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Existence results for some fractional order coupled systems with impulses and nonlocal conditions on the half line

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Abstract. In this paper, we deal with initial value problems for coupled systems of nonlinear fractional differential equations, subject to coupled nonlocal initial and impulsive conditions on the half line. Global existence-uniqueness results are obtained under weak conditions allowing the reaction part of the problem to increase indefinitely with time. Our approach relies mainly to some fixed point theorem of Perov's type in generalized gauge spaces. The obtained results improve, generalize and complement many existing results in the literature. An example illustrating our main finding is also given.

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1. Introduction

Recently, an intensive interest has been given to the investigation of differential equations of fractional order. This is motivated by the natural introduction of fractional operators in the modeling of several phenomena whose nonlocal dynamics involving long-term effects are taken into account. These models have been applied successfully in many fields such as in mechanics, bio-chemistry, electrical engineering, control, porous media, medicine, etc. (see [6, 11]).

On the other hand, differential equations involving impulse effects appear as an appropriate model for some evolutionary problems. It is the case of many real-world processes that are subject of abrupt of changes in certain moments of times

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and arising in a variety of disciplines, including biology, population dynamics, electric technology, control theory, engineering, etc. For more details on this subject, we reefer to the monographs [3, 12].

Banach's contractive principle is one of the most useful tools in nonlinear functional analysis that ensures the existence and uniqueness of a fixed point on complete metric spaces. One of the extensions of this principle for contractive mappings on spaces endowed with vector valued metrics, was done by Perov in [16] and Perov and Kibento in [17]. Many other generalizations in this direction have been investigated. In [15], Precup established the extension in Perov's sens of some fixed point theorem in spaces endowed with a family of pseudo-metrics. Many authors applied the vector version's fixed point theorems in the study of the existence of solutions for systems of differential and integral equations, see for example [4, 5, 9, 10, 20] and the references therein. In this line of research, we consider in this work, the following nonlinear coupled system of fractional differential equations:

$$\begin{cases} {}^{C}D^{\alpha}_{0^+}u(t) = f(t, u(t), v(t)), & t \in \mathcal{I}_i = \left]t_i, t_{i+1}\right], \, i \in \mathbb{N} \\ {}^{C}D^{\beta}_{0^+}v(t) = g(t, u(t), v(t)), & t \in \mathcal{I}_i = \left]t_i, t_{i+1}\right], \, i \in \mathbb{N} \end{cases} \tag{1.1}$$

with coupled nonlocal initial conditions:

$$\begin{cases} u(0) = \varphi(u, v), \\ v(0) = \psi(u, v), \end{cases} \tag{1.2}$$

and subject to coupled impulsive conditions:

$$\begin{cases} \Delta u(t_i) = I_i(u(t_i), v(t_i)), & i \in \mathbb{N}^* = \mathbb{N}\backslash\{0\} \\ \Delta v(t_i) = J_i(u(t_i), v(t_i)), & i \in \mathbb{N}^* = \mathbb{N}\backslash\{0\} \end{cases} \tag{1.3}$$

where ${}^{C}D^{\alpha}_{0+}$ and ${}^{C}D^{\beta}_{0+}$ denote the Caputo fractional derivative operators with the fixed lower limit equals zero, of order α and β in $]0, 1[$ respectively, $f, g : \mathbb{R}_+ \times \mathbb{R}^2 \longrightarrow \mathbb{R}$ are nonlinear continuous functions, $\Delta u(t_i) = u(t_i^+) - u(t_i^-)$, where $u(t_i^+)$ and $u(t_i^-)$ represent the right and left limits of u at $t = t_i$ and $\{t_i\}_{i\in\mathbb{N}^*}$ is a sequence of points in $\mathbb{R}_+$ such that $t_i < t_{i+1}$ for $i \in \mathbb{N}^*$, $I_i, J_i : \mathbb{R}^2 \longrightarrow \mathbb{R}$ are nonlinear continuous functions, $\phi, \psi : X \longrightarrow \mathbb{R}$ are nonlinear continuous functional where X is a generalized complete gauge space, which will be defined later.

It should be noted that the coupled nonlocal initial conditions (1.2) generalizes many other types of initial conditions considered in the literature, such as: classical initial conditions, multi-point conditions and integral conditions.

After converting (1.1)- (1.3) into an equivalent fixed point problem in generalized gauge space, we apply some fixed point theorem of Perov's type, established in [15]. Using this approach, we obtain a global existence-uniqueness results for (1.1)- (1.3) under weak conditions allowing the nonlinearity to increase indefinitely with time, which is not the case in many earlier results in the literature (see Remark 3.1). This study allows us also, to improve and generalize some other existence results in the literature for systems of fractional differential equations without impulses (see Remark 3.6).

The rest of the paper is organized as follows. In Section 2 we recall some definitions from fractional calculus. We introduce also the fixed point theorem in generalized gauge spaces, on which our result is based, as well as some related concepts. The main result concerning the global existence-uniqueness result for (1.1)- (1.3) is established in Section 3. Finally, in Section 4, we provide an illustrative example.

2. Preliminaries

Let us recall the notion of the fractional derivatives. For further details on some essential related properties, we refer to [6, 11].
Let n be a positive integer, α the positive real such that $n-1<\alpha\leq n$ and d^n/dt^n the classical derivative operator of order n.

