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Starlike and convex properties for Poisson distribution series

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Abstract. In this paper, we find the necessary and sufficient conditions, inclusion relations for Poisson distribution series belonging to the classes $\mathscr{S}^*(\alpha,\beta)$ and $\mathscr{C}^*(\alpha,\beta)$. Further, we consider an integral operator related to Poisson Distribution series.

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1. Introduction

Let $\mathscr{A}$ denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

which are analytic and univalent in the open disc $\mathbb{U} = \{z : z \in \mathbb{C}\ |z| < 1\}$. Let $\mathscr{T}$ be a subclass of $\mathscr{A}$ consisting of functions whose non-zero coefficients from second on is give by

$$f(z) = z - \sum_{n=2}^{\infty} |a_n| z^n, \qquad z \in \mathbb{U}. \tag{1.2}$$

In 2014, Porwal [4] introduced a power series whose coefficients are probabilities of Poisson distribution

$$\mathscr{K}(m,z) := z + \sum_{n=2}^{\infty} \frac{m^{n-1}}{(n-1)!} e^{-m} z^n, \quad z \in \mathbb{U},$$

where $m > 0$. By ratio test the radius of convergence of the above series is infinity. Further, Porwal [4] defined a series

$$\mathscr{F}(m,z) \quad = \quad 2z - \mathscr{K}(m,z) = z - \sum_{n=2}^{\infty} \frac{m^{n-1}}{(n-1)!} e^{-m} z^n, \qquad z \in \mathbb{U}.$$

Corresponding to the series $\mathscr{K}(m,z)$ using the Hadamard product for $f \in \mathscr{A}$, Porwal and Kumar [5] introduced a new linear operator $\mathscr{I}(m) : \mathscr{A} \to \mathscr{A}$ defined by

$$\begin{aligned}\mathscr{I}(m)f(z): &= \mathscr{K}(m,z) * f(z)\\ &= z + \sum_{n=2}^{\infty} \frac{m^{n-1}}{(n-1)!} e^{-m} a_n z^n, \qquad z \in \mathbb{U},\end{aligned}$$

where $*$ denotes the convolution (or Hadamard product) of two series

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \quad \text{and} \quad g(z) = \sum_{n=0}^{\infty} b_n z^n$$

is defined by

$$(f * g)(z) = \sum_{n=0}^{\infty} a_n b_n z^n.$$

Let $\mathscr{S}^*(\alpha,\beta)$ be the subclass of $\mathscr{T}$ consisting of functions which satisfy the condition:

$$\left| \frac{\frac{zf'(z)}{f(z)} - 1}{\frac{zf'(z)}{f(z)} + 1 - 2\alpha} \right| < \beta, \qquad z \in \mathbb{U},$$

where $0 \le \alpha < 1$ and $0 < \beta \le 1$.

Also, let $\mathscr{C}^*(\alpha,\beta)$ be the subclass of $\mathscr{T}$ consisting of functions which satisfy the condition:

$$\left| \frac{\frac{zf''(z)}{f'(z)}}{\frac{zf''(z)}{f'(z)} + 2(1-\alpha)} \right| < \beta, \qquad z \in \mathbb{U},$$

where $0 \le \alpha < 1$ and $0 < \beta \le 1$.

The classes $\mathscr{S}^*(\alpha,\beta)$ and $\mathscr{C}^*(\alpha,\beta)$, were introduced and studied by Gupta and Jain [2] (see [3]). Also, we note that for $\beta = 1$ the classes $\mathscr{S}^*(\alpha,\beta)$ and $\mathscr{C}^*(\alpha,\beta)$ reduce to the class of starlike and convex functions of order $\alpha (0 \le \alpha < 1)$ (see [6]).

A function $f \in \mathscr{A}$ is said to be in the class $\mathscr{R}^\tau(A,B)$, ($\tau \in \mathbb{C} \setminus \{0\}$, $-1 \le B < A \le 1$), if it satisfies the inequality

$$\left| \frac{f'(z) - 1}{(A-B)\tau - B[f'(z) - 1]} \right| < 1, \qquad z \in \mathbb{U}.$$

This class was introduced by Dixit and Pal [1].

