Stud. Univ. Babeş-Bolyai Math. 68(2023), No. 2, 307–317
DOI: 10.24193/subbmath.2023.2.07

Extension operators and Janowski starlikeness with complex coefficients

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Abstract. In this paper, we obtain certain generalizations of some results from [13] and [14]. Let $\Phi_{n,\alpha,\beta}$ be the extension operator introduced in [7] and let $\Phi_{n,Q}$ be the extension operator introduced in [16]. Let $a \in \mathbb{C}$, $b \in \mathbb{R}$ be such that $|1-a| < b \leq \mathrm{Re}\, a$. We consider the Janowski classes $S^*(a,b,\mathbb{B}^n)$ and $\mathcal{A}S^*(a,b,\mathbb{B}^n)$ with complex coefficients introduced in [4]. In the case $n=1$, we denote $S^*(a,b,\mathbb{B}^1)$ by $S^*(a,b)$ and $\mathcal{A}S^*(a,b,\mathbb{B}^1)$ by $\mathcal{A}S^*(a,b)$. We shall prove that the following preservation properties concerning the extension operator $\Phi_{n,\alpha,\beta}$ hold: $\Phi_{n,\alpha,\beta}(S^*(a,b)) \subseteq S^*(a,b,\mathbb{B}^n)$, $\Phi_{n,\alpha,\beta}(\mathcal{A}S^*(a,b)) \subseteq \mathcal{A}S^*(a,b,\mathbb{B}^n)$. Also, we prove similar results for the extension operator $\Phi_{n,Q}$:

$$\Phi_{n,Q}(S^*(a,b)) \subseteq S^*(a,b,\mathbb{B}^n),\ \Phi_{n,Q}(\mathcal{A}S^*(a,b)) \subseteq \mathcal{A}S^*(a,b,\mathbb{B}^n).$$

Mathematics Subject Classification (2010): 32H02, 30C45.

Keywords: g-Loewner chain, g-parametric representation, g-starlikeness, Janowski starlikeness, Janowski almost starlikeness, extension operator.

1. Preliminaries

Let $\mathbb{C}^n$ be the space of n complex variables equipped with the Euclidean inner product $\langle \cdot, \cdot \rangle$ and the Euclidean norm $\|\cdot\|$. Let $\mathbb{B}^n$ be the open unit ball in $\mathbb{C}^n$ and let U be the unit disc in $\mathbb{C}$. Also, let $H(\mathbb{B}^n)$ be the set of holomorphic mappings from $\mathbb{B}^n$ into $\mathbb{C}^n$. A mapping $f \in H(\mathbb{B}^n)$ is said to be normalized if $f(0) = 0$ and $Df(0) = I_n$. Let $J_f(z)$ be the complex Jacobian determinant of the Fréchet derivative $Df(z)$, i.e. $J_f(z) = detDf(z)$. A mapping $f \in H(\mathbb{B}^n)$ is locally biholomorphic mapping on $\mathbb{B}^n$ if $J_f(z) \neq 0$ for all $z \in \mathbb{B}^n$. We denote by $\mathcal{L}S_n$ the set of normalized locally biholomorphic mappings on the unit ball $\mathbb{B}^n$. In the case $n = 1$, we use the notation

Received 05 June 2022; Accepted 06 July 2022.

$\mathcal{L}S$ instead of $\mathcal{L}S_1$. Let $S(\mathbb{B}^n)$ be the set of normalized biholomorphic mappings on $\mathbb{B}^n$ and let S be the set of normalized univalent functions on U. Also, let $S^*(\mathbb{B}^n)$ be the set of normalized starlike mappings on $\mathbb{B}^n$.

Let $f, g \in H(\mathbb{B}^n)$. Then we say that $f \prec g$ if there exists a Schwarz mapping φ (i.e. $\varphi \in H(\mathbb{B}^n)$, $\|\varphi(z)\| \leq \|z\|$, $z \in \mathbb{B}^n$) such that $f = g \circ \varphi$ on $\mathbb{B}^n$. Moreover, if g is biholomorphic on $\mathbb{B}^n$, then the subordination condition $f \prec g$ is equivalent with $f(0) = g(0)$ and $f(\mathbb{B}^n) \subseteq g(\mathbb{B}^n)$.