Definition 2.1. The Riemann-Liouville fractional integral, and the Riemann-Liouville fractional derivative, of a real function u defined on $\mathbb{R}_+$ of order α, are defined respectively by

$$I_{0^+}^{\alpha}u(t):=\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}u(s)\,ds, \qquad t>0,$$

$$D_{0^+}^{\alpha}u(t):=\frac{d^n}{dt^n}I_{0^+}^{n-\alpha}u(t):=\frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dt^n}\int_0^t (t-s)^{n-\alpha-1}u(s)\,ds, \qquad t>0,$$

where $\Gamma(.)$ is the Gamma function, provided that the right hand sides exist point wise.

Definition 2.2. The Caputo fractional derivative of a real function u defined on $\mathbb{R}_+$ of order α, noted by ${}^{C}D_{0^+}^{\alpha}$, is defined by

$${}^{C}D_{0^+}^{\alpha}u(t):=\left(D_{0^+}^{\alpha}\left[u-\sum_{k=0}^{n-1}\frac{u^{(k)}(0)}{k!}(.)^k\right]\right)(t), \qquad t>0,$$

provided that the right hand side exists point wise.

We denote by $\mathcal{M}_n(\mathbb{R}_+)$, the set of all square matrices of order n with positive real elements, I the identity matrix of order n and by O the zero matrix of order n.

Definition 2.3. [18] A square real matrix M of order n, is said to be convergent to zero, if $M^k \longrightarrow O$, as $k\longrightarrow\infty$.

Definition 2.4. [18] Let $M\in\mathcal{M}_n(\mathbb{R}_+)$ with eigenvalues λ_i, $1\leq i\leq n$, that is $\lambda_i\in\mathbb{R}$ such that $det(M-\lambda_i I)=O$. Then

$$\rho(M)=\max_{1\leq i\leq n}|\lambda_i|$$

is called the spectral radius of M.

Lemma 2.5. [18] Let $M\in\mathcal{M}_n(\mathbb{R}_+)$. The following assumptions are equivalent.
(i) M is convergent to zero.
(ii) The matrix $I-M$ is non singular, and

$$(I-M)^{-1}=I+M+M^2+...+M^n+...,$$

(iii) $\rho(M) < 1$.

As it is pointed out in [13], the following lemma follows immediately from the characterization (iii) in Lemma 2.5.

Lemma 2.6. [13] If A is a square matrix that converges to zero and the elements of another matrix B are small enough, then $A + B$ also converges to zero.

We state now the extension of Gheorghiu's theorem for generalized contractions on complete generalized gauge spaces established in [15].

Let X be a generalized gauge space endowed with a complete gauge structure $\mathfrak{D} = \{D_\nu\}_{\nu\in\mathcal{N}}$, where $\mathcal{N}$ is an index set. For further details on gauge spaces and generalized gauge spaces we reefer to [7, 15].

Definition 2.7. [15] (Generalized contraction) Let $(X, \mathfrak{D})$ be a generalized gauge space with $\mathfrak{D} = \{D_\nu\}_{\nu\in\mathcal{N}}$. A map $T : D(T) \subset X \longrightarrow X$ is called a generalized contraction, if there exists a function $w : \mathcal{N} \longrightarrow \mathcal{N}$ and $M \in \mathcal{M}_n(\mathbb{R}_+)^{\mathcal{N}}$, $M = \{M_\nu\}_{\nu\in\mathcal{N}}$ such that

$$D_\nu(T(u), T(v)) \leq M_\nu D_{w(\nu)}(u, v), \;\; \forall u, v \in D(T), \, \forall \nu \in \mathcal{N} \tag{2.1}$$

and

$$\sum_{k=1}^{\infty} M_\nu M_{w(\nu)} ... M_{w^{k-1}(\nu)} D_{w^k(\nu)}(u, v) < \infty, \;\; \forall u, v \in D(T), \, \forall \nu \in \mathcal{N} \tag{2.2}$$

Theorem 2.8. [15, Theorem 2.1] Let $(X, \mathfrak{D})$ be a complete generalized gauge space and let $T : X \longrightarrow X$ be a generalized contraction. Then, T has a unique fixed point in X, which can be obtained by successive approximations starting from any element of X.

2.1. Equivalent system of integral equations

In the fractional case, there are two different approaches defining the concept of solutions for impulsive differential equations, which can be briefly described as follows (see [1, 2]):

Fractional derivatives with a fixed lower limit at the initial time. This approach (denoted respectively by V_2 in [1] and by A_1 in [2]) considers that the lower limit of the fractional derivative is kept equal to the initial time on any interval between two consecutive impulses, with only modified initial conditions.

Fractional derivatives with varying lower limits. This approach (denoted respectively by V_1 in [1] and by A_2 in [2]) neglects the lower limit of the fractional derivative at the initial time and moves it to each impulsive time.

In this work, we will adopt the case of fixed lower limit.

For any interval $\mathcal{I}$ of $\mathbb{R}_+$ (which may be unbounded), we denote by $\mathcal{C}(\mathcal{I})$ the set of all real continuous functions on $\mathcal{I}$ and by u_i the restriction of $u \in \mathcal{C}(\mathbb{R}_+)$ to $\mathcal{I}_i =]t_i, t_{i+1}]$, $(i \in \mathbb{N})$.

