.23) on fractional integral equations too, and for the nonlinear integral equation

$$x(t) = g(t) + \frac{1}{4\Gamma\left(\frac{1}{2}\right)} \int_0^1 |t-s|^{-\frac{1}{2}} [x(s)]^2 ds, \quad t \in [0,1] \tag{3.2}$$

with

$$\lambda = 1,\ b(t) = \frac{1}{4},\ \alpha = \frac{1}{2},$$

and

$$g(t) = \sqrt{t(1-t)} + \frac{1}{15\sqrt{\pi}} \left[t^{\frac{3}{2}}(4t-5) - (1-t)^{\frac{3}{2}}(4t+1) \right]$$

the exact solution is $x^*(t) = \sqrt{t(1-t)}$. The contraction condition

$$\frac{LM_bT^\alpha}{\Gamma(\alpha+1)} = \frac{1}{\sqrt{\pi}} < 1$$

is fulfilled and the iterative method (1.17)-(1.23) applied with $m = 30$, $n = 10$, $n = 50$, $n = 100$, $q = 1$ and $q = 4$, respectively, provides the results presented in Tables 3 and 4. The convergence is confirmed and we observed better results when we pass by $q = 1$ to $q = 4$. So, the Bernstein splines iterative method is better than the trapezoidal product integration method for fractional integral equations, too.

Table 3. Numerical results for (3.2) with $q = 1$

	$m = 30$ $q = 1$		
t_i/e_i	$n = 10$	$n = 50$	$n = 100$
$0,0$	$5,87E-04$	$2,48E-05$	$6,27E-06$
$0,2$	$8,15E-04$	$3,48E-05$	$8,81E-06$
$0,4$	$8,93E-04$	$3,80E-05$	$9,62E-06$
$0,6$	$8,93E-04$	$3,80E-05$	$9,62E-06$
$0,8$	$8,15E-04$	$3,48E-05$	$8,81E-06$
$0,7$	$8,64E-04$	$3,68E-05$	$9,32E-06$
$1,0$	$5,87E-04$	$2,48E-05$	$6,27E-06$

Table 4. Numerical results for (3.2) with $q = 4$

	$m = 30$ $q = 4$		
t_i/e_i	$n = 10$	$n = 50$	$n = 100$
$0,0$	$1,51E-04$	$6,26E-06$	$1,58E-06$
$0,2$	$2,10E-04$	$8,78E-06$	$2,22E-06$
$0,4$	$2,30E-04$	$9,60E-06$	$2,42E-06$
$0,6$	$2,30E-04$	$9,60E-06$	$2,42E-06$
$0,8$	$2,10E-04$	$8,78E-06$	$2,22E-06$
$1,0$	$1,51E-04$	$6,26E-06$	$1,58E-06$

Example 3.3. In order to make a comparison with other methods from the existing literature we present the results obtained on the following example. The linear weakly singular integral equation (Example 1. in [25], Example 6.2. in [32], Example 4. in [34])

$$x(t) = g(t) + \frac{1}{10}\int_0^1 |t-s|^{-\frac{1}{3}} x(s)\,ds, \quad t \in [0,1] \tag{3.3}$$

with $\lambda = \frac{1}{10}$, $b(t) = 1$, $\alpha = \frac{2}{3}$, and

$$g(t) = t^2(1-t)^2 - \frac{27}{30800}\left[t^{\frac{8}{3}}\left(54t^2 - 126t + 77\right) + (1-t)^{\frac{8}{3}}\left(54t^2 + 18t + 5\right)\right]$$

has the exact solution $x^*(t) = t^2(1-t)^2$. By applying the iterative method (1.17)-(1.23) with $m = 30$, $n = 10$, $n = 50$, $n = 100$, and taking $q = 1$ and $q = 4$, we obtain the results presented in Tables 5 and 6. Comparing the results between Table 6 ($n = 100$) and Table 1 in [25] (where the accuracy is $O\left(10^{-6}\right)$), we see better accuracy for our method. In Tables 5 and 6 we see that the accuracy is improved by passing from $n = 10$ to $n = 100$, that confirm the convergence of Bernstein splines method stated in Theorem 2.1. Moreover, for $q = 4$ the accuracy is better than those for $q = 1$, which means that again the Bernstein splines method provides better accuracy than the trapezoidal product integration method.

Table 5. Numerical results for (3.3) with $q = 1$

	$m = 30$ $q = 1$		
t_i/e_i	$n = 10$	$n = 50$	$n = 100$
$0,0$	$1,70E-05$	$8,94E-07$	$2,32E-07$
$0,1$	$1,74E-05$	$8,48E-07$	$2,19E-07$
$0,2$	$1,59E-06$	$4,51E-08$	$1,06E-08$
$0,3$	$1,99E-05$	$8,73E-07$	$2,22E-07$
$0,4$	$3,23E-05$	$1,43E-06$	$3,63E-07$
$0,5$	$3,67E-05$	$1,62E-06$	$4,13E-07$
$0,6$	$3,23E-05$	$1,43E-06$	$3,63E-07$
$0,7$	$1,99E-05$	$8,73E-07$	$2,22E-07$
$0,8$	$1,59E-06$	$4,51E-08$	$1,06E-08$
$0,9$	$1,74E-05$	$8,48E-07$	$2,19E-07$
$1,0$	$1,70E-05$	$8,94E-07$	$2,32E-07$