Lemma 1.1. [2] *A function $f(z)$ of the form* (1.2) *is in $\mathscr{S}^*(\alpha,\beta)$ if and only if*

$$\sum_{n=2}^{\infty} [n(1+\beta) - 1 + \beta(1-2\alpha)]\ |a_n| \le 2\beta(1-\alpha). \tag{1.3}$$

Lemma 1.2. [2] *A function $f(z)$ of the form* (1.2) *is in $\mathscr{C}^*(\alpha,\beta)$ if and only if*

$$\sum_{n=2}^{\infty} n[n(1+\beta) - 1 + \beta(1-2\alpha)]\ |a_n| \le 2\beta(1-\alpha). \tag{1.4}$$

To obtain our main results, we need the following lemmas:

Lemma 1.3. [1] *If* $f \in \mathscr{R}^\tau(A,B)$ *is of the form* (1.1)*, then*

$$|a_n| \le (A-B)\frac{|\tau|}{n}, \qquad n \in \mathbb{N}\setminus\{1\}. \tag{1.5}$$

In the present investigation, inspired by the works of Porwal [4] and Porwal and Kumar [5], we find the necessary and sufficient conditions for $\mathscr{F}(m,z)$ belonging to the classes $\mathscr{S}^*(\alpha,\beta)$ and $\mathscr{C}^*(\alpha,\beta)$. Also, we obtain inclusion relations for aforecited classes with $\mathscr{R}^\tau(A,B)$.

2. Necessary and sufficient conditions

Theorem 2.1. *If* $m>0$, $0\le\alpha<1$ *and* $0<\beta\le 1$, *then* $\mathscr{F}(m,z)\in\mathscr{S}^*(\alpha,\beta)$ *if and only if*

$$e^m m(1+\beta) \le 2\beta(1-\alpha). \tag{2.1}$$

Proof. Since

$$\mathscr{F}(m,z) = z - \sum_{n=2}^{\infty}\frac{m^{n-1}}{(n-1)!}e^{-m}z^n,$$

in view of Lemma 1.1, it is enough to show that

$$\sum_{n=2}^{\infty}[n(1+\beta)-1+\beta(1-2\alpha)]\,\frac{m^{n-1}}{(n-1)!}e^{-m} \le 2\beta(1-\alpha).$$

Let

$$T_1 = \sum_{n=2}^{\infty}[n(1+\beta)-1+\beta(1-2\alpha)]\,\frac{m^{n-1}}{(n-1)!}e^{-m}.$$

Now,

$$\begin{aligned}
T_1 &= \sum_{n=2}^{\infty}[n(1+\beta)-1+\beta(1-2\alpha)]\,\frac{m^{n-1}}{(n-1)!}e^{-m}\\
&= e^{-m}\sum_{n=2}^{\infty}[(n-1)(1+\beta)+2\beta(1-\alpha)]\,\frac{m^{n-1}}{(n-1)!}\\
&= e^{-m}\left[(1+\beta)\sum_{n=2}^{\infty}\frac{m^{n-1}}{(n-2)!}+2\beta(1-\alpha)\sum_{n=2}^{\infty}\frac{m^{n-1}}{(n-1)!}\right]\\
&= e^{-m}\left[(1+\beta)me^m+2\beta(1-\alpha)(e^m-1)\right]\\
&= (1+\beta)m+2\beta(1-\alpha)(1-e^{-m}).
\end{aligned}$$

But this last expression is bounded by $2\beta(1-\alpha)$, if and only if (2.1) holds. This completes the proof of Theorem 2.1. □

Theorem 2.2. *If* $m>0$, $0\le\alpha<1$ *and* $0<\beta\le 1$, *then* $\mathscr{F}(m,z)\in\mathscr{C}^*(\alpha,\beta)$ *if and only if*

$$e^m\left[(1+\beta)m^2+2(1+\beta(2-\alpha))m\right] \le 2\beta(1-\alpha). \tag{2.2}$$

Proof. Since

$$\mathscr{F}(m,z) = z - \sum_{n=2}^{\infty} \frac{m^{n-1}}{(n-1)!} e^{-m} z^n,$$

in view of Lemma 1.2, it is enough to show that

$$\sum_{n=2}^{\infty} n[n(1+\beta) - 1 + \beta(1-2\alpha)] \frac{m^{n-1}}{(n-1)!} e^{-m} \leq 2\beta(1-\alpha).$$

Let

$$T_2 = \sum_{n=2}^{\infty} n[n(1+\beta) - 1 + \beta(1-2\alpha)] \frac{m^{n-1}}{(n-1)!} e^{-m}.$$

