Stud. Univ. Babeş-Bolyai Math. 69(2024), No. 3, 613–637
DOI: 10.24193/subbmath.2024.3.09

Modified inertia Halpern method for split null point problem in Banach spaces

Hammed Anuoluwapo Abass, Godwin Ugwunnadi and Ojen Narain

Abstract. In this paper, we study split null point problem in reflexive Banach spaces. Using the Bregman technique together with a modified inertial Halpern method, we approximate a solution of split null point problem. Also, we establish a strong convergence result for approximating the solution of the aforementioned problems. It is worth mentioning that the iterative algorithm employ in this study is design in such a way that it does not require prior knowledge of operator norm. We display some numerical examples to illustrate the performance of the proposed iterative method. The result discuss in this paper extends and complements many related results in literature.

Mathematics Subject Classification (2010): 47H06, 47H09, 47J05, 47J25.

Keywords: Monotone variational inclusion problem, split feasibility problem, firmly nonexpansive-type mapping, fixed point problem, inertial method.

1. Introduction

Let E be a reflexive Banach space with E^* its dual and Q be a nonempty closed and convex subset of E. Let $g : E \to (-\infty, +\infty]$ be a proper, lower semicontinuous and convex function, then the Fenchel conjugate of g denoted as $g^* : E^* \to (-\infty, +\infty]$ is define as

$$g^*(x^*) = \sup\{\langle x^*, x\rangle - g(x) : x \in E\},\ x^* \in E^*.$$

Let the domain of g be denoted as $dom(g) = \{x \in E : g(x) < +\infty\}$, hence for any $x \in intdom(g)$ and $y \in E$, we define the right-hand derivative of g at x in the direction

Received 12 May 2022; Accepted 12 September 2022.

of y by

$$g^0(x,y) = \lim_{t\to 0^+} \frac{g(x+ty)-g(x)}{t}.$$

Let $g : E \to (-\infty, +\infty]$ be a function, then g is said to be:

(i) Gâteaux differentiable at x if $\lim_{t\to 0^+} \frac{g(x+ty)-g(x)}{t}$ exists for any y. In this case, $g^0(x,y)$ coincides with $\nabla g(x)$ (the value of the gradient ∇g of g at x);

(ii) Gâteaux differentiable, if it is Gâteaux differentiable for any $x \in intdomg$;

(iii) Fréchet differentiable at x, if its limit is attained uniformly in $\|y\| = 1$;

(iv) Uniformly Fréchet differentiable on a subset Q of E, if the above limit is attained uniformly for $x \in Q$ and $\|y\| = 1$.

(v) essentially smooth, if the subdifferential of g denoted as ∂g is both locally bounded and single-valued on its domain, where

$$\partial g(x) = \{w \in E : g(x) - g(y) \geq \langle w, y - x\rangle,\ y \in E\};$$

(vi) essentially strictly convex, if $(\partial g)^{-1}$ is locally bounded on its domain and g is strictly convex on every convex subset of $dom\ \partial g$;

(vii) Legendre, if it is both essentially smooth and essentially strictly convex. See [8, 9] for more details on Legendre functions.

Alternatively, a function g is said to be Legendre if it satisfies the following conditions:

(i) The $intdom(g)$ is nonempty, g is Gâteaux differentiable on $intdom(g)$ and $dom\nabla g = intdom(g)$;

(ii) The $intdomg^*$ is nonempty, g^* is Gâteaux differentiable on $intdomg^*$ and $dom\nabla g^* = intdom(g)$.

Let E be a Banach space and $B_s := \{z \in E : \|z\| \leq s\}$ for all $s > 0$. Then, a function $g : E \to \mathbb{R}$ is said to be uniformly convex on bounded subsets of E, [see pp. 203 and 221] [51] if $\rho_s t > 0$ for all $s, t > 0$, where $\rho_s : [0, +\infty) \to [0, \infty]$ is defined by

$$\rho_s(t) = \inf_{x,y\in B_s, \|x-y\|=t, \alpha\in(0,1)} \frac{\alpha g(x) + (1-\alpha)g(y) - g(\alpha(x) + (1-\alpha)y)}{\alpha(1-\alpha)},$$

for all $t \geq 0$, with ρ_s denoting the gauge of uniform convexity of g. The function g is also said to be uniformly smooth on bounded subsets of E, [see pp. 221] [51], if $\lim_{t\downarrow 0} \frac{\sigma_s}{t}$ for all $s > 0$, where $\sigma_s : [0, +\infty) \to [0, \infty]$ is defined by

$$\sigma_s(t) = \sup_{x\in B, y\in S_E, \alpha\in(0,1)} \frac{\alpha g(x) + (1-\alpha)ty) + (1-\alpha)g(x-\alpha ty) - g(x)}{\alpha(1-\alpha)},$$

for all $t \geq 0$, and uniformly convex if the function $\delta g : [0, +\infty) \to [0, +\infty)$ defined by

$$\delta g(t) := \sup\{\frac{1}{2}g(x) + \frac{1}{2}g(y) - g(\frac{x+y}{2}) : \|y - x\| = t\},$$

satisfies $\lim_{t\downarrow 0} \frac{\delta g(t)}{t} = 0$.

Definition 1.1. [11] Let E be a Banach space. A function $g : E \to (-\infty, \infty]$ is said to be proper if the interior of its domain $dom(g)$ is nonempty. Let $g : E \to (-\infty, \infty]$

be a convex and Gâteaux differentiable function. Then the Bregman distance corresponding to g is the function $D_g : dom(g) \times intdom(g) \to \mathbb{R}$ defined by

$$D_g(x, y) := g(x) - g(y) - \langle x - y, \nabla_E^g(y)\rangle, \ \forall \ x, y \in E. \tag{1.1}$$

It is clear that $D_g(x, y) \geq 0$ for all $x, y \in E$.

It is well-known that Bregman distance D_g does not satisfy all the properties of a metric function because D_g fail to satisfy the symmetric and triangular inequality property. However, the Bregman distance satisfies the following so-called three point identity: for any $x \in dom(g)$ and $y, z \in intdom(g)$,

$$D_g(x, z) = D_g(x, y) + D_g(y, z) + \langle x - y, \nabla_E^g(y) - \nabla_E^g(z)\rangle. \tag{1.2}$$

In particular,

$$D_g(x, y) = -D_g(y, x) + \langle y - x, \nabla_E^g(y) - \nabla_E^g(x)\rangle, \ \forall \ x, y \in E.$$

The relationship between D_g and $\|.\|$ is guaranteed when g is strongly convex with strong convexity constant $\rho > 0$ i.e.

$$D_g(x, y) \geq \frac{\rho}{2}\|x - y\|^2, \ \forall \ x \in dom(g), \ y \in intdom(g). \tag{1.3}$$

Let $g : E \to \mathbb{R}$ be a strictly convex and Gâteaux differentiable function and $T : Q \to intdom(g)$ be a mapping, a point $x \in Q$ is called a fixed point of T, if for all $x \in Q$, $Tx = x$. We denote by $Fix(T)$ the set of all fixed points of T. Furthermore, a point $p \in Q$ is called an asymptotic fixed point of T if Q contains a sequence $\{x_n\}$ which converges weakly to p such that $\lim_{n\to\infty} \|Tx_n - x_n\| = 0$. We denote by $\hat{Fix}(T)$ the set of asymptotic fixed points of T.
Let Q be a nonempty closed and convex subset of int(dom g), then we define an operator $T : Q \to int(domg)$ to be :

(i) Bregman relatively nonexpansive, if $Fix(T) \neq \emptyset$, and

$$D_f(p, Tx) \leq D_f(p, x), \ \forall \ p \in Fix(T), \ x \in Q \text{ and } \hat{Fix}(T) = Fix(T).$$