We recall that $f : \mathbb{B}^n \times [0, \infty) \to \mathbb{C}^n$ is a Loewner chain if $f(\cdot, t)$ is biholomorphic on $\mathbb{B}^n$, $f(0, t) = 0$, $Df(0, t) = e^t I_n$ for $t \geq 0$ and $f(\cdot, s) \prec f(\cdot, t)$ with $0 \leq s \leq t < \infty$ (see [17], [8]). The subordination condition $f(\cdot, s) \prec f(\cdot, t)$ is equivalent to the following statement: there is a unique biholomorphic Schwarz mapping $v = v(z, s, t)$ such that $f(z, s) = f(v(z, s, t), t)$, $z \in \mathbb{B}^n$, $0 \leq s \leq t$. The mapping $v = v(z, s, t)$ is called the *transition mapping* associated to $f(z, t)$ and satisfies the semigroup property: $v(z, s, u) = v(v(z, s, t), t, u)$, for all $z \in \mathbb{B}^n$, $0 \leq s \leq t \leq u$. In addition, $Dv(0, s, t) = e^{s-t} I_n$, $0 \leq s \leq t$ (see [17], [8]).

We recall that the following class of holomorphic mappings (see [17], [20]; see also [8]):

$$\mathcal{M} = \{h \in H(\mathbb{B}^n) : h(0) = 0, Dh(0) = I_n, \operatorname{Re} \langle h(z), z \rangle > 0, z \in \mathbb{B}^n \backslash \{0\}\}$$

is the generalization to higher dimensions ($n \geq 2$) of the Carathédory class of functions with positive real part on U.

We next give the definition of parametric representation on the unit ball in $\mathbb{C}^n$ (see [5], [8]).

Definition 1.1. We say that a mapping $f \in S(\mathbb{B}^n)$ has *parametric representation* if there exists a Loewner chain $f(z, t)$ such that f can be embedded as the first element of $f(z, t)$ and the family $\{e^{-t} f(\cdot, t)\}_{t \geq 0}$ is normal on $\mathbb{B}^n$.

Let $S^0(\mathbb{B}^n)$ be the family of mappings with parametric representation. This set has been introduced by Graham, Hamada and Kohr in [5]. Various results regarding this class can be found in [5], [9], [10] and the references therein.

In the following we consider a function $g : U \to \mathbb{C}$ which satisfies the following conditions (see [6]):

Assumption 1.2. Let $g : U \to \mathbb{C}$ be such that g is a univalent (i.e. holomorphic and injective) function on U, $g(0) = 1$ and g has positive real part on U.

For example, the function $g : U \to \mathbb{C}$ given by $g(\zeta) = \frac{1+\zeta}{1-\zeta}$, $\zeta \in U$, satisfies the requirements of Assumption 1.2.

In the following, let $g : U \to \mathbb{C}$ be an arbitrary function which satisfies the conditions of Assumption 1.2.

Let $\mathcal{M}_g$ be the following nonempty subset of $\mathcal{M}$ introduced by Graham, Hamada, Kohr and Kohr in [6] (see also [5], where the function g satisfies in addition the relation $g(\overline{\zeta}) = \overline{g(\zeta)}$, $z \in U$, and other conditions):

$$\mathcal{M}_g = \left\{ h \in H(\mathbb{B}^n) : h(0) = 0, Dh(0) = I_n, \left\langle h(z), \frac{z}{\|z\|^2} \right\rangle \in g(U), z \in \mathbb{B}^n \backslash \{0\} \right\}.$$

For $g(\zeta) = \frac{1-\zeta}{1+\zeta}$, $\zeta \in U$, we have that $\mathcal{M}_g = \mathcal{M}$.

Next, we recall the definition of a g-Loewner chain (see [6]; see also [5] and [9], for $g(\zeta) = \frac{1-\zeta}{1+\zeta}$, $\zeta \in U$).

Definition 1.3. Let $f(z,t) : \mathbb{B}^n \times [0,\infty) \to \mathbb{C}^n$. We say that $f(z,t)$ is a *g-Loewner chain* if $f(z,t)$ is a Loewner chain such that the family $\{e^{-t}f(\cdot,t)\}_{t\geq 0}$ is normal on $\mathbb{B}^n$ and the mapping $h(z,t)$ which occurs in the following Loewner differential equation:

$$\frac{\partial f}{\partial t} = Df(z,t)h(z,t), \text{ a.e. } t \geq 0, \ \forall z \in \mathbb{B}^n,$$

has the property $h(\cdot,t) \in \mathcal{M}_g$, for a.e. $t \geq 0$.

We remark that a normalized holomorphic mapping $f : \mathbb{B}^n \to \mathbb{C}^n$ has g-parametric representation if and only if there exists a g-Loewner chain $f(z,t)$ such that f can be embedded as the first element of the g-Loewner chain (see [6]; see also [5]).