Therefore,

$$\begin{aligned} T_2 &= e^{-m} \left[\sum_{n=2}^{\infty} (n-1)(n-2)(1+\beta) \frac{m^{n-1}}{(n-1)!} \right. \\ &\quad \left. + \sum_{n=2}^{\infty} (n-1)[3(1+\beta) - 1 + \beta(1-2\alpha)] \frac{m^{n-1}}{(n-1)!} + \sum_{n=2}^{\infty} 2\beta(1-\alpha) \frac{m^{n-1}}{(n-1)!} \right] \\ &= e^{-m} \left[(1+\beta) \sum_{n=3}^{\infty} \frac{m^{n-1}}{(n-3)!} \right. \\ &\quad \left. + 2[1+\beta(2-\alpha)] \sum_{n=2}^{\infty} \frac{m^{n-1}}{(n-2)!} + 2\beta(1-\alpha) \sum_{n=2}^{\infty} \frac{m^{n-1}}{(n-1)!} \right] \\ &= e^{-m} \left[(1+\beta) m^2 e^m + 2(1+\beta(2-\alpha)) m e^m + 2\beta(1-\alpha)(e^m - 1) \right] \\ &= (1+\beta) m^2 + 2(1+\beta(2-\alpha)) m + 2\beta(1-\alpha)(1-e^{-m}). \end{aligned}$$

But this last expression is bounded by $2\beta(1-\alpha)$, if and only if (2.2) holds. This completes the proof of Theorem 2.2. □

3. Inclusion results

Theorem 3.1. *Let $m > 0$, $0 \leq \alpha < 1$ and $0 < \beta \leq 1$. If $f \in \mathscr{R}^\tau(A,B)$, then $\mathscr{I}(m)f \in \mathscr{S}^*(\alpha,\beta)$ if and only if*

$$(A-B)|\tau| \left[(1+\beta)(1-e^{-m}) + \frac{(\beta(1-2\alpha)-1)}{m}(1-e^{-m}-me^{-m}) \right] \leq 2\beta(1-\alpha). \tag{3.1}$$

Proof. In view of Lemma 1.1, it suffices to show that

$$P_1 = \sum_{n=2}^{\infty} [n(1+\beta) - 1 + \beta(1-2\alpha)] \frac{m^{n-1}}{(n-1)!} e^{-m} |a_n| \leq 2\beta(1-\alpha).$$

Since $f \in \mathscr{R}^{\tau}(A,B)$, then by Lemma 1.3, we have

$$|a_n| \leq \frac{(A-B)|\tau|}{n}.$$

Therefore,

$$\begin{aligned}
P_1 &\leq \sum_{n=2}^{\infty}[n(1+\beta)-1+\beta(1-2\alpha)]\ \frac{m^{n-1}}{(n-1)!}e^{-m}\frac{(A-B)|\tau|}{n} \\
&= (A-B)|\tau|e^{-m}\sum_{n=2}^{\infty}[n(1+\beta)-1+\beta(1-2\alpha)]\ \frac{m^{n-1}}{n!} \\
&= (A-B)|\tau|e^{-m}\left[(1+\beta)\sum_{n=2}^{\infty}\ \frac{m^{n-1}}{(n-1)!}+\frac{(\beta(1-2\alpha)-1)}{m}\sum_{n=2}^{\infty}\frac{m^n}{n!}\right] \\
&= (A-B)|\tau|e^{-m}\left[(1+\beta)(e^m-1)+\frac{(\beta(1-2\alpha)-1)}{m}(e^m-1-m)\right] \\
&= (A-B)|\tau|\left[(1+\beta)[1-e^{-m}]+\frac{(\beta(1-2\alpha)-1)}{m}(1-e^{-m}-me^{-m})\right].
\end{aligned}$$

But this last expression is bounded by $2\beta(1-\alpha)$, if (3.1) holds. This completes the proof of Theorem 3.1. □

Theorem 3.2. *Let $m > 0$, $0 \leq \alpha < 1$ and $0 < \beta \leq 1$. If $f \in \mathscr{R}^{\tau}(A,B)$, then $\mathscr{I}(m)f \in \mathscr{C}^*(\alpha,\beta)$ if and only if*

$$(A-B)|\tau|\left[m(1+\beta)+2\beta(1-\alpha)(1-e^{-m})\right] \leq 2\beta(1-\alpha). \tag{3.2}$$

Proof. In view of Lemma 1.2, it suffices to show that

$$P_2 = \sum_{n=2}^{\infty} n[n(1+\beta)-1+\beta(1-2\alpha)]\ \frac{m^{n-1}}{(n-1)!}e^{-m}|a_n| \leq 2\beta(1-\alpha).$$