(ii) Bregman quasi-nonexpansive mapping if $Fix(T) \neq \emptyset$ and

$$D_f(p, Tx) \leq D_f(p, x), \forall \ x \in Q \text{ and } p \in Fix(T).$$

(iii) Bregman firmly nonexpansive (BFNE), if

$$\langle \nabla_E^g(Tx) - \nabla_E^g(Ty), Tx - Ty\rangle \leq \langle \nabla_E^g(x) - \nabla_E^g(y), Tx - Ty\rangle, \ \forall \ x, y \in E.$$

Definition 1.2. [20] Let Q be a nonempty, closed and convex subset of a reflexive Banach space E and $g : E \to (-\infty, +\infty]$ be a strongly coercive Bregman function. Let β and γ be real numbers with $\beta \in (-\infty, 1)$ and $\gamma \in [0, \infty)$, respectively. Then a mapping $T : Q \to E$ with $Fix(T) \neq \emptyset$ is called Bregman (β, γ)-demigeneralized if for any $x \in Q$ and $p \in Fix(T)$,

$$\langle x - p, \nabla_E^g(x) - \nabla_E^g(Tx)\rangle \geq (1 - \beta)D_g(x, Tx) + \gamma D_g(Tx, x), \ \forall \ x \in E \text{ and } p \in F(T).$$

For modelling inverse problems which arises from phase retrievals and medical image reconstruction, (see [12]), Censor and Elfving [17] introduced the Split Feasibility Problem (SFP) in 1994, which is to find

$$u^* \in C \text{ such that } Ku^* \in Q; \tag{1.4}$$

where C and Q are nonempty, closed and convex subsets of real Banach spaces E_1 and E_2 respectively, and $K : E_1 \to E_2$ is a bounded linear operator. The SFP have been well studied in the framework of real Hilbert spaces, uniformly convex and uniformly smooth Banach spaces, see ([2, 19, 24, 43] and other references contained in). Different optimization problems have been formulated in terms of SFP (1.4), for instance, If $Q = \{b\}$ in SFP (1.4) is a singleton, then we have the following convexly constrained linear inverse problem (in short, CCLIP) defined as follows:

$$\text{Find a point } u^* \in C \text{ such that } Ku^* = b.$$

The Split Null Point Problem (SNPP) introduced by Bryne et al. [13] is formulated as finding a point

$$x \in H_1 \text{ such that } 0 \in B_1(x) \text{ and } 0 \in B_2(Kx), \tag{1.5}$$

where H_1 and H_2 are real Hilbert spaces, $B_1 : H_1 \to 2^{H_1}$ and $B_2 : H_2 \to 2^{H_2}$ are multivalued mappings and $K : H_1 \to H_2$ are real Hilbert spaces.
In 2018, Jailoka and Suantai [23] introduced the following Halpern iterative method for approximating the split null point and fixed point problems for maximal monotone operators and multivalued demicontractive mapping T as follows:

$$\begin{cases} u, x_1 \in H_1, \\ y_n = J_{\lambda_n}^{B_1}(x_n + \gamma K^*(J_{\lambda_n}^{B_2} - I)Kx_n), \\ u_n = (1-\delta)y_n + \delta z_n, \\ x_{n+1} = \alpha_n u + (1-\alpha_n)u_n, \ n \geq 1, \end{cases}$$

where $z_n \in Ty_n$. Also, Oyewole et al. [33] introduced a new iterative method with self adaptive step-size for approximating solutions of a SFP for sum of two monotone operators and fixed point problem of a demimetric mapping in real Hilbert spaces. Strong convergence result was proved and numerical experiment to illustrate the performance of the algorithm were displayed.
In the framework of uniformly convex and smooth Banach spaces, Takahashi and Takahashi [45] introduced a shrinking projection method to approximate a solution of SNPP. Using their iterative method, they proved a strong convergence theorem.
Question: Can the results of [3, 6, 13, 22, 23, 32, 33, 45] be establish in a more general Banach spaces (reflexive Banach spaces)?

Let $B : E \to 2^{E^*}$ be a set-valued mapping. We define the domain and range of B by $domB = \{x \in E : Bx \neq \emptyset\}$ and $ranB = \bigcup_{x \in E} Bx$, respectively. The graph of B denoted by $G(B) = \{(x, x^*) \in E \times E^* : x^* \in Bx\}$. The mapping $B \subset E \times E^*$ is said to be monotone [38] if $\langle x - y, x^* - y^* \rangle \geq 0$ whenever $(x, x^*), (y, y^*) \in B$. It is also said to be maximal monotone [37] if its graph is not contained in the graph of any other monotone operator on E. If $B \subset E \times E^*$ is maximal monotone, then the set $B^{-1}(0) = \{z \in E : 0 \in Bz\}$ is closed and convex. Also, the resolvent associated with B and λ for any $\lambda > 0$ is the mapping $J^g_{\lambda B} : E \to 2^E$ with $Fix(J^g_{\lambda B}) = B^{-1}(0)$ defined by

$$J^g_{\lambda B} := (\nabla^g_E + \lambda B)^{-1} \circ \nabla^g_E.$$

It is worth mentioning that a mapping $B : E \to 2^{E^*}$ is called Bregman inverse strongly monotone (BISM) on the set C if

$$C \cap (domg) \cap (int\ dom\ g) \neq \emptyset,$$

and for any $x, y \in C \cap (int\ dom\ g),\ \eta \in Ax$ and $\xi \in Ay$, we have

$$\langle \eta - \xi, (\nabla^{g^*}_{E^*}(x) - \eta) - \nabla^{g^*}_{E^*}(\nabla^g_E(y) - \xi)\rangle \geq 0.$$

The anti-resolvent $B^g_\lambda : E \to 2^E$ associated with the mapping $b : E \to 2^{E^*}$ and $\lambda > 0$ is defined by

$$B^g_\lambda := \nabla^g_E \circ (\nabla^g_E - \lambda B). \tag{1.6}$$

Let $A : E \to E^*$ be a single-valued monotone mapping and $B : E \to 2^{E^*}$ be a multivalued monotone mapping. Then, the Monotone Variational Inclusion Problem (MVIP) (also known as the problem of finding a zero of sum of two monotone mappings) is to find $x \in E$ such that

$$0^* \in A(x) + B(x). \tag{1.7}$$

We denote by Ω, the solution set of problem (1.7).
A simple and efficient method for solving (1.7) is the forward-backward splitting method introduced by Lions and Mercier [26] in a Hilbert space H. It is known that this method converges weakly to an element in (1.7) under the assumption that A is α-inverse strongly monotone. Note that the inverse strongly monotonicity of A is a strict assumption. To avoid this assumption, Tseng [48] introduced the following algorithm which is known as Tseng's splitting algorithm for solving (1.7) as follows:

$$\begin{cases} x_1 \in H, \\ y_n = J^B_{\lambda_n}(x_n - \lambda_n A x_n), \\ x_{n+1} = y_n - \lambda_n(Ay_n - Ax_n),\ \forall\ n \geq 1, \end{cases} \tag{1.8}$$

where $A : H \to H$ is monotone and L-Lipschitz continuous and $\{\lambda_n\}$ is the sequence of suitable stepsize in $(0, \frac{1}{L})$. He proved that the sequence $\{x_n\}$ generated by (1.8) converges weakly to an element in (1.7). It is well-known that the step size of Tseng's splitting method requires prior knowledge of the Lipschitz constant of the mapping. However, from a practical point of view, the Lipschitz constant is very difficult to approximate.
It is well known that many interesting problems arising from mechanics, economics, finance, nonlinear programming, applied sciences, optimization such as equilibrium and variational inequality problems can be solved using MVIP. Considerable efforts have been devoted to develop efficient iterative method to approximate solutions of MVIP in which the resolvent operator technique is one of the vital technique.
Many authors have considered approximating solutions of (1.7) together with fixed