Let $\mathcal{PC}(\mathbb{R}_+)$ be the set of all real valued piece-wise continuous functions on $\mathbb{R}_+$:

$$\mathcal{PC}(\mathbb{R}_+) = \{u : \mathbb{R}_+ \to \mathbb{R} : u_i \in \mathcal{C}(\mathcal{I}_i) \text{ and } u(t_i^+) \text{ exist for every } i \in \mathbb{N}\} \tag{2.3}$$

endowed with the saturated family $\{d_\nu : \nu \in \mathcal{N}\}$ of pseudo-metrics, generating its topology, defined by

$$d_\nu(u,v) = \max_{t\in\nu}\left\{e^{-\lambda t}|u(t)-v(t)|\right\}, \forall u,v \in \mathcal{PC}(\mathbb{R}_+), \tag{2.4}$$

where ν runs over the set of all compact subsets of $\mathbb{R}_+$ denoted by $\mathcal{N}$, and λ is a positive real number to be specified later.

In what follows, we consider $X = \mathcal{PC}(\mathbb{R}_+) \times \mathcal{PC}(\mathbb{R}_+)$, endowed with the generalized complete gauge structure $\mathfrak{D} = \{D_\nu\}_{\nu\in\mathcal{N}}$ defined for $W_1 = (u_1, v_1), W_2 = (u_2, v_2) \in X$ by:

$$D_\nu(W_1, W_2) = \begin{pmatrix} d_\nu(u_1,u_2) \\ \\ d_\nu(v_1,v_2) \end{pmatrix}, \tag{2.5}$$

where d_ν is the pseudo-metric on $\mathcal{PC}(\mathbb{R}_+)$ given in (2.4).

Reproducing the proof of [14, Lemma 1], in addition of [8, Lemma 2.6] with a slight adaptation, we get the system of integral equations equivalent to (1.1)-(1.2) given by the following lemma.

Lemma 2.9. Let f, g, I_i, J_i $(i \in \mathbb{N}^*)$ be continuous functions and φ, ψ continuous functionals such that:

$$\begin{array}{c} \forall (u,v), (\tilde{u},\tilde{v}) \in X \text{ if } u = \tilde{u} \text{ and } \ v = \tilde{v} \text{ on } \ [0,t_1[, \text{ then} \\ \varphi(u,v) = \varphi(\tilde{u},\tilde{v}) \quad \text{and} \quad \psi(u,v) = \psi(\tilde{u},\tilde{v}) \end{array} \tag{2.6}$$

Then, $(u,v) \in X$ is a solution of (1.1)-(1.3) if and only if (u,v) is a solution of the following system of integral equations

$$\begin{cases} u(t) = \begin{cases} \varphi(u,v) + \displaystyle\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s,u(s),v(s))ds, & t \in \mathcal{I}_0 \\ \varphi(u,v) + \displaystyle\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s,u(s),v(s))ds + \sum_{j=1}^{i} I_j(u(t_j),v(t_j)), & t \in \mathcal{I}_i \end{cases} \\ v(t) = \begin{cases} \psi(u,v) + \displaystyle\int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} g(s,u(s),v(s))ds, & t \in \mathcal{I}_0 \\ \psi(u,v) + \displaystyle\int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} g(s,u(s),v(s))ds + \sum_{j=1}^{i} J_j(u(t_j),v(t_j)), & t \in \mathcal{I}_i \end{cases} \end{cases} \tag{2.7}$$

For $i = 1, 2$, let $T_i : X \to \mathcal{PC}(\mathbb{R}_+)$ be the operators defined for every $W := (u,v) \in X$ by

$$T_1(W)(t) = \varphi(u,v) + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s,u(s),v(s))ds + \sum_{t_j<t} I_j(u(t_j),v(t_j)) \tag{2.8}$$

$$T_2(W)(t) = \psi(u,v) + \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} g(s,u(s),v(s))ds + \sum_{t_j<t} J_j(u(t_j),v(t_j)) \tag{2.9}$$

Let us consider the operator: $T : X \to X$ defined by

$$T(u,v) = (T_1(u,v), T_2(u,v)), \quad \forall (u,v) \in X, \tag{2.10}$$

where T_1 and T_2 are given respectively by (2.8) and (2.9).

Thus, according to Lemma 2.9, the solutions of (1.1)-(1.3) can be regarded as fixed points of T.

3. Main results

In this section, we will prove a global existence-uniqueness result for (1.1)-(1.3), to this end, we consider the following assumptions:

(H_1) There exist continuous positive real valued functions $A_i, B_i : i = 1, 2$ defined on $\mathbb{R}_+$, and satisfying

(i) $|f(t, \xi_1, \eta_1) - f(t, \xi_2, \eta_2)| \leq A_1(t)|\xi_1 - \xi_2| + A_2(t)|\eta_1 - \eta_2|$
$|g(t, \xi_1, \eta_1) - g(t, \xi_2, \eta_2)| \leq B_1(t)|\xi_1 - \xi_2| + B_2(t)|\eta_1 - \eta_2|$,
whenever the left hand sides are defined.

