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Remarks on fixed point sets

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Dedicated to Professor Ioan A. Rus on the occasion of his 80th anniversary

Abstract. This is a collection of examples illustrating various properties of fixed point sets under regularity assumptions on mappings.

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Working for many years on the fixed point theory I came across several original questions asked by the newcomers to the field and young researchers. Answering very often required constructions of various examples. Here, samples are recalled, answered and discussed. Presented examples concern geometrical and topological properties of the fixed point sets of mappings satisfying certain regularity conditions. If known, are formulated and shown with some modifications and novelties.

For the whole text, let X denotes a Banach space with norm $\|\cdot\|$ and C be a convex, closed and bounded subset of X. H stands for Hilbert space. We shall deal with the classical situation of a mapping $T : C \to C$ satisfying the Lipschitz condition (lipschitzian mappings),

$$\|Tx - Ty\| \leq k \|x - y\| . \tag{1}$$

The fixed point set for T is defined as,

$$FixT = [x \in C : x = Tx] .$$

If T is a *contraction,* $k < 1$, $FixT$ consists of exactly one point, say z, and for any $x_0 \in C$, the sequence of iterates $x_n = T^n x_0$ converges, $\lim_{n\to\infty} x_n = z$.

If T is *nonexpansive,* $k = 1$, the set $FixT$ may be empty unless the set C catisfies some additional regularity condition. However, always *the minimal displacement* by T satisfies,

$$d(T) = \inf [\|x - Tx\| : x \in C] = 0.$$

Even if $FixT \neq \emptyset$ the iterates $T^n x_0$, in general, do not converge. If the space X, or the set C, show some special conditions, it may have influence on the regularity of $FixT$. For example, if X is strictly convex, then $FixT$ is convex. If C is weakly compact and T has a fixed point in every $T-$invarint closed convex subset of C, then $FixT$ is *a nonexpansive retract of* C. The last means that there exists a nonexpansive mapping

$R : C \to FixT$ such that $FixR = FixT = R(C)$. Without regularity conditions the fixed point sets can be very bizzare (see below and [1]).

If $k > 1, FixT$ is in many cases empty unless C is compact. If C is not compact, it may be that $d(T) > 0$.

The above standard facts in fixed point theory can be found, among other places, in books [2], [5], [6] see also [3].

The first to discuss is:

Question 1. Given a Banach space X and a nonempty, closed set $F \subset X$. Is there a way, procedure, algorithm to find out whether F is a fixed point set for a nonexpansive mapping $T : X \to X$ or $T : C \to C$ for a convex C containing F?

The situation is clear if $X = H$ is a Hilbert space. The fixed point sets of nonexpansive mappings are convex. Since for any nonempty closed convex suset $C \subset H$ the nearest point retraction P_C: $X \to C$ assigning to each $x \in H$ the unique point

$P_C(x) \in C$ such that

$$\|x - P_C x\| = \min[\|x - y\| : y \in C]$$

is nonexpansive, convexity is the necessary and sufficient condition for a closed set to be of the form $C = FixT$ for nonexpansive T.

The nearest point projection P is not the only one nonexpansive mapping having C as the fixed point set. The reflection with respect to $P, S = 2P - I$ is also nonexpansive with $FixS = C$. There are more examples in [4].

In other, even very regular, spaces convexity is not enough.

Example 1. Let for $p \in (1, \infty)$, l_3^p be the three dimensional space $\mathbb{R}^3$ furnished with the norm

$$\|x\|_p = \|(x_1, x_2, x_3)\|_p = (|x_1|^p + |x_2|^p + |x_3|^p)^{\frac{1}{p}}.$$

Let K be the triangle with vertices at unit vectors, $e^1 = (0,0,1), e^2 = (0,1,0), e^3 = (0,0,1)$. There is the unique closed ball containing K having minimal radius. Simple calculus shows that this is $B(z_p, r_p)$ centered at $z_p = (t_p, t_p, t_p)$ with

$$\begin{aligned} t_p &= \left(2^{\frac{1}{p-1}} + 1\right)^{-1} \\ r_p &= 2^{\frac{1}{p}} \left(2^{\frac{1}{p-1}} + 1\right)^{\frac{1-p}{p}}. \end{aligned}$$

For two extremal cases, $p = 1$ and $p = \infty$, we have $z_1 = \lim_{p \to 1} z_p = (0,0,0), r_1 = 1$ and $z_\infty = \lim_{p \to \infty} z_p = \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right), r_\infty = \frac{1}{2}$. Observe that $z_p \in K$ if and only if $p = 2$. Let $C \subset l_3^p$ be a closed convex set containing K and such that $z_p \in C$. Assume that $T : C \to C$ is nonexpansive and such that $FixT = K$. For any $x \in K$ we have

$$\|Tz_p - Tx\| \leq \|z_p - x\| \leq r_p.$$

Thus $K \subset B(Tz_p, r_p)$. In view that the ball of radius r_p containing K is unique, it must be $z_p = Tz_p$. For $p \neq 2, z_p \notin K$ and thus, we have the contradiction $K \neq FixT$.