Let $S_g^0(\mathbb{B}^n)$ be the set of mappings with g-parametric representation on $\mathbb{B}^n$. Then $S_g^0(\mathbb{B}^n) \subseteq S^0(\mathbb{B}^n)$ (see [6]).

If $g(\zeta) = \frac{1-\zeta}{1+\zeta}$, $\zeta \in U$, then any g-Loewner chain is a Loewner chain and the set $S_g^0(\mathbb{B}^n)$ becomes $S^0(\mathbb{B}^n)$ (see [6]; see also [5]). In the case $n \geq 2$, there exists Loewner chains that are not g-Loewner chains when $g(\zeta) = \frac{1-\zeta}{1+\zeta}$, $\zeta \in U$. For example, when $n = 2$, the mapping $p(z,t) : \mathbb{B}^2 \times [0,\infty) \to \mathbb{C}^n$ given by

$$p(z,t) = \left(\frac{e^t z_1}{(1-z_1)^2}, \frac{e^t z_2}{(1-z_2)^2} + \frac{e^{2t} z_1^2}{(1-z_1)^4}\right), \ z = (z_1,z_2) \in \mathbb{B}^2, \ t \geq 0,$$

is a Loewner chain, but the family $\{e^{-t}p(\cdot,t)\}_{t\geq 0}$ is not normal on $\mathbb{B}^2$. Thus, $p(\cdot,t)$ is not a g-Loewner chain for $g(\zeta) = \frac{1-\zeta}{1+\zeta}$, $\zeta \in U$ (see [5]).

In the next part, we shall refer to the following univalent function g on U with $g(0) = 1$ and positive real part on U:

Assumption 1.4. Let $g : U \to \mathbb{C}$ be a holomorphic function on U given by

$$g(\zeta) = \frac{1 + A\zeta}{1 + B\zeta}, \ \zeta \in U, \tag{1.1}$$

where $A, B \in \mathbb{C}$, $A \neq B$ and g has positive real part on U.

This function was considered in [4].

Imposing the condition that the function g given by Assumption 1.4 to have positive real part implies certain conditions on the complex parameters A and B. These conditions are illustrated in the following remark due to Curt [4].

Remark 1.5. [4] Let $g : U \to \mathbb{C}$ be a function described by Assumption 1.4. Then one of the following two conditions holds:

$$|B| < 1, \ |A| \leq 1 \text{ and } \operatorname{Re}(1 - A\overline{B}) \geq |A - B|, \tag{1.2}$$

or

$$|B| = 1, \ |A| \leq 1 \text{ and } -1 \leq A\overline{B} < 1. \tag{1.3}$$

In this context, we remark that the function g maps the unit disc onto the open disc of center $a := \frac{1-A\overline{B}}{1-|B|^2}$ and radius $b := \frac{|A-B|}{1-|B|^2}$, for $|B| < 1$. It is immediate that $|1-a| < b \leq \text{Re } a$. If $|B| = 1$ then g maps the unit disc onto the half-plane $\{z \in \mathbb{C} : \text{Re}z > \frac{1+A\overline{B}}{2}\}$.

Moreover, we have that g is convex on U.

Next, we present the following subclasses of starlike mappings on $\mathbb{B}^n$ introduced by Curt [4]:

Definition 1.6. Let $a \in \mathbb{C}$, $b \in \mathbb{R}$ be such that $|1-a| < b \leq \text{Re } a$. Let

$$S^*(a, b, \mathbb{B}^n) = \left\{ f \in \mathcal{L}S_n : \left| \frac{\|z\|^2}{\langle [Df(z)]^{-1} f(z), z \rangle} - a \right| < b, \ z \in \mathbb{B}^n \backslash \{0\} \right\},$$

be the set of Janowski starlike mappings on $\mathbb{B}^n$ and let

$$\mathcal{A}S^*(a, b, \mathbb{B}^n) = \left\{ f \in \mathcal{L}S_n : \left| \frac{\langle [Df(z)]^{-1} f(z), z \rangle}{\|z\|^2} - a \right| < b, \ z \in \mathbb{B}^n \backslash \{0\} \right\},$$

be the set of Janowski almost starlike mappings on $\mathbb{B}^n$.

For $a \in \mathbb{R}$ (which is equivalent to Re $a = a$), the above sets become the classes mentioned in [3]. In the case $n = 1$, we denote $S^*(a, b, \mathbb{B}^1)$ by $S^*(a, b)$, respectively $\mathcal{A}S^*(a, b, \mathbb{B}^1)$ by $\mathcal{A}S^*(a, b)$.