point problems in real Hilbert and Banach spaces, see [3, 1, 5, 33, 42].
For instance, Okeke and Izuchukwu [32] studied and analysed an iterative method for approximating split feasibility problem and variational inclusion problem in p-uniformly convex Banach spaces which are uniformly smooth, they proved a strong convergence result for approximating the solution of the aforementioned problems. Shehu [40] considered the splitting method for finding zeros of the sum of maximal monotone operator and Lipschitz continuous monotone operator in Banach space. He proved weak and strong convergence results and give some applications of his main result. In the framework of 2-uniformly convex real Banach spaces which are also uniformly smooth, Abass et al. [4] investigated a shrinking algorithm for finding zeros of the sum of maximal monotone operators and Lipschitz continuous monotone operators which is also a common fixed point for finite family of relatively quasi-nonexpansive mappings.
Suppose $A = 0$ in (1.7), then (1.7) reduces to the following Monotone Inclusion Problem (MIP), which is to find $x \in E$ such that

$$0^* \in B(x). \tag{1.9}$$

Many results on MIP have been extended by authors from real Hilbert spaces to more general Banach spaces. For instance, Reich and Sabach [36] introduced some iterative algorithms and proved two strong convergence results for approximating a common solution of a finite family of MIP (1.9) in a reflexive Banach spaces. Recently, Timnak et al. [47] introduced a new Halpern-type iterative scheme for finding a common zero of finitely many maximal monotone mappings in a reflexive Banach spaces and prove the following strong convergence theorem.

Theorem 1.3. *Let E be a reflexive Banach space and $f : E \to \mathbb{R}$ be a strongly coercive Bregman function which is bounded on bounded subsets and uniformly convex and uniformly smooth on bounded subset of E. Let $A_i : E \to 2^{E^*}, i = 1, 2, ...,$ be N maximal monotone operators such that $Z := \cap_{i=1}^{N} A_i^{-1}(0^*) \neq \emptyset$. Let $\{\alpha_n\}_{n\in\mathbb{N}}$ and $\{\beta_n\}_{n\in\mathbb{N}}$ be two sequences in $(0,1)$ satisfying the following control conditions:*
(i) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$;
(ii) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$.
Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence generated by

$$\begin{cases} u \in E,\ x_1 \in E \text{ chosen arbitrarily}, \\ y_n = \nabla f^*[\beta_n \nabla f(x_n) + (1-\beta_n)\nabla f(Res^f_{\lambda_n A_N}) \cdots (Res^f_{r_1 A_1}(x_n))], \\ x_{n+1} = \nabla f^*[\alpha_n \nabla f(u) + (1-\alpha_n)\nabla f(y_n)], \end{cases} \tag{1.10}$$

for $n \in \mathbb{N}$, where ∇f is the gradient of f. If $r_i > 0$, for each $i = 1, 2, ..., N$, then the sequence $\{x_n\}_{n\in\mathbb{N}}$ defined in (1.10) converges strongly to $proj^f_Z u$ as $n \to \infty$.

Very recently, Ogbuisi and Izuchukwu [30] introduced an iterative algorithm and obtained a strong convergence result for approximating a zero of sum of two maximal monotone operators which is also a fixed point of a Bregman strongly nonexpansive mapping in the framework of a reflexive Banach spaces.

We will also like to emphasize that approximating a common solution of SNPP have some possible applications to mathematical models whose constraints can be expressed as SNPP. In fact, this happens in practical problems like signal processing, network resource allocation, image recovery, to mention a few, (see [21]). It is worth mentioning that the problem considered in this article generalizes the ones in [6, 18, 29].

Inspired by the results discussed above, we introduce an iterative algorithm which does not require the prior knowledge of operator norm as this may give difficulty in computing, to approximate a solution of split null point problem involving single-valued, multi-valued monotone and Lipschitz continuous monotone mappings in reflexive Banach spaces. Using our iterative algorithm, we prove a strong convergence result for approximating solutions of the aforementioned problems. Finally, we illustrate some numerical experiments to show the performance and behavior of our main result. The result discussed in this paper complements and extends many related results in literature.
We state our contributions in this article as follows:

1. The main result in this paper generalizes the results in [10], [**?**] and [32] from p-uniformly Banach spaces which are also uniformly smooth to reflexive Banach spaces and [5, 6, 18, 29, 31, 32, 47] from real Hilbert spaces to a reflexive Banach spaces.

2. The iterative method defined in this article is design in such a way that it does not depend on the operator norm, see [20, 33].

3. We proved a strong convergence result which is more desirable than the weak convergence result obtained in [44].

4. The sequence of stepsizes of our algorithms is chosen without the prior knowledge of the Lipschitz constant and the uniform smoothness constant of the mapping, see [40].

2. Preliminaries

We state some known and useful results which will be needed in the proof of our main result. In the sequel, we denote strong and weak convergence by "$\to$" and "$\rightharpoonup$", respectively.

Definition 2.1. A function $g : E \to \mathbb{R}$ is said to be strongly coercive if

$$\lim_{\|x\|\to\infty} \frac{g(x)}{\|x\|} = \infty.$$

Definition 2.2. A mapping $T : C \to E$ is said to be demiclosed at p if $\{x_n\}$ is a sequence in C such that $\{x_n\}$ converges weakly to some $x^* \in C$ and $\{Tx_n\}$ converges strongly to p, then $Tx^* = p$.

Lemma 2.3. [47] *Let E be a Banach space, $s > 0$ be a constant, ρ_s be the gauge of uniform convexity of g and $g : E \to \mathbb{R}$ be a strongly coercive Bregman function. Then,*
(i) For any $x, y \in B_s$ and $\alpha \in (0,1)$, we have

$$D_g\big(x, \nabla_{E^*}^{g^*}[\alpha \nabla_E^g \nabla_E^g(y) + (1-\alpha)\nabla_E^g(z)]\big) \leq \alpha D_g(x,y) + (1-\alpha)D_g(x,z) - \alpha(1-\alpha)\rho_s(\|\nabla_E^g(y) - \nabla_E^g(z)\|),$$

(ii) For any $x, y \in B_s := \{z \in E : \|z\| \leq s\},\ s > 0$,

$$\rho_s(\|x-y\|) \leq D_g(x,y).$$

Lemma 2.4. [16] *Let E be a reflexive Banach space, $g : E \to \mathbb{R}$ be a strongly coercive Bregman function and V be a function defined by*

$$V(x,x^*) = g(x) - \langle x, x^*\rangle + g^*(x^*),\ x \in E,\ x^* \in E^*.$$

The following assertions also hold:

$$D_g(x, \nabla_{E^*}^{g^*}(x^*)) = V(x,x^*),\ \textit{for all}\ x \in E\ \textit{and}\ x^* \in E^*.$$

$$V(x,x^*) + \langle \nabla_{E^*}^{g^*}(x^*) - x, y^*\rangle \leq V(x, x^*+y^*)\ \textit{for all}\ x \in E \textit{and}\ x^*, y^* \in E^*.$$

Also, following a similar approach as in Lemma 2.4 and for any $x \in E, y^*,\ z^* \in B_r$ and $\alpha \in (0,1)$, we have

$$V_g(x, \alpha y^* + (1-\alpha)z^*) \leq \alpha V_g(x,y^*) + (1-\alpha)V_g(x,z^*) - \alpha(1-\alpha)\rho_r^*(\|y^* - x^*\|). \tag{2.1}$$