(ii) For $\lambda > 0, \mu > 1$, $q := 1 + 1/\alpha$ and $\tilde{q} := 1 + 1/\beta$, we have

$$S_{\lambda,\mu} := \int_0^{+\infty} A_1^q(s) e^{\frac{-q\lambda s}{\mu}} ds < \infty \text{ and } \tilde{S}_{\lambda,\mu} := \int_0^{+\infty} A_2^q(s) e^{\frac{-q\lambda s}{\mu}} ds < \infty$$

$$R_{\lambda,\mu} := \int_0^{+\infty} B_1^{\tilde{q}}(s) e^{\frac{-\tilde{q}\lambda s}{\mu}} ds < \infty \text{ and } \tilde{R}_{\lambda,\mu} := \int_0^{+\infty} B_2^{\tilde{q}}(s) e^{\frac{-\tilde{q}\lambda s}{\mu}} ds < \infty$$

(H_2) There exist fixed compacts $K_i, \tilde{K}_j$ and non negative real numbers $L_i, \tilde{L}_i, M_j, \tilde{M}_j, (1 \leq i \leq l, 1 \leq j \leq m)$, satisfying what follows for every $(u_1, v_1), (u_2, v_2) \in X$:

$$|\varphi(u_1, v_1) - \varphi(u_2, v_2)| \leq \sum_{i=1}^{l} \Big(L_i d_{K_i}(u_1 - u_2) + \tilde{L}_i d_{K_i}(v_1 - v_2) \Big)$$
$$|\psi(u_1, v_1) - \psi(u_2, v_2)| \leq \sum_{j=1}^{m} \Big(M_j d_{\tilde{K}_j}(u_1 - u_2) + \tilde{M}_j d_{\tilde{K}_j}(v_1 - v_2) \Big)$$

(H_3) There exist positive real sequences $\{h_i\}$,$\{\tilde{h}_i\}$,$\{k_i\}$ and $\{\tilde{k}_i\}$ that converge to $H, \tilde{H}, K$ and $\tilde{K}$ respectively and satisfying for every $\xi_1, \xi_2, \eta_1, \eta_2 \in \mathbb{R}$ and $i \in \mathbb{N}^*$, the following estimations:

$$|I_i(\xi_1, \eta_1) - I_i(\xi_2, \eta_2)| \leq h_i |\xi_1 - \xi_2| + \tilde{h}_i |\eta_1 - \eta_2|$$
$$|j_i(\xi_1, \eta_1) - j_i(\xi_2, \eta_2)| \leq k_i |\xi_1 - \xi_2| + \tilde{k}_i |\eta_1 - \eta_2|$$

Remark 3.1. It is not hard to see that hypothesis (H_1) includes as special cases the Lipschitz condition with constant or integrable arguments, widely used in the literature (see for example [9, 20, 19, 5]). This being said, we emphasize here that hypothesis $(H_1.(ii))$ allows the nonlinearity to increase indefinitely with time, which can not be covered by the previous special cases (that is when A_i, B_i are constants or $A_i, B_i \in L^1(\mathbb{R}_+)$). Therefore, our work generalizes and complements many existing results in the literature.

For $\lambda > 0$ and $\mu > 1$, let $M_{\alpha,\beta}(\lambda,\mu)$ be the square matrix defined by:

$$M_{\alpha,\beta}(\lambda,\mu) := \begin{pmatrix} \sum_{i=1}^{l} L_i + \Lambda^{\alpha}_{\lambda,\mu} + H & \sum_{i=1}^{l} \tilde{L}_i + \tilde{\Lambda}^{\alpha}_{\lambda,\mu} + \tilde{H} \\ \sum_{i=1}^{m} M_i + \Lambda^{\beta}_{\lambda,\mu} + K & \sum_{i=1}^{m} \tilde{M}_i + \tilde{\Lambda}^{\beta}_{\lambda,\mu} + \tilde{K} \end{pmatrix} \tag{3.1}$$

Where

$$\begin{aligned} \Lambda^{\alpha}_{\lambda,\mu} &= \tfrac{1}{\Gamma(\alpha)\lambda^{\alpha}} \left(\tfrac{1}{(\alpha+1)^{\alpha^2}}\Gamma(\alpha^2)\right)^{\frac{1}{1+\alpha}} \left(S_{\lambda,\mu}\right)^{\frac{\alpha}{1+\alpha}} \\ \tilde{\Lambda}^{\alpha}_{\lambda,\mu} &= \tfrac{1}{\Gamma(\alpha)\lambda^{\alpha}} \left(\tfrac{1}{(\alpha+1)^{\alpha^2}}\Gamma(\alpha^2)\right)^{\frac{1}{1+\alpha}} \left(\tilde{S}_{\lambda,\mu}\right)^{\frac{\alpha}{1+\alpha}} \\ \Lambda^{\beta}_{\lambda,\mu} &= \tfrac{1}{\Gamma(\beta)\lambda^{\beta}} \left(\tfrac{1}{(\beta+1)^{\beta^2}}\Gamma(\beta^2)\right)^{\frac{1}{1+\beta}} \left(R_{\lambda,\mu}\right)^{\frac{\beta}{1+\beta}} \\ \tilde{\Lambda}^{\beta}_{\lambda,\mu} &= \tfrac{1}{\Gamma(\beta)\lambda^{\beta}} \left(\tfrac{1}{(\beta+1)^{\beta^2}}\Gamma(\beta^2)\right)^{\frac{1}{1+\beta}} \left(\tilde{R}_{\lambda,\mu}\right)^{\frac{\beta}{1+\beta}} \end{aligned} \tag{3.2}$$

Theorem 3.2. Let $(H_1)-(H_3)$ and (2.6) hold true. Then, the system (1.1)-(1.3) admits a unique global solution in X provided that: there exist $\lambda > 0$ and $\mu > 1$ such that

$$\text{The matrix } M_{\alpha,\beta}(\lambda,\mu) \text{ given in (3.1), converges to zero.} \tag{3.3}$$

Proof. Recall that the solutions of (1.1)-(1.3) are the fixed points of the operator T defined in (2.10). We shall prove that T is a generalized contraction in the sens of Definition 2.7, to deduce the result from Theorem 2.8. To this end, let us define a mapping $w : \mathcal{N} \longrightarrow \mathcal{N}$ as follows:

$$w(\nu) = \left[0, \max_{1\le i\le l, 1\le j\le n} \{\nu^m, K_i^m, \tilde{K}_j^m\}\right], \tag{3.4}$$

where ν^m denotes $\max \nu$ and K_i, $\tilde{K}_j$ are the compacts given by (H_2).