Table 6. Numerical results for (3.3) with $q = 4$

	$m = 30$ $q = 4$		
t_i/e_i	$n = 10$	$n = 50$	$n = 100$
$0,0$	$4,48E-06$	$2,25E-07$	$5,81E-08$
$0,1$	$4,40E-06$	$2,13E-07$	$5,48E-08$
$0,2$	$4,13E-07$	$1,13E-08$	$2,66E-09$
$0,3$	$5,03E-06$	$2,19E-07$	$5,56E-08$
$0,4$	$8,16E-06$	$3,58E-07$	$9,10E-08$
$0,5$	$9,26E-06$	$4,06E-07$	$1,03E-07$
$0,6$	$8,16E-06$	$3,58E-07$	$9,10E-08$
$0,7$	$5,03E-06$	$2,19E-07$	$5,56E-08$
$0,8$	$4,13E-07$	$1,13E-08$	$2,66E-09$
$0,9$	$4,40E-06$	$2,13E-07$	$5,48E-08$
$1,0$	$4,48E-06$	$2,25E-07$	$5,81E-08$

4. Conclusions

The iterated Bernstein splines method was applied to second kind weakly singular and fractional Fredholm integral equations and in Theorem 2.1 the convergence of this method was proved providing the order of convergence $O\left(h^{\alpha}\right)$. The condition that ensures the convergence is the same as the contraction condition and therefore, the applicability of this method is limited by the contraction condition. On the other hand the accuracy of this method is better than those provided by the trapezoidal product integration, as was observed in the previously presented numerical examples and, on some cases, provides better accuracy than the existing methods from literature.

References

[1] Agarwal, R.P., Benchohra, M., Hamani, S., *A survey on existence results for boundary value problems of nonlinear fractional differential equations and inclusions*, Acta Appl. Math., **109**(2010), 973-1033.

[2] Allouch, C., Sablonnière, P., Sbibih, D., Tahrichi, M., *Product integration methods based on discrete spline quasi-interpolants and application to weakly singular integral equations*, J. Comput. Appl. Math., **233**(2010), 2855-2866.

[3] Amin, R., Alrabaiah, H., Mahariq, I., Zeb, A., *Theoretical and computational results for mixed type Volterra-Fredholm fractional integral equations*, Fractals, **30**(2022), no. 1, 2240035.

[4] András, S., *Weakly singular Volterra and Fredholm-Volterra integral equations*, Studia Univ. Babeş-Bolyai Math., **48**(2003), no. 3, 147-155.

[5] Atkinson, K.E., *The numerical solution of an Abel integral equation by a product trapezoidal method*, SIAM J. Numer. Anal., **11**(1974), no. 1, 97–101.

[6] Atkinson, K.E., *An Introduction to Numerical Analysis*, 2nd ed., John Wiley & Sons, New York, 1989.

[7] Bagley, R.L., Calico, R.A., *Fractional order state equations for the control of viscoelastically damped structures*, J. Guid. Contr. Dynam., **14**(1991), 304-311.

[8] Baleanu, D., Diethelm, K., Scalas, E., Trujillo, J.J., *Fractional Calculus: Models and Numerical Methods*, in: Series on Complexity, Nonlinearity and Chaos, vol. **3**, World Scientific Publishers, Co., N. Jersey, London, Singapore, 2012.

[9] Brunner, H., *The numerical solution of weakly singular Volterra integral equations by collocation on graded meshes*, Math. Comp., **45**(172)(1985), 417–437.

[10] Brunner, H., Pedas, A., Vainikko, G., *The piecewise polynomial collocation method for nonlinear weakly singular Volterra equations*, Math. Comp., **68**(227)(1999), 1079–1095.

[11] Cao, Y., Huang, M., Liu, L., Xu, Y., *Hybrid collocation methods for Fredholm integral equations with weakly singular kernels*, Appl. Numer. Math., **57**(2007), 549–561.

[12] Diethelm, K., *The Analysis of Fractional Differential Equations. Lecture Notes In Mathematics*, vol. **2004**, Springer-Verlag Berlin, Heidelberg, 2010.

[13] Diogo, T., *Collocation and iterated collocation methods for a class of weakly singular Volterra integral equations*, J. Comput. Appl. Math., **229**(2009), 363–372.

[14] Du, M., Wang, Z., Hu, H., *Measuring memory with order of fractional derivative*, Sci. Rep., **3**(2013), 3431.

[15] Ezzat, M.A., Sabbah, A.S., El-Bary, A.A., Ezzat, S.M., *Thermoelectric viscoelastic fluid with fractional integral and derivative heat transfer*, Adv. Appl. Math. Mech., **7**(2015), 528-548.

[16] Garrappa, R., *Numerical solution of fractional differential equations: A survey and software tutorial*, Mathematics, **6**(2018), 16.