Lemma 2.5. [20] *Let E_1 and E_2 be two Banach spaces. Let $F : E_1 \to E_2$ be a bounded linear operator and $T : E_2 \to E_2$ be a Bregman (ϕ, σ)-demigeneralized for some $\phi \in (-\infty, 1)$ and $\sigma \in [0, \infty)$. Suppose that $K = ran(A) \cap Fix(T) \neq \emptyset$ (where $ran(A)$ denotes the range of (A). Then for any $(x,q) \in E_1 \times K$,*

$$\begin{aligned}\langle x - q, F^*(\nabla_{E_2}^{g_2}(T(Fx)))\rangle &\geq (1-\phi)D_{g_2}(Fx, T(Fx)) + \sigma D_{g_2}(T(Fx), Fx)\\ &\geq (1-\phi)D_{g_2}(Fx, T(Fx)).\end{aligned} \tag{2.2}$$

So, given any real numbers ξ_1 and ξ_2, the mapping $L_1 : E_1 \to [0,\infty)$ and $L_2 : E_2 \to [0.\infty)$ formulated for $x \in E_1$ as

$$L_1(x) = \begin{cases} \frac{D_{g_2}(Fx,TFx)}{D_{g_1}^*(F^*(\nabla_{E_2}^{g_2}(Fx)), F^*(\nabla_{E_2}^{g_2}(TFx))}, & \textit{if} \quad (I-T)Fx \neq 0,\\ \xi_1, & \textit{otherwise,}\end{cases} \tag{2.3}$$

and

$$L_2(x) = \begin{cases} \frac{D_{g_1}^*(\nabla_{E_1}^{g_1}(x) - \gamma F^*(\nabla_{E_2}^{g_2}(Fx) - \nabla_{E_2}^{g_2}(TFx)), \nabla_{E_1}^{g_1}(x))}{D_{g_1}^*(F^*(\nabla_{E_2}^{g_2}(Fx)), F^*(\nabla_{E_2}^{g_2}(TFx))}, & \textit{if}\ , \quad (I-T)Fx \neq 0,\\ \xi_2, & \textit{otherwise,}\end{cases} \tag{2.4}$$

are well-defined, where γ is any nonnegative real number. Moreover, for any $(x,p) \in E_1 \times K$, we have

$$D_{g_1}(q,y) \leq D_{g_1}(q,x) - (\gamma(1-\phi)L_1(x) - L_2(x))D_{g_1^*}(F^*(\nabla_{E_2}^{g_2}(Fx)), F^*(\nabla_{E_2}^{g_2}(TFx)), \tag{2.5}$$

where

$$y = (\nabla^{g_1}_{E_1})^{-1}[\nabla^{g_1}_{E_1}(x) - \gamma F^*(\nabla^{g_2}_{E_2}(Fx) - \nabla^{g_2}_{E_2}(TFx))].$$

Remark 2.6. From Definition 2.2 of [20], It can be seen that $J^g_{\lambda B}$ is $(0,1)-$ demigeneralized. Therefore, we conclude from (2.5) that

$$D_{g_1}(q,y) \leq D_{g_1}(q,x) - (\gamma L_1(x) - L_2(x))D_{g_1^*}(F^*(\nabla^{g_2}_{E_2}(Fx)), F^*(\nabla^{g_2}_{E_2}(J^g_{\lambda B}Fx)), \tag{2.6}$$

where $T = J^g_{\lambda B}$ *and* $B : E \to 2^{E^*}$ *is a maximal monotone operator.*

Lemma 2.7. [16] *Let E be a Banach space and $g : E \to \mathbb{R}$ a Gâteaux differentiable function which is uniformly convex on bounded subsets of E. Let $\{x_n\}_{n\in\mathbb{N}}$ and $\{y_n\}_{n\in\mathbb{N}}$ be bounded sequences in E. Then,*

$$\lim_{n\to\infty} D_g(y_n, x_n) = 0 \Rightarrow \lim_{n\to\infty} \|y_n - x_n\| = 0.$$

Lemma 2.8. [7] *Let $A : E \to E^*$ be a monotone, hemicontinuous and bounded operator, and $B : E \to 2^{E^*}$ be a maximal monotone operator. Then $A+B$ is maximal monotone.*

Lemma 2.9. [36] *Let $g : E \to \mathbb{R}$ be a Gâteaux differentiable and totally convex function. If $x_0 \in E$ and the sequence $\{D_g(x_n, x_0)\}$ is bounded, then the sequence $\{x_n\}$ is also bounded.*

Definition 2.10. Let C be a nonempty closed and convex subset of a reflexive Banach space E and $g : E \to (-\infty, +\infty]$ be a strongly coercive Bregman function. A Bregman projection of $x \in int(dom(g))$ onto $C \subset int(domg)$ is the unique vector $P^g_C(x) \in C$ satisfying

$$D_g(P^g_C(x), x) = int\{D_g(y, x) : y \in C\}.$$

Lemma 2.11. [34] *Let C be a nonempty closed and convex subset of a reflexive Banach space E and $x \in E$. Let $g : E \to \mathbb{R}$ be a strongly coercive Bregman function. Then,*
(i) $z = P^g_C(x)$ if and only if $\langle \nabla^g_E(x) - \nabla^g_E(z), y - z\rangle \leq 0, \ \forall\ y \in C$.
(ii) $D_g(y, P^g_C(x)) + D_g(P^g_C(x), x) \leq D_g(y, x), \ \forall\ y \in C$.

Lemma 2.12. [50] *Let $\{a_n\}$, $\{\gamma_n\}$, $\{\delta_n\}$ and $\{t_n\}$ be sequences of nonnegative real numbers satisfying the following relation:*

$$a_{n+1} \leq (1 - t_n - \gamma_n)a_n + \gamma_n n a_{n-1} + t_n s_n + \delta_n, \ \forall n \geq 0,$$

where $\sum_{n=n_0}^{\infty} t_n = +\infty$, $\sum_{n=n_0}^{\infty} \delta_n < +\infty$, for each $n \geq n_0$ (where n_0 is a positive integer) and $\{\gamma_n\} \subset [0, \frac{1}{2}]$, $\limsup_{n\to\infty} s_n \leq 0$. Then, the sequence $\{a_n\}$ converges weakly to zero.

Lemma 2.13. [27] *Let Γ_n be a sequence of real numbers that does not decrease at infinity, in the sense that there exists a subsequence $\{\Gamma_{n_k}\}_{k\geq 0}$ of $\{\Gamma_n\}$ which satisfies $\Gamma_{n_k} \leq \Gamma_{n_j+1}$ for all $j \geq 0$. Also, consider a sequence of integers $\{\tau(n)\}_{n\geq n_0}$ defined by*

$$\tau(n) := \max\{k \leq n \mid \Gamma_{n_k} \leq \Gamma_{n_k+1}\}.$$

Then $\{\tau(n)\}_{n\geq n_0}$ *is a nondecreasing sequence satisfying* $\lim_{n\to\infty}\tau(n)=\infty$. *If it holds that* $\Gamma_{\tau(n)}\leq\Gamma_{\tau(n)+1}$ *for all* $n\geq n_0$ *then we have*

$$\Gamma_\tau(n)\leq\Gamma_{\tau(n)+1}.$$

3. Main result

Throughout this section, we assume that

Assumption 3.1.