The following remark provides a connection between Janowski starlikeness, respectively Janowski almost starlikeness with complex coefficients and g-starlikeness on $\mathbb{B}^n$ (see [4]).

Remark 1.7. Let $a \in \mathbb{C}$, $b \in \mathbb{R}$ be such that $|1-a| < b \leq \text{Re } a$.

(i) If $g(\zeta) = \frac{1+(\overline{a}-1)/b\zeta}{1+(|a|^2-b^2-a)/b\zeta}$, $\zeta \in U$, then $S_g^*(\mathbb{B}^n)$ becomes $S^*(a, b, \mathbb{B}^n)$.

(ii) If $g(\zeta) = \frac{1+(a-|a|^2+b^2)/b\zeta}{1+(1-\overline{a})/b\zeta}$, $\zeta \in U$, then $S_g^*(\mathbb{B}^n)$ becomes $\mathcal{A}S^*(a, b, \mathbb{B}^n)$.

(iii) If $b = a \in R$ $(b = a > 0)$, then we have that

$$\mathcal{A}S^*\left(a, a, \mathbb{B}^n\right) = S^*_{\frac{1}{2a}}(\mathbb{B}^n) \text{ and } S^*\left(a, a, \mathbb{B}^n\right) = \mathcal{A}S^*_{\frac{1}{2a}}(\mathbb{B}^n).$$

Note that the functions mentioned in Remark 1.7(i), (ii) satisfy the conditions of Assumption 1.4.

Next, we consider the following extension operator introduced by Graham, Hamada, Kohr and Suffridge in [7].

Definition 1.8. Let $\alpha \geq 0$, $\beta \geq 0$ and $n \geq 2$. Let $\Phi_{n,\alpha,\beta} : \mathcal{L}S \to \mathcal{L}S_n$ be given by

$$\Phi_{n,\alpha,\beta}(f)(z) = \left(f(z_1), \tilde{z} \left(\frac{f(z_1)}{z_1} \right)^{\alpha} (f'(z_1))^{\beta} \right), \ z = (z_1, \tilde{z}) \in \mathbb{B}^n, \tag{1.4}$$

where

$$\left(\frac{f(z_1)}{z_1} \right)^{\alpha} \Big|_{z_1=0} = 1, \ (f'(z_1))^{\beta} \big|_{z_1=0} = 1.$$

For $\alpha = 0$ and $\beta = 1/2$, the extension operator $\Phi_{n,\alpha,\beta}$ reduces to Roper-Suffridge extension operator $\Phi_n : \mathcal{L}S \to \mathcal{L}S_n$ given by (see [19])

$$\Phi_n(f)(z) = \left(f(z_1), \tilde{z}\sqrt{f'(z_1)}\right), \ z = (z_1, \tilde{z}) \in \mathbb{B}^n,$$

where the branch of the square root is chosen such that $\sqrt{f'(z_1)}|_{z_1=0} = 1$.

The extension operator $\Phi_{n,\alpha,\beta}$ satisfies important preservation properties for $\alpha \in [0,1]$, $\beta \in [0,1/2]$, $\alpha + \beta \leq 1$. In [7], it was shown that $\Phi_{n,\alpha,\beta}(f)(S) \subseteq S^0(\mathbb{B}^n)$ and $\Phi_{n,\alpha,\beta}(f)(S^*) \subseteq S^*(\mathbb{B}^n)$. In the same paper, the authors proved that $\Phi_{n,\alpha,\beta}$ conserves convexity only if $(\alpha, \beta) = (0, 1/2)$. Also, $\Phi_{n,\alpha,\beta}$ conserves starlikeness of order $\gamma \in (0,1)$ (see [11]), spirallikeness of type $\gamma \in (-\pi/2, \pi/2)$ and order $\delta \in (0,1)$ (see [12]; see also [1]) and almost starlikeness of type $\gamma \in (0,1)$ and order $\delta \in [0,1)$ (see [1]). More recent preservation results regarding this extension operator and Bloch mappings, in the case of complex Banach spaces, are obtained in [6].

We next present the definition of the Muir extension operator $\Phi_{n,Q}$ (see [16]).

Definition 1.9. Assume that $Q : \mathbb{C}^{n-1} \to \mathbb{C}$ is a homogeneous polynomial of degree 2 and $n \geq 2$. Let $\Phi_{n,Q} : \mathcal{L}S \to \mathcal{L}S_n$ be such that

$$\Phi_{n,Q}(f)(z) = (f(z_1) + Q(\tilde{z})f'(z_1), \tilde{z}\sqrt{f'(z_1)}), \ z = (z_1, \tilde{z}) \in \mathbb{B}^n, \tag{1.5}$$

where $\sqrt{f'(z_1)}|_{z_1=0} = 1$.