Note that according to (3.4), it follows that

$$\text{For every } \nu \in \mathcal{N} : w^n(\nu) = w(\nu), \quad \forall n \ge 2 \tag{3.5}$$

Let $\nu \in \mathcal{N}$ and $t \in \nu$. Using $(H_1(i))$, (H_2), (H_3), we get:

$$|T_1(u_1,v_1)(t) - T_1(u_2,v_2)(t)| \le$$

$$\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \{A_1(s)|u_1(s)-u_2(s)| + A_2(s)|v_1(s)-v_2(s)|\} ds$$

$$+\sum_{t_i<t} \left\{h_i|u_1(t_i)-u_2(t_i)| + \tilde{h}_i|v_1(t_i)-v_2(t_i)|\right\}$$

$$+\sum_{i=1}^{l} \left\{L_i d_{K_i}(u_1-u_2) + \tilde{L}_i d_{K_i}(v_1-v_2)\right\}$$

$$\leq \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \left\{ A_1(s) e^{\lambda s \frac{\mu-1}{\mu}} \max_{\sigma\in[0,t]} e^{-\lambda\sigma} \left| u_1(\sigma) - u_2(\sigma) \right| \right.$$

$$\left. + A_2(s)\, e^{\lambda s \frac{\mu-1}{\mu}} \max_{\sigma\in[0,t]} e^{-\lambda\sigma} \left| v_1(\sigma) - v_2(\sigma) \right| \right\} ds$$

$$+ \sum_{i=1}^{l} \left\{ L_i d_{K_i}(u_1 - u_2) + \tilde{L}_i d_{K_i}(v_1 - v_2) \right\}$$

$$+ H e^{\lambda t} \max_{\sigma\in[0,t]} e^{-\lambda\sigma} \left| u_1(\sigma) - u_2(\sigma) \right| + \tilde{H} e^{\lambda t} \max_{\sigma\in[0,t]} e^{-\lambda\sigma} \left| v_1(\sigma) - v_2(\sigma) \right|,$$

where λ is the positive parameter introduced in (2.4) and $\mu > 1$. Note that according to (3.4), the compacts $[0,t]$ and K_i $(1 \leq i \leq l)$ are included in $w(\nu)$. Hence

$$|T_1(u_1,v_1)(t) - T_1(u_2,v_2)(t)| \leq \left\{ \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} A_1(s) e^{\lambda s \frac{\mu-1}{\mu}} ds \right\} d_{w(\nu)}(u_1 - u_2)$$

$$+ \left\{ \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} A_2(s) e^{\lambda s \frac{\mu-1}{\mu}} ds \right\} d_{w(\nu)}(v_1 - v_2)$$

$$+ \sum_{i=1}^{l} \left\{ L_i d_{w(\nu)}(u_1 - u_2) + \tilde{L}_i d_{w(\nu)}(v_1 - v_2) \right\}$$

$$+ H e^{\lambda t} d_{w(\nu)}(u_1 - u_2) + \tilde{H} e^{\lambda t} d_{w(\nu)}(v_1 - v_2)$$

Now, multiplying the above inequality by $e^{-\lambda t}$, we get:

$$e^{-\lambda t} |T_1(u_1,v_1)(t) - T_1(u_2,v_2)(t)| \leq$$

$$\left\{ \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} e^{-\lambda(t - s\frac{\mu-1}{\mu})} A_1(s) ds \right\} d_{w(\nu)}(u_1 - u_2)$$

$$+ \left\{ \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} e^{-\lambda(t - s\frac{\mu-1}{\mu})} A_2(s) ds \right\} d_{w(\nu)}(v_1 - v_2) \tag{3.6}$$

$$+ \sum_{i=1}^{l} \left\{ L_i d_{w(\nu)}(u_1 - u_2) + \tilde{L}_i d_{w(\nu)}(v_1 - v_2) \right\}$$

$$+ H d_{w(\nu)}(u_1 - u_2) + \tilde{H} d_{w(\nu)}(v_1 - v_2)$$

Let us find estimates for the integrals in (3.6):

$$I := \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} e^{-\lambda(t - s\frac{\mu-1}{\mu})} A_1(s) ds = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} e^{-\lambda(t-s)} A_1(s) e^{\frac{-\lambda s}{\mu}} ds$$

Performing the change of variable $X = \lambda(t-s)$, we get:

$$I = \frac{1}{\Gamma(\alpha)\lambda^{\alpha}} \int_0^{\lambda t} X^{\alpha-1} e^{-X} A_1\left(t - \frac{X}{\lambda}\right) e^{-\frac{\lambda}{\mu}(t - \frac{X}{\lambda})} dX$$

In view of $(H_1.(ii))$, Hölder's inequality gives:

$$I \leq \frac{1}{\Gamma(\alpha)\lambda^\alpha}\left\{\int_0^{\lambda t}\left(X^{\alpha-1}e^{-X}\right)^{1+\alpha}dX\right\}^{\frac{1}{1+\alpha}} \times$$

$$\left\{\int_0^{\lambda t}\left(A_1\left(t-\frac{X}{\lambda}\right)e^{-\frac{\lambda}{\mu}(t-\frac{X}{\lambda})}\right)^{1+\frac{1}{\alpha}}dX\right\}^{\frac{\alpha}{1+\alpha}}$$