1. E_1 *and* E_2 *be two reflexive Banach spaces,* $g_1 : E_1 \to (-\infty,+\infty]$ *and* $g_2 : E_2 \to (-\infty,+\infty]$ *be strongly coercive Bregman functions which are bounded on bounded subsets and uniformly convex and uniformly smooth on bounded subsets of* E_1 *and* E_2 *with constant* $\beta>0$*, respectively.*

2. $\nabla^{g_1}_{E_1}$ *and* $\nabla^{g_2}_{E_2}$ *be the gradients of* E_1 *dependent on* g_1 *and* E_2 *dependent on* g_2 *respectively.*

3. $A_1 : E_1 \to E_1^*$ *be a monotone and L-Lipschitz continuous mapping,* $B_1 : E_1 \to 2^{E_1^*}$ *and* $B_2 : E_2 \to 2^{E_2^*}$ *are maximal monotone mappings respectively, and* $J^{g_2}_{\lambda B_2}$ *be the resolvent of* g_2 *on* B_2 *for* $\lambda>0$*, and* $\lambda_n=\rho l^{m_n}$ *where* m_n *is the smallest nonnegative integer such that*
$$\lambda_n\|A_1z_n-A_1y_n\|\leq\mu\|z_n-y_n\|. \tag{3.1}$$

4. *Suppose that* $K : E_1 \to E_2$ *is a bounded linear operator such that* $K\neq 0$ *and* $K^* : E_2^* \to E_1^*$ *be the adjoint of* K*. Given that* $\rho>0$, $l\in(0,1)$, $\mu\in(0,\sigma)$*, where* σ *is a constant given by* (1.3).

5. *The control sequence* $\{\alpha_n\},\{\beta_n\}$ *and* $\{\delta_n\}$ *are sequences in* $(0,1)$ *such that* $\alpha_n+\beta_n+\delta_n=1,\{\theta_n\}\subset[0,\frac{1}{2}]$ *and the following conditions are satisfied:*

(i) $\lim\limits_{n\to\infty}\alpha_n=0,\ \sum\limits_{n=1}^{\infty}\alpha_n=\infty,$

(ii) $0<\liminf\limits_{n\to\infty}\beta_n\leq\limsup\limits_{n\to\infty}\beta_n<1,$

(iii) $0<a\leq\theta_n<\delta_n\leq\frac{1}{2},\ \forall\ n\geq 1.$

Algorithm 3.2. *Define a sequence* $\{x_n\}_{n=1}^\infty$ *generated arbitrarily by chosen* $x_0,x_1\in E_1$ *and any fixed* $u\in E_1$*, such that*

$$\begin{cases} w_n=(\nabla^{g_1}_{E_1})^{-1}[\nabla^{g_1}_{E_1}(x_n)+\theta_n(\nabla^{g_1}_{E_1}(x_{n-1})-\nabla^{g_1}_{E_1}(x_n))],\\ z_n=(\nabla^{g_1}_{E_1})^{-1}\big[\nabla^{g_1}_{E_1}(w_n)-\gamma K^*(\nabla^{g_2}_{E_2}(Kw_n)-\nabla^{g_2}_{E_2}(J^{g_2}_{\lambda B_2}Kw_n))\big]\\ y_n=J^{g_1}_{\lambda_n B_1}\big[(\nabla^{g_1}_{E_1})(z_n)-\lambda_nA_1z_n\big]\\ u_n=(\nabla^{g_1}_{E_1})^{-1}\big[\nabla^{g_1}_{E_1}(y_n)-\lambda_n(A_1y_n-A_1z_n)\big],\\ x_{n+1}=(\nabla^{g_1}_{E_1})^{-1}\big[\alpha_n\nabla^{g_1}_{E_1}(u)+\beta_n\nabla^{g_1}_{E_1}(x_n)+\delta_n\nabla^{g_1}_{E_1}(u_n)\big]. \end{cases} \tag{3.2}$$

Suppose that $\Omega := \{p \in (A_1 + B_1)^{-1}(0) : Kp \in B_2^{-1}(0)\} \neq \emptyset$, *let* $\gamma > 0$, *let the sequences* $\{\xi_{1,n}\}_{n\in\mathbb{N}}$ *and* $\{\xi_{2,n}\}_{n\in\mathbb{N}}$ *satisfy the following conditions:*

(i) *there exists a positive real number* ϕ_1 *such that*

$$0 < \phi_1 < \liminf_{n\to\infty} \frac{\xi_{2,n}}{\xi_{1,n}} < \gamma,$$

where

$$\xi_{1,n} = \begin{cases} \frac{D_{g_2}(Kw_n, J^{g_2}_{\lambda_n B_2} w_n)}{D^*_{g_1}(K^*(\nabla^{g_2}_{E_2}(Kw_n)), K^*(\nabla^{g_2}_{E_2}(J^{g_2}_{\lambda_n B_2} Kw_n))}, & \text{if} \quad (I - J^{g_2}_{\lambda_n B_2})Kw_n \neq 0, \\ \xi_1, & \text{otherwise}, \end{cases}$$

and

$$\xi_{2,n} = \begin{cases} \frac{D^*_{g_1}(\nabla^{g_1}_{E_1}(w_n) - \gamma K^*(\nabla^{g_2}_{E_2}(Kw_n) - \nabla^{g_2}_{E_2}(J^{g_2}_{\lambda_n B_2} Kw_n)), \nabla^{g_1}_{E_1}(w_n))}{D^*_{g_1}(K^*(\nabla^{g_2}_{E_2}(Kw_n)), K^*(\nabla^{g_2}_{E_2}(J^{g_2}_{\lambda_n B_2} Kw_n))}, \\ \quad \text{if } (I - J^{g_2}_{\lambda_n B_2})Kw_n \neq 0, \\ \xi_2, \quad \text{otherwise}. \end{cases}$$

Then, the sequence $\{x_n\}$ *generated iteratively converges strongly to* $z = P^{g_1}_\Omega u$, *where* $P^{g_1}_\Omega$ *is the Bregman projection of* E_1 *onto* Ω.

Proof. It can be seen in Lemma 3.2 of [44] that the Armijo linesearch rule defined by (3.1) is well-defined and

$$\min\{\rho, \frac{\mu l}{L}\} \leq \lambda_n \leq \rho.$$

Now, let $x^* \in \Omega$ then, using definition of u_n in (3.2) we have from (1.1) that

$$\begin{aligned} D_{g_1}(x^*, u_n) &= D_{g_1}\big(x^*, (\nabla^{g_1}_{E_1})^{-1}[\nabla^{g_1}_{E_1}(y_n) - \lambda_n(A_1 y_n - A_1 z_n)]\big) \\ &= g_1(x^*) - g_1(u_n) - \langle x^* - u_n, \nabla g_1(y_n) - \lambda_n(A_1 y_n - A_1 z_n)\rangle \\ &= g_1(x^*) - g_1(u_n) - \langle x^* - u_n, \nabla g_1(y_n)\rangle + \lambda_n\langle x^* - u_n, A_1 y_n - A_1 z_n\rangle \\ &= g_1(x^*) - g_1(y_n) - \langle x^* - y_n, \nabla g_1(y_n)\rangle + \langle x^* - y_n, \nabla g_1(y_n)\rangle \\ &+ g_1(y_n) - g_1(u_n) - \langle x^* - u_n, \nabla g_1(y_n)\rangle + \lambda_n\langle x^* - u_n, A_1 y_n - A_1 z_n\rangle \\ &= g_1(x^*) - g_1(y_n) - \langle x^* - y_n, \nabla g_1(y_n)\rangle - g_1(u_n) + g_1(y_n) \\ &+ \langle u_n - y_n, \nabla g_1(y_n)\rangle + \lambda_n\langle x^* - u_n, A_1 y_n - A_1 z_n\rangle \\ &= D_{g_1}(x^*, y_n) - D_{g_1}(u_n, y_n) + \lambda_n\langle x^* - u_n, A_1 y_n - A_1 z_n\rangle. \end{aligned} \tag{3.3}$$

Using (1.2), we get

$$D_{g_1}(x^*, u_n) = D_{g_1}(x^*, z_n) - D_{g_1}(y_n, z_n) + \langle x^* - y_n, \nabla g_1(z_n) - \nabla g_1(y_n)\rangle. \tag{3.4}$$