In general, no topological propertis can characterize whether a closed set F is, or not is, of the form $F = FixT$ for nonexpansive T . There may be the case that the set $C \subset X$ is not the fixed point set of of any nanexpansive mapping $T : X \to X$, but for certain larger space $Y, X \subset Y$ there is even an isometry $G : Y \to Y$ having having $FixG$ isometric to F.

Example 2. Let X be an arbitrary space and let $F \subset X$ be nonempty and closed. Consider the space $Z = C \times c_0$ normed for $(x, y) \in Z, x \in X, y = (y_1, y_2, ...) \in c_0$, by

$$\|(x, y)\| = \max\left[\|x\|, \|y\|_{c_0}\right].$$

Since for any x_1, x_2 in X

$$|dist\,(x_1, F) - dist\,(x_2, F)| \leq \|x_1 - x_2\|,$$

the mapping $T : Z \to Z$, defined as

$$T\,(x, y) = T\,(x, (y_1, y_2, ...)) = (x, (dist\,(x, F)\,, y_1, y_2, ...))$$

is not only nonexpansive but isometric. If $(x, y) = T\,(x, y)$, then

$$(y_1, y_2, ...) = (dist\,(x, F)\,, y_1, y_2, ...)$$

which implies

$$dist\,(x, F) = y_1 = y_2 = ... = 0.$$

Thus, $Fix\,(T) \subset Z$ is

$$Fix\,(T) = F \times \{0\}$$

which is an isometric the copy of $F \subset X$.

Summing up, any closed subset of any Banach space can be considered as a fixed point set of an isometry.

Next two questions concern mappings which are lipschitzian with $k > 1$.

Question 2. Do there exist some characteristics of fixed point sets for Lipschitz mapping with constant $k > 1$? Do they depennd on the size of k or the regularity of the space?

Question 3. The same as above with the additional assumption that T has convergent iterates, for any x there exists $T_\infty x = \lim_{n\to\infty} T^n x$.

The first answer is that there is no dependence on the size of k. If C is a closed and convex set and $T : C \to C$ is $k-$lipschitzian, then for any $\alpha \in (0, 1)$ the mapping $T_\alpha = (1 - \alpha)\,I + \alpha T$ satisfies the Lipschitz condition wit a constant $k_\alpha \leq (1 - \alpha + \alpha k)$ and has $FixT_\alpha = FixT$. Since $\lim_{\alpha\to 0} k_\alpha = 1$ possible characteristics may not depend on the size of k.

The rest is answered by the following fact.

Claim 1. Let $C \subset X$ be a nonempty closed convex and bounded. Suppose $F \subset C$ is nonempty and closed. For any $\varepsilon > 0$ there exists a mapping $T : C \to C$ such that T is $k-$lipschtzian with $k \leq 1 + \varepsilon, FixT = F$.
Moreover, for any $x \in C, T_\infty x = \lim_{n\to\infty} T^n x$ does exist.

Proof. Let F and C be as assumed. Fix a point $z \in F$ and $0 < \varepsilon < 1$. Define the mapping $T : C \to C$ as

$$Tx = x + \varepsilon \frac{dist\,(x, F)\,(z - x)}{2diamC}.$$

Observe that $FixT = F$. Now it is easy to see that

$$\|dist\,(x, F)\,(z - x) - dist\,(x, F)\,(z - y)\| \leq 2diamC\,\|x - y\|$$

and consequently,

$$\|Tx - Ty\| \leq (1 + \varepsilon)\,\|x - y\|\,.$$

Finally, for any $x \notin F$ the iterates $\{x, Tx, T^2x, ...\}$ form on the segment $[x, z]$ the sequence monotone in the way that for any $n = 0, 1, 2, ...,$ $T^{n+1}x \in [T^n x, z]$. It implies convergence. $\square$

Additional information can be obtained that $T_\infty x = \lim_{n\to\infty} T^n x$ is the point in $F \cap [x, z]$ closest to x, $\|x - T^\infty x\| = dist\,(x, F \cap [x, z])$. Obviously, the above construction does not imply the continuity of T_∞. The simpliest example of it is, if we consider F consisting of only two distinct points.

The next question concerns retracts. Let us remain in our standartd assumptions for $F \subset C$. The set F is said to be the retract (lipschitzian, nonexpansive) if there is a continuous (lipschitzian, nonexpansive) mapping $R : C \to F$ such that $R\,(C) = FixR = F$. Any such R is said to be a retraction of C onto F. For a given retract F of C, there may be many retractions of various regularity. Retractions are characterized by the condition $R^2 = R$ which implies that $R\,(C)$ is the retract of C. For any $x \in C$ the iterates of each retraction R stabilize after one step, $R^n x = Rx, n = 1, 2, 3, ...$.