Consequently:

$$\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}e^{-\lambda(t-s\frac{\mu-1}{\mu})}A_1(s)ds \leq \Lambda^\alpha_{\lambda,\mu} \tag{3.7}$$

In the same way, we can prove that

$$\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}e^{-\lambda(t-s\frac{\mu-1}{\mu})}A_2(s)ds \leq \tilde{\Lambda}^\alpha_{\lambda,\mu} \tag{3.8}$$

In view of (3.7) and (3.8), and after taking the maximum on ν, the estimation (3.6) can be rewritten as:

$$\begin{aligned} d_\nu\left(T_1(u_1,v_1),T_1(u_2,v_2)\right) \leq{}& \Lambda^\alpha_{\lambda,\mu}\, d_{w(\nu)}(u_1-u_2)+\tilde{\Lambda}^\alpha_{\lambda,\mu}\, d_{w(\nu)}(v_1-v_2) \\ &+\sum_{i=1}^{l}\left\{L_i d_{w(\nu)}(u_1-u_2)+\tilde{L}_i d_{w(\nu)}(v_1-v_2)\right\} \\ &+ H d_{w(\nu)}(u_1-u_2)+\tilde{H} d_{w(\nu)}(v_1-v_2) \end{aligned} \tag{3.9}$$

Similarly, we prove that the following inequality holds true for every $(u_1,v_1),(u_2,v_2) \in X$ and every $\nu \in \mathcal{N}$:

$$\begin{aligned} d_\nu\left(T_2(u_1,v_1),T_2(u_2,v_2)\right) \leq{}& \Lambda^\beta_{\lambda,\mu}\, d_{w(\nu)}(u_1-u_2)+\tilde{\Lambda}^\beta_{\lambda,\mu}\, d_{w(\nu)}(v_1-v_2) \\ &+\sum_{i=1}^{m}\left\{M_i d_{w(\nu)}(u_1-u_2)+\tilde{M}_i d_{w(\nu)}(v_1-v_2)\right\} \\ &+ H d_{w(\nu)}(u_1-u_2)+\tilde{K} d_{w(\nu)}(v_1-v_2) \end{aligned} \tag{3.10}$$

Now, (3.9) together with (3.10) lead to what follows for every $(u_1,v_1),(u_2,v_2) \in X$ and every $\nu \in \mathcal{N}$:

$$D_\nu\left(T(u_1,v_1),T(u_2,v_2)\right) \leq M_{\alpha,\beta}\left(\lambda,\mu\right) D_{w(\nu)}\left(\left(u_1,u_2\right),\left(v_1,v_2\right)\right) \tag{3.11}$$

That is (2.1) holds true with $M_\nu = M_{\alpha,\beta}\left(\lambda,\mu\right)$, which is independent of ν. Consequently the series (2.2) turns in our case into

$$\sum_{n=0}^{\infty} M^{n+1}_{\alpha,\beta}\left(\lambda,\mu\right) D_{w^n(\nu)}\left(u,v\right) \tag{3.12}$$

According to (3.5), we have:

$$sup\left\{D_{w^n(\nu)}\left(u,v\right): n=0,1,2,\dots\right\} = sup\left\{D_\nu\left(u,v\right), D_{w(\nu)}\left(u,v\right)\right\} < \infty.$$

Since, moreover $M_{\alpha,\beta}(\lambda,\mu)$ is convergent to zero, then the series in (3.12) converges too. That is T is a generalized contraction and the result follows so, from Theorem 2.8 □

Remark 3.3. In view of Lemme 2.5, the following condition is equivalent to (3.3)

$$\sqrt{\left(\sum_{i=1}^{l} L_i+\Lambda^{\alpha}_{\lambda,\mu}+H-\sum_{i=1}^{m}\tilde{M}_i-\tilde{\Lambda}^{\beta}_{\lambda,\mu}-\tilde{K}\right)^2+4\left(\sum_{i=1}^{l}\tilde{L}_i+\tilde{\Lambda}^{\alpha}_{\lambda,\mu}+\tilde{H}\right)\left(\sum_{i=1}^{m}M_i+\Lambda^{\beta}_{\lambda,\mu}+K\right)} \\ +\sum_{i=1}^{l} L_i+\Lambda^{\alpha}_{\lambda,\mu}+H+\sum_{i=1}^{m}\tilde{M}_i+\tilde{\Lambda}^{\beta}_{\lambda,\mu}+\tilde{K}<2 \tag{3.13}$$

The following Corollary provides a global existence-uniqueness result for a particular class of (1.1)-(1.3).