On substituting (3.4) into (3.3), we obtain

$$\begin{aligned} D_{g_1}(x^*, u_n) &= D_{g_1}(x^*, z_n) - D_{g_1}(y_n, z_n) - D_{g_1}(u_n, y_n) \\ &+ \langle x^* - y_n, \nabla g_1(z_n) - \nabla g_1(y_n)\rangle + \lambda_n \langle x^* - u_n, A_1 y_n - A_1 z_n\rangle \\ &= D_{g_1}(x^*, z_n) - D_{g_1}(y_n, z_n) - D_{g_1}(u_n, y_n) + \langle x^* - y_n, \nabla g_1(z_n) - \nabla g_1(y_n)\rangle \\ &+ \lambda_n \langle y_n - u_n, A_1 y_n - A_1 z_n\rangle - \lambda_n \langle y_n - x^*, A_1 y_n - A_1 z_n\rangle \\ &= D_{g_1}(x^*, z_n) - D_{g_1}(y_n, z_n) - D_{g_1}(u_n, y_n) + \lambda_n \langle y_n - u_n, A_1 y_n - A_1 z_n\rangle \\ &- \langle y_n - x^*, \nabla g_1(z_n) - \nabla g_1(y_n) - \lambda_n (A_1 z_n - A_1 y_n)\rangle. \qquad (3.5) \end{aligned}$$

By applying the definition of y_n, we have $\nabla g_1(z_n) - \lambda_n A_1 z_n \in \nabla g_1(y_n) + \lambda_n B_1$. Since $B_1 : E_1 \to 2^{E_1^*}$ is a maximal monotone mapping, there exists $a_n \in B_1 y_n$ such that $\nabla g_1(z_n) - \lambda_n A_1 z_n = \nabla g_1(y_n) + \lambda_n a_n$, it follows that

$$a_n = \frac{1}{\lambda_n}(\nabla g_1(z_n) - \nabla g_1(y_n) - \lambda_n A_1 z_n). \qquad (3.6)$$

Since $0 \in (A_1 + B_1)x^*$ and $A_1 y_n + a_n \in (A_1 + B_1) y_n$, it follows from Lemma 2.8 that $A_1 + B_1$ is maximal monotone, hence

$$\langle y_n - x^*, A_1 y_n + a_n\rangle \geq 0. \qquad (3.7)$$

On substituting (3.6) into (3.7), we get

$$\frac{1}{\lambda_n}\langle y_n - x^*, \nabla g_1(z_n) - \nabla g_1(y_n) - \lambda_n A_1 z_n + \lambda_n A_1 y_n\rangle \geq 0.$$

That is

$$\langle y_n - x^*, \nabla g_1(z_n) - \nabla g_1(y_n) - \lambda_n (A_1 z_n - A_1 y_n)\rangle \geq 0. \qquad (3.8)$$

Combining (3.5) and (3.8), and using (1.3), we have

$$\begin{aligned} D_{g_1}(x^*, u_n) &\leq D_{g_1}(x^*, z_n) - D_{g_1}(y_n, z_n) - D_{g_1}(u_n, y_n) + \lambda_n \langle y_n - u_n, A_1 y_n - A_1 z_n\rangle \\ &\leq D_{g_1}(x^*, z_n) - D_{g_1}(y_n, z_n) - D_{g_1}(u_n, y_n) + \lambda_n ||y_n - u_n||\ ||A_1 y_n - A_1 z_n|| \\ &\leq D_{g_1}(x^*, z_n) - D_{g_1}(y_n, z_n) - D_{g_1}(u_n, y_n) + \mu \|y_n - u_n\|\ \|y_n - z_n\| \\ &\leq D_{g_1}(x^*, z_n) - D_{g_1}(y_n, z_n) - D_{g_1}(u_n, y_n) + \mu\big(\|y_n - u_n\|^2 + \|y_n - z_n\|^2\big) \\ &\leq D_{g_1}(x^*, z_n) - \big(1 - \frac{\mu}{\sigma}\big) D_{g_1}(y_n, z_n) - \big(1 - \frac{\mu}{\sigma}\big) D_{g_1}(y_n, u_n) \qquad (3.9) \\ &\leq D_{g_1}(x^*, z_n). \qquad (3.10) \end{aligned}$$

Also, from (2.6) and (3.2), we get

$$\begin{aligned}
D_{g_1}(x^*, z_n) &= D_{g_1}\Bigg((\nabla^{g_1}_{E_1})^{-1}\big(\nabla^{g_1}_{E_1}(w_n) - \gamma K^*(\nabla^{g_2}_{E_2}(Kw_n - \nabla^{g_2}_{E_2}(J^{g_2}_{\lambda B_2}Kw_n)))\big)\Bigg) \\
&\leq D_{g_1}(x^*, w_n) - (\gamma\xi_{1,n} - \xi_{2,n}) D_{g_1^*}\big(K^*(\nabla^{g_2}_{E_2}(Kw_n)), K^*(\nabla^{g_2}_{E_2}(J^{g_2}_{\lambda B_2}Kw_n))\big) \qquad (3.11) \\
&\leq D_{g_1}(x^*, w_n) \qquad (3.12) \\
&= D_{g_1}\Bigg(x^*, (\nabla^{g_1}_{E_1})^{-1}\big(\nabla^{g_1}_{E_1}(x_n) + \theta_n(\nabla^{g_1}_{E_1}(x_{n-1}) - \nabla^{g_1}_{E_1}(x_n))\big)\Bigg) \\
&\leq (1-\theta_n) D_{g_1}(x^*, x_n) + \theta_n D_{g_1}(x^*, x_{n-1}). \qquad (3.13)
\end{aligned}$$