Since $f \in \mathscr{R}^{\tau}(A,B)$, then by Lemma 1.3, we have

$$|a_n| \leq \frac{(A-B)|\tau|}{n}.$$

Therefore,

$$\begin{aligned} P_2 &\leq \sum_{n=2}^{\infty} n[n(1+\beta)-1+\beta(1-2\alpha)]\ \frac{m^{n-1}}{(n-1)!}e^{-m}\frac{(A-B)|\tau|}{n} \\ &= (A-B)|\tau|e^{-m}\sum_{n=2}^{\infty}[n(1+\beta)-1+\beta(1-2\alpha)]\ \frac{m^{n-1}}{(n-1)!} \\ &= (A-B)|\tau|e^{-m}\sum_{n=2}^{\infty}[(n-1)(1+\beta)+2\beta(1-\alpha)]\ \frac{m^{n-1}}{(n-1)!} \\ &= (A-B)|\tau|e^{-m}\left[\sum_{n=2}^{\infty}(1+\beta)\ \frac{m^{n-1}}{(n-2)!}+2\beta(1-\alpha)\sum_{n=2}^{\infty}\frac{m^{n-1}}{(n-1)!}\right] \\ &= (A-B)|\tau|e^{-m}\left[(1+\beta)\sum_{n=2}^{\infty}\ \frac{m^{n-1}}{(n-2)!}+2\beta(1-\alpha)\sum_{n=2}^{\infty}\frac{m^{n-1}}{(n-1)!}\right] \\ &= (A-B)|\tau|e^{-m}\left[me^m(1+\beta)+2\beta(1-\alpha)(e^m-1)\right]. \end{aligned}$$

But this last expression is bounded by $2\beta(1-\alpha)$, if (3.2) holds. This completes the proof of Theorem 3.2. □

4. An integral operator

Theorem 4.1. *If $m > 0$, $0 \leq \alpha < 1$ and $0 < \beta \leq 1$, then*

$$\mathscr{G}(m,z) = \int_0^z \frac{\mathscr{F}(m,t)}{t}dt$$

is in $\mathscr{C}^*(\alpha,\beta)$ if and only if inequality (2.1) is satisfied.

Proof. Since

$$\mathscr{G}(m,z) = z - \sum_{n=2}^{\infty}\frac{e^{-m}m^{n-1}}{(n-1)!}\frac{z^n}{n} = z - \sum_{n=2}^{\infty}\frac{e^{-m}m^{n-1}}{n!}z^n$$

by Lemma 1.2, we need only to show that

$$\sum_{n=2}^{\infty} n[n(1+\beta)-1+\beta(1-2\alpha)]\ \frac{m^{n-1}}{n!}e^{-m} \leq 2\beta(1-\alpha).$$

Let

$$Q_1 = \sum_{n=2}^{\infty} n[n(1+\beta)-1+\beta(1-2\alpha)]\ \frac{m^{n-1}}{n!}e^{-m}.$$

Now,

$$\begin{aligned} Q_1 &= \sum_{n=2}^{\infty}[n(1+\beta)-1+\beta(1-2\alpha)]\ \frac{m^{n-1}}{(n-1)!}e^{-m} \\ &= e^{-m}\sum_{n=2}^{\infty}[(n-1)(1+\beta)+2\beta(1-\alpha)]\ \frac{m^{n-1}}{(n-1)!} \\ &= e^{-m}\left[\sum_{n=2}^{\infty}(n-1)(1+\beta)\ \frac{m^{n-1}}{(n-1)!}+\sum_{n=2}^{\infty}2\beta(1-\alpha)\ \frac{m^{n-1}}{(n-1)!}\right] \\ &= e^{-m}\left[(1+\beta)\sum_{n=2}^{\infty}\ \frac{m^{n-1}}{(n-2)!}+2\beta(1-\alpha)\sum_{n=2}^{\infty}\ \frac{m^{n-1}}{(n-1)!}\right] \\ &= e^{-m}\left[(1+\beta)me^{m}+2\beta(1-\alpha)(e^{m}-1)\right] \\ &= (1+\beta)m+2\beta(1-\alpha)(1-e^{-m}). \end{aligned}$$

But this last expression is bounded by $2\beta(1-\alpha)$, if and only if (2.1) holds. This completes the proof of Theorem 4.1. □

Theorem 4.2. *If* $m>0$, $0\le\alpha<1$ *and* $0<\beta\le1$, *then*

$$\mathscr{G}(m,z)=\int_0^z\frac{\mathscr{F}(m,t)}{t}dt$$

is in $\mathscr{S}^*(\alpha,\beta)$ *if and only if*

$$(1+\beta)(1-e^{-m})+\frac{(\beta(1-2\alpha)-1)}{m}(1-e^{-m}-me^{-m})\le 2\beta(1-\alpha).$$

The proof of Theorem 4.2 is lines similar to the proof of Theorem 4.1, so we omitted the proof of Theorem 4.2.

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