Corollary 3.4. Assume that in addition of $(H_1)-(H_3)$ and (2.6), the following hypothesis holds true:

$$\forall\epsilon>0,\ \exists\lambda>0,\ \mu>1,\text{ such that: } S_{\lambda,\mu},\ \tilde{S}_{\lambda,\mu},\ R_{\lambda,\mu},\ \tilde{R}_{\lambda,\mu}<\epsilon. \tag{3.14}$$

Then, (1.1)-(1.3) admits a unique global solution in X provided that:

$$Q:=\begin{pmatrix} \sum_{i=1}^{l} L_i+H & \sum_{i=1}^{l}\tilde{L}_i+\tilde{H} \\ \\ \sum_{i=1}^{m} M_i+K & \sum_{i=1}^{m}\tilde{M}_i+\tilde{K}\end{pmatrix},\ \text{converges to zero} \tag{3.15}$$

Proof. Note first that $M_{\alpha,\beta}(\lambda,\mu)=P_{\alpha,\beta}(\lambda,\mu)+Q$, where

$$P_{\alpha,\beta}(\lambda,\mu):=\begin{pmatrix}\Lambda^{\alpha}_{\lambda,\mu} & \tilde{\Lambda}^{\alpha}_{\lambda,\mu} \\ \\ \Lambda^{\beta}_{\lambda,\mu} & \tilde{\Lambda}^{\beta}_{\lambda,\mu}\end{pmatrix}$$

It is not hard to see that under hypothesis (3.14), the elements of $P_{\alpha,\beta}(\lambda,\mu)$ are small enough.

Hence, in view of (3.15) together with Lemma 2.6, $M_{\alpha,\beta}(\lambda,\mu)$ is convergent to zero and the result follows so from Theorem 3.2. □

When $I_i=J_i=0$ for every $i\in\mathbb{N}^*$, that is by omitting the impulsive condition (1.3), then the problem (1.1)-(1.3) is reduced to:

$$\begin{cases} {}^{C}D^{\alpha}_{0^+}u(t)=f(t,u(t),v(t)), & t>0\\ {}^{C}D^{\beta}_{0^+}v(t)=g(t,u(t),v(t)), & t>0\\ u(0)=\varphi(u,v), \\ v(0)=\psi(u,v), \end{cases} \tag{3.16}$$

In this particular case, we have:

$$\Delta u(t_i):=u(t_i^+)-u(t_i^-)=I_i(u(t_i),v(t_i))=0$$

and

$$\Delta v(t_i):=v(t_i^+)-v(t_i^-)=J_i(u(t_i),v(t_i))=0$$

Which means that the space $X = \mathcal{PC}(\mathbb{R}_+) \times \mathcal{PC}(\mathbb{R}_+)$, where $\mathcal{PC}(\mathbb{R}_+)$ is defined by (2.3), becomes $\mathcal{C}(\mathbb{R}_+) \times \mathcal{C}(\mathbb{R}_+)$. So, as particular cases of Theorem 3.2 and Corollary 3.4, we have the following Corollary.

Corollary 3.5. Under hypotheses $(H_1) - (H_2)$, the system (3.16) admits a unique global solution in $\mathcal{C}(\mathbb{R}_+) \times \mathcal{C}(\mathbb{R}_+)$ provided that: there exist $\lambda > 0$ and $\mu > 1$ such that:

$$\tilde{M}_{\alpha,\beta}(\lambda,\mu) = \begin{pmatrix} \sum_{i=1}^{l} L_i + \Lambda^{\alpha}_{\lambda,\mu} & \sum_{i=1}^{l} \tilde{L}_i + \tilde{\Lambda}^{\alpha}_{\lambda,\mu} \\ \sum_{i=1}^{m} M_i + \Lambda^{\beta}_{\lambda,\mu} & \sum_{i=1}^{m} \tilde{M}_i + \tilde{\Lambda}^{\beta}_{\lambda,\mu} \end{pmatrix} \text{ converges to zero} \tag{3.17}$$

where $\Lambda^{\alpha}_{\lambda,\mu}, \tilde{\Lambda}^{\alpha}_{\lambda,\mu}, \Lambda^{\beta}_{\lambda,\mu}, \tilde{\Lambda}^{\beta}_{\lambda,\mu}$ are given by (3.2). If in addition, (3.14) holds true, then (3.17) is weakened to

$$\tilde{Q} := \begin{pmatrix} \sum_{i=1}^{l} L_i & \sum_{i=1}^{l} \tilde{L}_i \\ \sum_{i=1}^{m} M_i & \sum_{i=1}^{m} \tilde{M}_i \end{pmatrix} \text{ converges to zero.} \tag{3.18}$$

Remark 3.6. Note that (3.14) includes the Lipschitz condition with constant and integrable arguments. In this case, the matrices Q in (3.15) and $\tilde{Q}$ in (3.18) are independent of A_i, B_i $(i = 1, 2)$. Moreover, with the classical initial conditions, $\sum_{i=1}^{l} L_i$, $\sum_{i=1}^{l} \tilde{L}_i$, $\sum_{i=1}^{m} M_i$ and $\sum_{i=1}^{m} \tilde{M}_i$ vanish. All this, allows us to see clearly that Corollary 3.4 and Corollary 3.5 provide significant improvements and generalizations of many recent results in the literature, such as [9, Theorem 15], [19, Theorem 3.3], [20, Theorem 3.1], [20, Theorem 3.2] and [5, Theorem 3.2].