From (2.1), (3.2), (3.9) and (3.10), we get

$$\begin{aligned}
D_{g_1}(x^*, x_{n+1}) &\leq D_{g_1}\Bigg(x^*, (\nabla^{g_1}_{E_1})^{-1}\big(\alpha_n \nabla^{g_1}_{E_1}(u) + \beta_n \nabla^{g_1}_{E_1}(x_n) + \delta_n \nabla^{g_1}_{E_1}(u_n)\big)\Bigg) \\
&\leq V_{g_1}\Bigg(x^*, \alpha_n \nabla^{g_1}_{E_1}(u) + \beta_n \nabla^{g_1}_{E_1}(x_n) + \delta_n \nabla^{g_1}_{E_1}(u_n)\Bigg) \\
&= g_1(x^*) - \langle x^*, \alpha_n \nabla^{g_1}_{E_1}(u) + \beta_n \nabla^{g_1}_{E_1}(x_n) + \delta_n \nabla^{g_1}_{E_1}(u_n)\rangle \\
&+ g_1^*\big(\alpha_n \nabla^{g_1}_{E_1}(u) + \beta_n \nabla^{g_1}_{E_1}(x_n) + \delta_n \nabla^{g_1}_{E_1}(u_n)\big) \\
&\leq \alpha_n g_1(x^*) + \beta_n g_1(x^*) + \delta_n g_1(x^*) - \beta_n \langle x^*, \nabla^{g_1}_{E_1}(x_n)\rangle \\
&- \delta_n \langle x^*, \nabla^{g_1}_{E_1}(u_n)\rangle - \alpha_n \langle x^*, \nabla^{g_1}_{E_1}(u)\rangle + \beta_n g_1^*(\nabla^{g_1}_{E_1}(x_n)) \\
&+ \delta_n g_1^*(\nabla^{g_1}_{E_1}(u_n)) + \alpha_n g_1^*(\nabla^{g_1}_{E_1}(u)) - \beta_n\delta_n \rho_r^*(\|\nabla^{g_1}_{E_1}(x_n) - \nabla^{g_1}_{E_1}(u_n)\|) \\
&- \beta_n\delta_n \rho_r^*(\|\nabla^{g_1}_{E_1}(x_n) - \nabla^{g_1}_{E_1}(u)\|) \\
&\leq \beta_n\Bigg(g_1(x^*) - \langle x^*, \nabla^{g_1}_{E_1}(x_n)\rangle + g_1^*(\nabla^{g_1}_{E_1}(x_n))\Bigg) \\
&+ \delta_n\Bigg(g_1(x^*) - \langle x^*, \nabla^{g_1}_{E_1}(u_n)\rangle + g_1^*(\nabla^{g_1}_{E_1}(u_n))\Bigg) \\
&+ \alpha_n\Bigg(g_1(x^*) - \langle x^*, \nabla^{g_1}_{E_1}(u)\rangle + g_1^*\Bigg(\nabla^{g_1}_{E_1}(u)\Bigg)\Bigg) \\
&- \beta_n\delta_n \rho_r^*\big(\|\nabla^{g_1}_{E_1}(x_n) - \nabla^{g_1}_{E_1}(u_n)\|\big) \\
&= \beta_n V_{g_1}(x^*, \nabla^{g_1(x_n)}_{E_1}) + \delta_n V_{g_1}(x^*, \nabla^{g_1}_{E_1}(u_n)) + \alpha_n V_{g_1}(x^*, \nabla^{g_1}_{E_1}(u)) \\
&- \beta_n\delta_n \rho_r^*\big(\|\nabla^{g_1}_{E_1}(x_n) - \nabla^{g_1}_{E_1}(u_n)\|\big) \\
&\leq \beta_n D_{g_1}(x^*, x_n) + \delta_n D_{g_1}(x^*, u_n) + \alpha_n D_{g_1}(x^*, u) \\
&- \beta_n\delta_n \rho_r^*\big(\|\nabla^{g_1}_{E_1}(x_n) - \nabla^{g_1}_{E_1}(u_n)\|\big) \\
&\leq \beta_n D_{g_1}(x^*, x_n) \\
&+ \delta_n\Bigg(D_{g_1}(x^*, w_n) - \big(1 - \frac{\mu}{\sigma}\big) D_{g_1}(y_n, z_n) - \big(1 - \frac{\mu}{\sigma}\big) D_{g_1}(y_n, u_n) \\
&- (\gamma\xi_{1,n} - \xi_{2,n}) D_{g_1^*}\big(K^*(\nabla^{g_2}_{E_2}(Kw_n)), K^*(\nabla^{g_2}_{E_2}(J^{g_2}_{\lambda B_2}Kw_n))\big)
\end{aligned}$$

$$
\begin{aligned}
&+ \alpha_n D_{g_1}(x^*, u) - \beta_n \delta_n \rho_r^* \big(\|\nabla_{E_1}^{g_1}(x_n) - \nabla_{E_1}^{g_1}(u_n)\|\big) \\
&\leq \beta_n D_{g_1}(x^*, x_n) + \delta_n(1-\theta_n)\big(D_{g_1}(x^*, x_n) + \delta_n\theta_n D_{g_1}(x^*, x_{n-1})\\
&- \delta_n\big(1 - \frac{\mu}{\sigma}\big)\big(D_{g_1}(y_n, z_n) - D_{g_1}(y_n, u_n)\big)\\
&- \delta_n(\gamma\xi_{1,n} - \xi_{2,n}) D_{g_1^*}\big(K^*(\nabla_{E_2}^{g_2}(Kw_n)), K^*(\nabla_{E_2}^{g_2}(J_{\lambda B_2}^{g_2} Kw_n))\big)\\
&+ \alpha_n D_{g_1}(x^*, u) - \beta_n \delta_n \rho_r^* \big(\|\nabla_{E_1}^{g_1}(x_n) - \nabla_{E_1}^{g_1}(u_n)\|\big) \\
&\leq (1 - \alpha_n - \delta_n\theta_n) D_{g_1}(x^*, x_n) + \delta_n\theta_n D_{g_1}(x^*, x_{n-1}) + \alpha_n D_{g_1}(x^*, u)\\
&- \delta_n\big(1 - \frac{\mu}{\sigma}\big)\big(D_{g_1}(y_n, z_n) - D_{g_1}(y_n, u_n)\big)\\
&- \delta_n(\gamma\xi_{1,n} - \xi_{2,n}) D_{g_1^*}\big(K^*(\nabla_{E_2}^{g_2}(Kw_n)), K^*(\nabla_{E_2}^{g_2}(J_{\lambda B_2}^{g_2} Kw_n)\big)\\
&- \beta_n \delta_n \rho_r^* \big(\|\nabla_{E_1}^{g_1}(x_n) - \nabla_{E_1}^{g_1}(u_n)\|\big) \qquad (3.14)\\
&\leq (1 - \alpha_n - \delta_n\theta_n) D_{g_1}(x^*, x_n) + \delta_n\theta_n D_{g_1}(x^*, x_{n-1}) + \alpha_n D_{g_1}(x^*, u)\\
&\leq \max\{D_{g_1}(x^*, x_n), D_{g_1}(x^*, x_{n-1}), D_{g_1}(x^*, u)\}, \ \forall\, n \geq 1. \qquad (3.15)
\end{aligned}
$$

By induction, we obtain that

$$
D_{g_1}(x^*, x_n) \leq \max\{D_{g_1}(x^*, x_1), D_{g_1}(x^*, x_0), D_{g_1}(x^*, u)\}.
$$

Hence, $\{D_{g_1}(x^*, x_n)\}$ is bounded and therefore we conclude that from Lemma 2.9 that $\{x_n\}$ is bounded. More so, $\{w_n\}, \{z_n\}, \{y_n\}$ and $\{u_n\}$ are bounded. The remaining proof is divided into two cases.

Case A: If there exists $n_0 \in \mathbb{N}$ such that $\{D_{g_1}(x^*, x_n)\}_{n=n_0}^{N}$ is decreasing, then $\{D_{g_1}(x^*, x_n)\}_{n\in\mathbb{N}}$ is convergent. Thus, we have that $D_{g_1}(x^*, x_n) - D_{g_1}(x^*, x_{n+1}) \to 0$, as $n \to \infty$. Hence, from (3.14), we have that

$$
\begin{aligned}
&\delta_n\big(1 - \frac{\mu}{\sigma}\big)\Bigg(D_{g_1}(y_n, z_n) - D_{g_1}(y_n, u_n)\Bigg)\\
&- \delta_n(\gamma\xi_{1,n} - \xi_{2,n}) D_{g_1^*}\big(K^*(\nabla_{E_2}^{g_2}(Kw_n)), K^*(\nabla_{E_2}^{g_2}(J_{\lambda B_2}^{g_2} Kw_n))\big)\\
&\leq (1 - \alpha_n) D_{g_1}^*(x^*, x_n) - D_{g_1}(x^*, x_{n+1}) + \delta_n\theta_n\big(D_{g_1}(x^*, x_{n-1}) - D_{g_1}(x^*, x_n)\big)\\
&+ \alpha_n D_{g_1}(x^*, u). \qquad (3.16)
\end{aligned}
$$

On applying condition (i) and (ii), we obtain that

$$
\lim_{n\to\infty} D_{g_1}(y_n, z_n) = 0 = \lim_{n\to\infty} D_{g_1}(y_n, u_n). \tag{3.17}
$$

From Lemma 2.7, we get that

$$
\lim_{n\to\infty} \|y_n - z_n\| = 0 = \lim_{n\to\infty} \|y_n - u_n\|. \tag{3.18}
$$

Since g_1 is bounded and uniformly smooth on bounded sets of E_1, it follows that $\nabla_{E_1}^{g_1}$ is uniformly continuous on bounded subsets of E_1. Thus, we conclude from (3.18) that

$$
\lim_{n\to\infty} \|\nabla_{E_1}^{g_1}(y_n) - \nabla_{E_1}^{g_1}(z_n)\| = 0. \tag{3.19}
$$