For $Q \equiv 0$, the extension operator $\Phi_{n,Q}$ reduces to the extension operator Φ_n.

The extension operator $\Phi_{n,Q}$ preserves parametric representation and starlikeness if $\|Q\| \leq 1/4$ (see [10]), convexity if $\|Q\| \leq 1/2$ (see [16]) and starlikeness of order $\alpha \in (0,1)$ if $\|Q\| \leq \frac{1-|2\alpha-1|}{8\alpha}$ (see [21]; see also [2]). In a recent study, there has been investigated results concerning extended Loewner chains and this extension operator, as well as other preservation results (see [15]). Also, modifications of the Muir extension operator were considered in [6].

Assume that $a \in \mathbb{C}$, $b \in \mathbb{R}$ such that $|1-a| < b \leq \operatorname{Re} a$. In the next part, we aim to show that the extension operators $\Phi_{n,\alpha,\beta}$ and $\Phi_{n,Q}$ map a function $f \in S^*(a,b)$ into a mapping from $S^*(a,b,\mathbb{B}^n)$. Also, $\Phi_{n,\alpha,\beta}$ and $\Phi_{n,Q}$ map a function $f \in \mathcal{A}S^*(a,b)$ into a mapping from $\mathcal{A}S^*(a,b,\mathbb{B}^n)$. Therefore, the extension operators $\Phi_{n,\alpha,\beta}$ and $\Phi_{n,Q}$ preserve the Janowski starlikeness and Janowski almost starlikeness with complex coefficients from the case of one complex variable to several complex variables.

2. Main results

In [6], I. Graham, H. Hamada, G. Kohr and M. Kohr proved that g-parametric presentation and g-starlikeness is preserved through the extension operators $\Phi_{n,\alpha,\beta}$ and $\Phi_{n,Q}$, when the function g is convex on U and satisfies the conditions of Assumption 1.2. This result was obtained in a more general case, namely on the unit ball of a complex Banach space.

All along this section we assume that $n \geq 2$.

We state in the next two results the preservation of g-starlikeness under $\Phi_{n,\alpha,\beta}$ and $\Phi_{n,Q}$, when the function g is convex on U satisfying Assumption 1.2.

Theorem 2.1. [6] *Let $g : U \to \mathbb{C}$ be a univalent holomorphic function on U, with $g(0) = 1$, $\mathrm{Re} g(\zeta) > 0$, $\zeta \in U$, and g is convex on U. Also, let $\alpha \in [0, 1]$, $\beta \in [0, 1/2]$, $\alpha + \beta \leq 1$. If $f \in S_g^*$ then $F = \Phi_{n,\alpha,\beta}(f) \in S_g^*(\mathbb{B}^n)$.*

In the next result, let be the distance from 1 to $\partial g(U)$, denoted by $d(1, \partial g(U))$, and equal to $\inf_{\zeta\in\partial g(U)} |\zeta - 1|$.

Theorem 2.2. [6] *Let $g : U \to \mathbb{C}$ be a univalent function on U, with $g(0) = 1$, $\mathrm{Re} g(\zeta) > 0$, $\zeta \in U$, and g is convex on U. Also, let $\|Q\| \leq d(1, \partial g(U))/4$, where Q is a homogeneous polynomial of degree 2 from $\mathbb{C}^{n-1}$ to $\mathbb{C}$. If $f \in S_g^*$ then*

$$F = \Phi_{n,Q}(f) \in S_g^*(\mathbb{B}^n).$$

It is clear that, for the function g defined by Assumption 1.4, the above statements hold.

In addition, we have the following result.

Remark 2.3. Let g be a function satisfying the conditions from Assumption 1.4. Then

$$d(1, \partial g(U)) = \frac{|A - B|}{1 + |B|}.$$

Proof. Since the function g satisfies the requirements of Assumption 1.4, then, in view of Remark 1.5, the complex coefficients A and B satisfy one of the following two relations:

$$|B| < 1, \ |A| \leq 1 \text{ and } \mathrm{Re}(1 - A\overline{B}) \geq |A - B|,$$

or

$$|B| = 1, \ |A| \leq 1 \text{ and } -1 \leq A\overline{B} < 1.$$

We shall analyze the above two cases.