[17] Gorenflo, R., Vessella, S., *Abel Integral Equations: Analysis and Applications*, in: Lecture Notes in Mathematics, vol. **1461**, Springer Verlag, Berlin, 1991.

[18] Graham, I., *Galerkin methods for second kind integral equations with singularities*, Math. Comp., **39**(1982), no. 160, 519-533.

[19] Kaneko, H., Noren, R., Xu, Y., *Regularity of the solution of Hammerstein equations with weakly singular kernel*, Integral Equations Operator Theory, **13**(1990), 660-670.

[20] Kilbas, A.A., Srivastava, H.M., Trujillo, J.J., *Theory and Applications of Fractional Differential Equations*, Elsevier Science B.V., Amsterdam, 2006.

[21] Lakshmikantham, V., Leela, S., Vasundhara, J., *Theory of Fractional Dynamic Systems*, Cambridge Academic Publishers, Cambridge, 2009.

[22] Lorentz, G.G., *Bernstein Polynomials*, Toronto, Univ. Toronto Press, 1953

[23] Lubich, C., *Runge-Kutta theory for Volterra and Abel integral equations of the second kind*, Math. Comp., **41**(163)(1983), 87-102.

[24] Maleknejad, K., Mollapourasl, R., Ostadi, A., *Convergence analysis of Sinc-collocation methods for nonlinear Fredholm integral equations with a weakly singular kernel*, J. Comput. Appl. Math., **278**(2015), 1–11.

[25] Maleknejad, K., Nosrati, M., Najafi, E., *Wavelet Galerkin method for solving singular integral equations*, Comput. Appl. Math., **31**(2012), no. 2, 373-390.

[26] Mandal, M., Nelakanti, G., *Superconvergence results of Legendre spectral projection methods for weakly singular Fredholm–Hammerstein integral equations*, J. Comput. Appl. Math., **349**(2019), 114–131.

[27] Micula, S., *An iterative numerical method for fractional integral equations of the second kind*, J. Comput. Appl. Math., **339**(2018), 124–133.

[28] Micula, S., *A numerical method for weakly singular nonlinear Volterra integral equations of the second kind*, Symmetry, **12**(2020), 1862.

[29] Miller, K.S., Ross, B., *An Introduction to the Fractional Calculus and Differential Equations*, John Wiley, New York, 1993.

[30] Mohammad, M., Trounev, A., *Implicit Riesz wavelets based-method for solving singular fractional integro-differential equations with applications to hematopoietic stem cell modeling*, Chaos Solitons Fractals, **138**(2020), 109991.

[31] Muskhelishvili, N.I., Radok, J.R.M., *Singular Integral Equations: Boundary Problems of Function Theory and their Application to Mathematical Physics*, Courier Corporation, Chelmsford, 2008.

[32] Okayama, T., Matsuo, T., Sugihara, M., *Sinc-collocation methods for weakly singular Fredholm integral equations of the second kind*, J. Comput. Appl. Math., **234**(2010), 1211-1227.

[33] Podlubny, I., *Fractional Differential Equation*, Academic Press, San Diego, 1999.

[34] Ren, Y., Zhang, B., Qiao, H., *A simple Taylor-series expansion method for a class of second kind integral equations*, J. Comput. Appl. Math., **110**(1999), 15-24.

[35] Schneider, C., *Regularity of the solution to a class of weakly singular Fredholm integral equations of the second kind*, Integral Equations Operator Theory, **2**(1979), 62-68.

[36] Schneider, C., *Product integration for weakly singular integral equations*, Math. Comp., **36**(153)(1981), 207-213.

[37] Srivastava, H.M., Dubey, V.P., Kumar, R., Singh, J., Kumar, D., Baleanu, D., *An efficient computational approach for a fractional-order biological population model with carrying capacity*, Chaos Solitons Fractals, **138**(2020), 109880.

[38] Torvik, P.J., Bagley, R.L., *On the appearance of the fractional derivative in the behavior of real materials*, J. Appl. Mech., **51**(1984), 294–298.

[39] Usta, F., *Numerical analysis of fractional Volterra integral equations via Bernstein approximation method*, J. Comput. Appl. Math., **384**(2021), 113198.

[40] Wu, G.C., Baleanu, D., *Variational iteration method for fractional calculus – A universal approach by Laplace transform*, Adv. Differential Equations, **2013**(18)(2013), 1–9.

[41] Yang, Y., Tang, Z., Huang, Y., *Numerical solutions for Fredholm integral equations of the second kind with weakly singular kernel using spectral collocation method*, Appl. Math. Comput., **349**(2019), 314–324.

[42] Young, A., *Approximate product-integration*, Proc. R. Soc. Lond., Ser. A, **224**(1954), 552–561.

[43] Yousefi, S.A., *Numerical solution of Abel's integral equation by using Legendre wavelets*, Appl. Math. Comput., **175**(2006), 574-580.

[44] Yousefi, A., Javadi, S., Babolian, E., *A computational approach for solving fractional integral equations based on Legendre collocation method*, Math. Sciences, **13**(2019), 231–240.

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