4. Example

Let us consider the following system:

$$\begin{cases} {}^{C}D^{\frac{15}{20}}_{0^+} u(t) = \dfrac{1}{10} e^{\frac{t}{80}} \left(2u(t) + v(t)\right), & t > 0,\ t \neq t_i = 10^i,\ i \in \mathbb{N}^* \\ {}^{C}D^{\frac{13}{20}}_{0^+} v(t) = \dfrac{1}{10} e^{\frac{t}{80}} \left(u(t) + v(t)\right), & t > 0,\ t \neq t_i = 10^i,\ i \in \mathbb{N}^* \\ \Delta u(t_i) = \frac{7}{25i(i+1)(1+|u(t_i)|)} + \frac{5}{25i(i+1)(1+|v(t_i)|)}, & i \in \mathbb{N}^* \\ \Delta v(t_i) = \frac{6}{25\times 2^i(1+|u(t_i)|)} + \frac{9}{25\times 2^i(1+|v(t_i)|)}, & i \in \mathbb{N}^* \\ u(0) = \frac{1}{10} \sup\limits_{t\in[0,\,1]} u(t) + \frac{1}{5} \sup\limits_{t\in[0,\,\frac{1}{2}]} v(t) \\ v(0) = \frac{1}{5} sin\left(u(\frac{1}{6}) + v(\frac{1}{3})\right) \end{cases} \tag{4.1}$$

The problem (4.1) is identified to (1.1)-(1.3), with:

$$\alpha = \tfrac{15}{20}, f\left(t, \xi, \eta\right) = \frac{1}{10} e^{\frac{t}{80}} \left(2\xi + \eta\right),$$

$$I_i\left(\xi, \eta\right) = \frac{7}{25i(i+1)\left(1 + |\xi|\right)} + \frac{5}{25i(i+1)\left(1 + |\eta|\right)}$$

$$\beta = \tfrac{13}{20}, g\left(t, \xi, \eta\right) = \frac{1}{10} e^{\frac{t}{80}} \left(\xi + \eta\right),$$

$$J_i\left(\xi, \eta\right) = \frac{6}{25 \times 2^i\left(1 + |\xi|\right)} + \frac{9}{25 \times 2^i\left(1 + |\eta|\right)}$$

$$\varphi(u, v) = \tfrac{1}{10} \sup_{t \in [0,\, 1]} u\left(t\right) + \tfrac{1}{5} \sup_{t \in [0,\, \frac{1}{2}]} v\left(t\right), \;\; \psi(u, v) = \tfrac{1}{5} sin\left(u(\tfrac{1}{6}) + v(\tfrac{1}{3})\right)$$

It is not hard to see that $(H_1.(i))$ is satisfied with:

$$A_1(t) = \frac{1}{5} e^{\frac{t}{80}}, \;\; A_2(t) = B_1(t) = B_2(t) = \frac{1}{10} e^{\frac{t}{80}}$$

A straightforward computation leads to:

$$S_{\lambda,\mu} = \frac{7\lambda}{30\mu}, \tilde{S}_{\lambda,\mu} = \frac{\lambda}{10\mu}, R_{\lambda,\mu} = \tilde{R}_{\lambda,\mu} = \frac{23\lambda}{260\mu}$$

Which means that $(H_1.(ii))$ is satisfied too.
It can be easily seen that (H_2) is satisfied with:

$$l = 2, L_1 = \frac{1}{10} e^{\lambda}, L_2 = 0, \tilde{L}_1 = 0, \tilde{L}_2 = \frac{1}{5} e^{\frac{\lambda}{2}}, K_1 = [0, 1], K_2 = [0, \tfrac{1}{2}]$$

$$m = 2, M_1 = \frac{1}{10} e^{\frac{\lambda}{6}}, M_2 = 0, \tilde{M}_1 = 0, \tilde{M}_2 = \frac{1}{5} e^{\frac{\lambda}{3}}, \tilde{K}_1 = [0, \tfrac{1}{6}], \;\tilde{K}_2 = [0, \tfrac{1}{3}]$$

For all $i \in \mathbb{N}^*$ we have:

$$\left|I_i\left(\xi_1, \eta_1\right) - I_i\left(\xi_2, \eta_2\right)\right| \leq \frac{7}{25i(i+1)} |\xi_1 - \xi_2| + \frac{5}{25i(i+1)} |\eta_1 - \eta_2|$$

$$\left|J_i\left(\xi_1, \eta_1\right) - J_i\left(\xi_2, \eta_2\right)\right| \leq \frac{6}{25 \times 2^i} |\xi_1 - \xi_2| + \frac{9}{25 \times 2^i} |\eta_1 - \eta_2|$$

That is, (H_3) is satisfied with:

$$\{h_i\} = \left\{\tfrac{7}{25i(i+1)}\right\}, \{\tilde{h}_i\} = \left\{\tfrac{5}{25i(i+1)}\right\}, \{k_i\} = \left\{\tfrac{6}{25 \times 2^i}\right\}, \{\tilde{k}_i\} = \left\{\tfrac{9}{25 \times 2^i}\right\}$$

$$H = \tfrac{7}{25}, \;\; \tilde{H} = \tfrac{5}{25}, \;\; K = \tfrac{6}{25}, \tilde{K} = \tfrac{9}{25}$$

If we choose $\lambda = \frac{1}{2}$ and $\mu = 20$, the matrix $M_{\alpha,\beta}\left(\lambda, \mu\right)$ given in (3.1), becomes in this case:

$$M_{\alpha,\beta}\left(\lambda, \mu\right) := \begin{pmatrix} 0.486788 & 0.498721 \\ \\ 0.474757 & 0.495513 \end{pmatrix},$$

which admits the following eigenvalues: $\lambda_1 = 0.977761 < 1$ and $\lambda_2 = 0.00453945 < 1$ and consequently $M_{\alpha,\beta}(\lambda, \mu)$ converges to zero.

Hence, all conditions of Theorem 3.2 are fulfilled, and therefore the system (4.1) admits a unique global solution in $\mathcal{PC}(\mathbb{R}_+) \times \mathcal{PC}(\mathbb{R}_+)$.

Note that f and g in (4.1) increase indefinitely with time, and therefore many existing results in the literature fail to be applicable.

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