• Assume that $|B| = 1$, $|A| \leq 1$ and $\mathrm{Re}(1 - A\overline{B}) \geq |A - B|$. In this case, we have $g(U) = \{z \in \mathbb{C} : \mathrm{Re}\, z > \frac{1+A\overline{B}}{2}\}$. Thus,

$$\partial g(U) = \{z \in \mathbb{C} : z = \frac{1 + A\overline{B}}{2} + iy, \ y \in \mathbb{R}\}.$$

Let $\zeta \in \partial g(U)$. Then $\zeta = \frac{1+A\overline{B}}{2} + iy$, where $y \in \mathbb{R}$. We have that

$$|\zeta - 1| = \left|\frac{1 + A\overline{B}}{2} + iy - 1\right| = \left|\frac{-1 + A\overline{B}}{2} + iy\right|.$$

Using the above relation and the fact that $-1 \leq A\overline{B} < 1$, we have that

$$\inf_{\zeta\in\partial g(U)} |\zeta - 1| = \inf_{y\in\mathbb{R}} \left|\frac{-1 + A\overline{B}}{2} + iy\right| = \inf_{y\in\mathbb{R}} \sqrt{\left(\frac{1 - A\overline{B}}{2}\right)^2 + y^2} = \frac{1 - A\overline{B}}{2}.$$

Note that, for $|B| = 1$ and since $-1 \leq A\overline{B} < 1$, we have the following equivalence:

$$\frac{1 - A\overline{B}}{2} = \frac{|1 - A\overline{B}|}{2} = \frac{\left||B|^2 - A\overline{B}\right|}{1 + |B|} = \frac{|\overline{B}| \cdot |A - B|}{1 + |B|} = \frac{|A - B|}{1 + |B|}.$$

• Assume that $|B| = 1$, $|A| \leq 1$ and $-1 \leq A\overline{B} < 1$. Then

$$g(U) = U\left(\frac{1 - A\overline{B}}{1 - |B|^2}, \frac{|A - B|}{1 - |B|^2}\right).$$

Thus,

$$\partial g(U) = \left\{z \in C : z = \frac{1 - A\overline{B}}{1 - |B|^2} + \lambda\frac{|A - B|}{1 - |B|^2},\ |\lambda| = 1\right\}.$$

Let $\zeta \in \partial g(U)$. Then there exists $\lambda \in C$ with $|\lambda| = 1$ such that

$$\zeta = \frac{1 - A\overline{B}}{1 - |B|^2} + \lambda\frac{|A - B|}{1 - |B|^2}.$$

Further, an elementary computation implies that:

$$\begin{aligned}
|\zeta - 1| &= \left|\frac{1 - A\overline{B}}{1 - |B|^2} + \lambda\frac{|A - B|}{1 - |B|^2} - 1\right| \\
&= \frac{\Big||B|^2 - A\overline{B} + \lambda|A - B|\Big|}{1 - |B|^2} \\
&= \frac{\Big|\lambda|A - B| - \overline{B}(A - B)\Big|}{1 - |B|^2} \\
&\geq \frac{\Big||A - B| - |\overline{B}| \cdot |A - B|\Big|}{1 - |B|^2} \\
&= \frac{|A - B| \cdot |1 - |\overline{B}||}{1 - |B|^2} \\
&= \frac{|A - B| \cdot |1 - |B||}{1 - |B|^2} \\
&= \frac{|A - B|}{1 + |B|}.
\end{aligned}$$

Note that the equality is attained in the above inequality when

$$\lambda_0 = \frac{\overline{B}(A - B)}{|\overline{B}(A - B)|}\ (|\lambda_0| = 1).$$

In this case, we get

$$\inf_{\zeta \in \partial g(U)} |\zeta - 1| = \inf_{|\lambda| = 1}\left|\frac{1 - A\overline{B}}{1 - |B|^2} + \lambda\frac{|A - B|}{1 - |B|^2} - 1\right| = \frac{|A - B|}{1 + |B|}.$$

Taking into account the both cases analyzed above, we conclude that

$$d(1, \partial g(U)) = \inf_{\zeta \in \partial g(U)} |\zeta - 1| = \frac{|A - B|}{1 + |B|}.$$ □

In view of Theorem 2.1 and Remark 1.7, we deduce the following consequence.