Question 4. What about mappings having iterates stabilized after k steps, meaning $T^{k+1}x = T^k x$?

Under this condition for all $n \geq k$, we have $T^n = T^k$. In other notation, for any $x \in C$,

$$T^k x = T^{k+1} x = T^{k+2} x = ... \,.$$

Thus $T^{2k} = T^k$ and $R = T^k$ is a retraction. Hence the fixed point sets of such mappings are the retracts of C. There are simple example of such mappings.

Example 3. Let $B \subset X$ be the unit ball. It is the retract of X. The standart retraction $Q : X \to B$ is the radial projection mapping

$$Qx = \begin{cases} x & \text{if } \|x\| \leq 1 \\ \frac{x}{\|x\|} & \text{if } \|x\| > 1. \end{cases}.$$

Let $S = 2Q - I$. Since Q satisfies Lipschitz condition with constant $k \in [1, 2]$, so S is also lipschitzian. More precisely,

$$Sx = \begin{cases} x & \text{if } \|x\| \leq 1 \\ \frac{2-\|x\|}{\|x\|}x & \text{if } \|x\| > 1 \end{cases},$$

and

$$\|Sx\| = \begin{cases} \|x\| & \text{if } \|x\| \leq 1 \\ |\|x\| - 2| & \text{if } \|x\| > 1 \end{cases}.$$

For any x we have $S^{n+1}x = S^n x$ begining from $n = Ent\frac{\|x\|}{2}$. Consequently for any $r > 1$, $S : rB \to rB$ for $n = Ent\frac{r}{2}, S^n$ is a lipschitzian retraction of rB onto B.

It is easy to observe that there are more Lipschitz retractions of the whole space X on B other than Q . Such are for example mappings of the form,

$$R_n = Q \circ S^n, n = 1, 2, \ldots .$$

In the most interesting case of the Hilbert space H, both mappings Q and S are nonexpansive. So, nonexpansive are all $R_n, n = 1, 2, \ldots$.

There is an advice for interested reader to prove:

Claim 2. Suppose $F, C,\ F \subset C \subset H$ are closed, convex and bounded. Assume that F (so, also C) has nonempty interior. Let $P = P_F : C \to F$ be the closest point projection and $S = 2P - I$. Then there exists $n \geq 1$ such that S^n is a nonexpansive retraction of C onto F.

The last is the following:

Question 5. What can be said about mappings similar to presented above if we only assume that for each individual point x the sequence $T^n x$ stabilizes but not necessarily on on the same level?

The situation is unclear. This condition does not bring much of regularity. There are sets $F, C, F \subset C$ such that F is not the retract of C or even are disconnected which satisfy the following.

For any $\varepsilon > 0$ there is a $(1 + \varepsilon)$ −lipschitzian mapping $T : C \to F$ having $F = FixT$ and such that for each $x \in C, T^{k+1}x = T^k x$ for certain k depending on $x, k = k(x)$. It looks like it is unknown how to characterize such sets. Here we present only a few simple examples.

Example 4. Consider the plain $\mathbb{R}^2$ with standard Eucledean norm, or any Hilbert space. Let B, S denote the unit ball and the unit sphere and let P be the radial retraction on B. Then the mapping $T : B \to B$,

$$Tx = P((1 + \varepsilon)x), \tag{2}$$

is $(1 + \varepsilon)$ −lipschitzian, $FixT = S \cup \{0\}$ and for any $x \in B$ the iterates of T stabilize. The level of stabilization $k = k(x)$ for $x \notin FixT$ depends only on the norm of x and grows to infinity as $x \to 0$.

It is only a technicality to construct similar examples with $F = FixT = S \cup \frac{1}{2}S \cup \{0\}$ or $F = rB \cup S, 0 < r < 1$ and other modifications as,

Example 5. Again for the Euclidean plane $\mathbb{R}^2$ plain consider the square

$$K = [(x_1, x_2) : |x_1| \leq 1, |x_2| \leq 1].$$

Let $P = P_K$ be the closest point projection of $\mathbb{R}^2$ on K,

$$P\ (x_1, x_2) = (\alpha(x_1), \alpha(x_2)),$$

where

$$\alpha(t) = \begin{cases} -1 & \text{if} \quad t < -1 \\ t & \text{if} \quad -1 \leq t \leq 1 \\ 1 & \text{if} \quad t > 1 \end{cases} .$$

Let again the mapping $T : K \to K$ be defined by (2). Then T fulfils required conditions and has the fixed point set consisting of nine points: the origin, positive and negative unit vectors and vertices of the square.

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