Theorem 2.4. *Let $a \in \mathbb{C}$, $b \in \mathbb{R}$ be such that $|1 - a| < b \leq \operatorname{Re} a$. Also, let $\alpha \in [0, 1]$, $\beta \in [0, 1/2]$, $\alpha + \beta \leq 1$. Then the following properties hold:*

(i) *if* $f \in S^*(a,b)$ *then* $\Phi_{n,\alpha,\beta}(f) \in S^*(a,b,\mathbb{B}^n)$,
(ii) *if* $f \in \mathcal{A}S^*(a,b)$ *then* $\Phi_{n,\alpha,\beta}(f) \in \mathcal{A}S^*(a,b,\mathbb{B}^n)$.

Proof. (i) If we take the function g as in Remark 1.7 (i), then $S_g^* = S^*(a,b)$ and $S_g^*(\mathbb{B}^n) = S^*(a,b,\mathbb{B}^n)$. Therefore, in view of Theorem 2.1, we deduce that

$$\Phi_{n,\alpha,\beta}(S^*(a,b)) \subseteq S^*(a,b,\mathbb{B}^n).$$

(ii) Let the function g be given as in Remark 1.7 (ii). In this case, we have that $S_g^* = \mathcal{A}S^*(a,b)$ and $S_g^*(\mathbb{B}^n) = \mathcal{A}S^*(a,b,\mathbb{B}^n)$. From Theorem 2.1, we obtain that

$$\Phi_{n,\alpha,\beta}(\mathcal{A}S^*(a,b)) \subseteq \mathcal{A}S^*(a,b,\mathbb{B}^n).$$

This completes the proof. □

In the case $a, b \in \mathbb{R}$ with $|1-a| < b \le a = \text{Re } a$, the above result was obtained in [13].

The next two results are consequences of Theorem 2.2 and Remark 1.7.

Theorem 2.5. *Let* $a \in \mathbb{C}$, $b \in \mathbb{R}$ *be such that* $|1-a| < b \le \text{Re } a$. *Let* $Q : \mathbb{C}^{n-1} \to \mathbb{C}$ *be a homogeneous polynomial of degree* 2*, such that*

$$\|Q\| \le \frac{b^2 - (1-a)(1-\overline{a})}{4(b + ||a|^2 - b^2 - a|)}.$$

If $f \in S^*(a,b)$, *then* $\Phi_{n,Q}(f) \in S^*(a,b,\mathbb{B}^n)$.

Proof. Let g be the function from Remark 1.7 (i). Thus, we get that S_g^* becomes $S^*(a,b)$ and $S_g^*(\mathbb{B}^n)$ becomes $S^*(a,b,\mathbb{B}^n)$. Then the asserted property of the Muir extension operator $\Phi_{n,Q}$ follows from Theorem 2.1, i.e.

$$\Phi_{n,Q}(S^*(a,b)) \subseteq S^*(a,b,\mathbb{B}^n). \tag{2.1}$$

The function g has the form from Assumption 1.4, where

$$A = \frac{\overline{a} - 1}{b} \text{ and } B = \frac{|a|^2 - b^2 - a}{b}.$$

Moreover, we have that:

$$\begin{aligned}\frac{|A-B|}{4(1+|B|)} &= \frac{\left|\overline{a} - 1 - |a|^2 + b^2 + a\right|}{4\left|b + ||a|^2 - b^2 - a|\right|} \\ &= \frac{|b^2 - (|a|^2 - 2\text{Re } a + 1)|}{4(b + ||a|^2 - b^2 - a|)} \\ &= \frac{|b^2 - (1-a)(1-\overline{a})|}{4(b + ||a|^2 - b^2 - a|)} \\ &= \frac{b^2 - (1-a)(1-\overline{a})}{4(b + ||a|^2 - b^2 - a|)},\end{aligned}$$

since $|a|^2 - 2\text{Re } a + 1 = (1-a)(1-\overline{a}) \in \mathbb{R}$ and $b > |1-a| = |1-\overline{a}|$.

Therefore, the assumption

$$\|Q\| \le \frac{b^2-(1-a)(1-\overline{a})}{4(b+||a|^2-b^2-a|)}$$

shows that the relation (2.1) holds, as asserted. □

If we assume that $a\in\mathbb{R}$ in the hypothesis of the above result, then we deduce the preservation property concerning the extension operator $\Phi_{n,Q}$ and the class $S^*(a,b)$ with real coefficients obtained in [14].

Let us now refer to the Muir extension operator $\Phi_{n,Q}$ and state the following property.

Theorem 2.6. *Let $a\in\mathbb{C}$, $b\in\mathbb{R}$ be such that $|1-a|<b\le \mathrm{Re}\ a$. Let $Q:\mathbb{C}^{n-1}\to\mathbb{C}$ be a homogeneous polynomial of degree 2, such that*

$$\|Q\| \le \frac{b^2-(1-a)(1-\overline{a})}{4(b+|1-\overline{a}|)}.$$

If $f\in\mathcal{A}S^*(a,b)$ then $\Phi_{n,Q}(f)\in\mathcal{A}S^*(a,b,\mathbb{B}^n)$.

Proof. We consider the function g as in Remark 1.7 (ii). It is clear that $S_g^*=\mathcal{A}S^*(a,b)$ and $S_g^*(\mathbb{B}^n)=\mathcal{A}S^*(a,b,\mathbb{B}^n)$. Taking into account Theorem 2.1, we deduce that the following relation is true:

$$\Phi_{n,Q}(\mathcal{A}S^*(a,b))\subseteq \mathcal{A}S^*(a,b,\mathbb{B}^n). \tag{2.2}$$

The function g can be also written in the form given in Assumption (1.4), where

$$A=\frac{a-|a|^2+b^2}{b} \text{ and } B=\frac{1-\overline{a}}{b}.$$

Next, we evaluate the following quantity:

$$\begin{aligned}\frac{|A-B|}{4(1+|B|)} &= \frac{\left|a-|a|^2+b^2-1+\overline{a}\right|}{4|b+|1-\overline{a}||}\\ &= \frac{|b^2-(|a|^2-2\mathrm{Re}\ a+1)|}{4(b+|1-\overline{a}|)}\\ &= \frac{|b^2-(1-a)(1-\overline{a})|}{4(b+|1-\overline{a}|)}\\ &= \frac{b^2-(1-a)(1-\overline{a})}{4(b+|1-\overline{a}|)},\end{aligned}$$

using the fact that $|a|^2-2\mathrm{Re}\ a+1=(1-a)(1-\overline{a})\in\mathbb{R}$ and $b>|1-a|=|1-\overline{a}|$. Consequently, the condition

$$\|Q\| \le \frac{b^2-(1-a)(1-\overline{a})}{4(b+|1-\overline{a}|)}$$

implies that the relation (2.2) holds, as asserted. □

For $a,b\in\mathbb{R}$ where $|1-a|<b\le a=\mathrm{Re}\ a$, the above property was obtained in [14].

Question 2.7. Assume that $n \geq 2$. Let $\Psi_n : \mathcal{L}S_n \to \mathcal{L}S_{n+1}$ be the Pfaltzgraff-Suffridge extension operator given by (see [18]):

$$\Psi_n(f)(z) = \left(f(\tilde{z}), z_{n+1}[J_f(\tilde{z})]^{\frac{1}{n+1}}\right),\ z = (\tilde{z}, z_{n+1}) \in \mathbb{B}^{n+1},$$

were $[J_f(\tilde{z})]^{\frac{1}{n+1}}\Big|_{\tilde{z}=0} = 1$. We wonder if it is possible that Janowski (almost) starlikeness with complex coefficients to be preserved under the extension operator Ψ_n from the unit ball $\mathbb{B}^n$ to the unit ball $\mathbb{B}^{n+1}$. If it is true, under which conditions does this property hold?

Conclusions. In this paper, we have considered g-parametric representation and g-starlikeness on the Euclidean unit ball $\mathbb{B}^n$, when the function $g : U \to \mathbb{C}$ is univalent on U, $g(0) = 1$ and has positive real part on U (see [6]). Then we have referred to the property of preservation of g-starlikeness under the extension operator $\Phi_{n,\alpha,\beta}$, when g is convex on U and $\alpha \in [0,1]$, $\beta \in [0,1/2]$, $\alpha + \beta \leq 1$ (see [6]). For the same conditions imposed on g, we have stated that the Muir extension operator $\Phi_{n,Q}$ preserves g-starlikeness when $\|Q\| \leq d(1, \partial g(U))/4$ (see [6]).

Assume $a \in \mathbb{C}$, $b \in \mathbb{R}$ such that $|1-a| < b \leq \operatorname{Re} a$. Using the connection between the Janowski classes $S^*(a,b)$, $\mathcal{A}S^*(,b)$ and g-starlikeness, for a particular choice of g depending on the parameters a, b, we have proved that $\Phi_{n,\alpha,\beta}$ preserves these classes for $\alpha \in [0,1]$, $\beta \in [0,1/2]$, $\alpha + \beta \leq 1$. By making use of the same idea, we also prove that $\Phi_{n,Q}$ conserves these classes when $\|Q\| \leq M(a,b)$, where $M(a,b)$ is a constant depending on the parameters a and b. These results generalize the properties obtained in [13, 14], for the Janowski classes with real parameters.

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