From (3.18), we have

$$\lim_{n\to\infty} \|u_n - z_n\| = 0. \tag{3.20}$$

Also, from (3.16), we have

$$\lim_{n\to\infty} \beta_n\delta_n\rho_r^*\Bigg(\|\nabla_{E_1}^{g_1}(x_n) - \nabla_{E_1}^{g_1}(u_n)\|\Bigg) = 0 \tag{3.21}$$

$$= \lim_{n\to\infty} \delta_n(\gamma\xi_{1,n} - \xi_{2,n})D_{g_1^*}\big(K^*(\nabla_{E_2}^{g_2}(Kw_n)), K^*(\nabla_{E_2}^{g_2}(J_{\lambda B_2}^{g_2}Kw_n))\big). \tag{3.22}$$

By Lemma 2.7 and from properties of the functions $\rho_r,\ D_{g_1}^*$ and K, we have

$$\lim_{n\to\infty} \|K^*(\nabla_{E_2}^{g_2}(Kw_n)) - K^*(\nabla_{E_2}^{g_2}(J_{\lambda B_2}^{g_2}Kw_n))\| = 0, \tag{3.23}$$

and

$$\lim_{n\to\infty} (\|\nabla_{E_1}^{g_1}(x_n) - \nabla_{E_1}^{g_1}(u_n)\|) = 0. \tag{3.24}$$

Employing Lemma 2.7, we arrive at

$$\lim_{n\to\infty} \|Kw_n - J_{\lambda B_2}^{g_2}Kw_n) = 0. \tag{3.25}$$

and

$$\lim_{n\to\infty} \|u_n - x_n\| = 0. \tag{3.26}$$

In view of (3.2), we obtain that

$$\lim_{n\to\infty} \|z_n - w_n\| = 0. \tag{3.27}$$

From (3.20) and (3.26), we ge that

$$\lim_{n\to\infty} \|u_n - x_n\| = 0. \tag{3.28}$$

From (3.2), it is easy to see that

$$\begin{aligned}\|\nabla_{E_1}^{g_1}(x_{n+1}) - \nabla_{E_1}^{g_1}(x_n)\| &\le \alpha_n\|\nabla_{E_1}^{g_1}(u) - \nabla_{E_1}^{g_1}(x_n)\| + \beta_n\|\nabla_{E_1}^{g_1}(x_n) - \nabla_{E_1}^{g_1}(x_n)\| \\ &\quad + \delta_n\|\nabla_{E_1}^{g_1}(u_n) - \nabla_{E_1}^{g_1}(x_n)\|.\end{aligned} \tag{3.29}$$

Hence, we have from (3.29) and condition (i) of (3.2) that

$$\lim_{n\to\infty} \|\nabla_{E_1}^{g_1}(x_{n+1}) - \nabla_{E_1}^{g_1}(x_n)\| = 0. \tag{3.30}$$

Since $\nabla_{E_1}^{g_1}$ is norm to norm uniformly continuous on bounded subset of E_1^*, we have

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0. \tag{3.31}$$

From (3.18) and (3.26), we get that

$$\lim_{n\to\infty} \|y_n - x_n\| = 0. \tag{3.32}$$

From (3.2), we obtain from (3.31)

$$\|\nabla_{E_1}^{g_1}(w_n) - \nabla_{E_1}^{g_1}(x_n)\| = \theta_n\|\nabla_{E_1}^{g_1}(x_{n-1}) - \nabla_{E_1}^{g_1}(x_n)\| \to 0, \text{ as } n\to\infty.. \tag{3.33}$$

Using the fact that $\nabla^{g_1}_{E_1}$ is norm to norm uniformly continuous on bounded subset of E_1^*, we have

$$\lim_{n\to\infty} \|w_n - x_n\| = 0. \tag{3.34}$$

Lastly, with (3.27) and (3.34), we arrive at

$$\lim_{n\to\infty} \|z_n - x_n\| = 0. \tag{3.35}$$

Since $\{x_n\}_{n\in\mathbb{N}}$ is bounded and E_1 is reflexive, we deduce that there exists a subsequence $\{x_{n_j}\}_{j\in\mathbb{N}}$ of $\{x_n\}_{n\in\mathbb{N}}$ which converges weakly to z. Also, from (3.28), (3.32), (3.34) and (3.35), we have that there exist subsequences $\{u_{n_j}\}_{j\in\mathbb{N}}$ of $\{u_n\}_{n\in\mathbb{N}}$, $\{y_{n_j}\}_{j\in\mathbb{N}}$ of $\{y_n\}_{n\in\mathbb{N}}$, $\{w_{n_j}\}_{j\in\mathbb{N}}$ of $\{w_n\}_{n\in\mathbb{N}}$ and $\{z_{n_j}\}_{j\in\mathbb{N}}$ of $\{z_n\}_{n\in\mathbb{N}}$ converge weakly to z respectively. Hence, from (3.25) and the demiclosedness principle we have that $J^{g_2}_{\lambda B_2}(Kz) = Kz$, therefore we conclude that $Kz \in B_2^{-1}(0)$. To show that $z \in (A_1 + B_1)^{-1}(0)$. Let $(v, w) \in G(A_1 + B_1)$, we have $w - A_1 v \in B_1 v$. From the definition of y_n, we observe that

$$\nabla^{g_1}_{E_1}(z_n) - \lambda_n A_1 z_n \in \nabla^{g_1}_{E_1}(y_n) + \lambda_n B_1 y_n,$$

or equivalently

$$\frac{1}{\lambda_n}(\nabla^{g_1}_{E_1}(z_n) - \nabla^{g_1}_{E_1}(y_n) - \lambda_n A_1 z_n) \in B_1 y_n.$$

By the maximal monotonicity of B_1, we get

$$\langle v - y_n, w - A_1 v + \frac{1}{\lambda_n}(\nabla^{g_1}_{E_1}(z_n) - \nabla^{g_1}_{E_1}(y_n) - \lambda_n A_1 z_n)\rangle \geq 0.$$

Also, from the monotonicity of A_1, we have

$$\begin{aligned}
\langle v - y_n, w\rangle &\geq \langle v - y_n, A_1 v + \frac{1}{\lambda_n}(\nabla^{g_1}_{E_1}(z_n) - \nabla^{g_1}_{E_1}(y_n) - \lambda_n A_1 z_n)\rangle \\
&= \langle v - y_n, A_1 v - A_1 z_n\rangle + \frac{1}{\lambda_n}\langle v - y_n, \nabla^{g_1}_{E_1}(z_n) - \nabla^{g_1}_{E_1}(y_n)\rangle \\
&= \langle v - y_n, A_1 v - A_1 y_n\rangle + \langle v - y_n, A_1 y_n - A_1 z_n\rangle \\
&+ \frac{1}{\lambda_n}\langle v - y_n, \nabla^{g_1}_{E_1}(z_n) - \nabla^{g_1}_{E_1}(y_n)\rangle \\
&\geq \langle v - y_n, A_1 y_n - A_1 z_n\rangle + \frac{1}{\lambda_n}\langle v - y_n, \nabla^{g_1}_{E_1}(z_n) - \nabla^{g_1}_{E_1}(y_n)\rangle. \qquad (3.36)
\end{aligned}$$

Since A_1 is Lipschitz continuous and $y_{n_j} \rightharpoonup z$, it follows from (3.18) and (3.19) that

$$\langle v - z, w\rangle \geq 0.$$

By the monotonicity of $A_1 + B_1$, we get $0 \in (A_1 + B_1)z$, that is $z \in (A_1 + B_1)^{-1}(0)$. Hence $z \in \Omega$.
Next, we show that $\{x_n\}$ converges strongly to z, where $z = P^{g_1}_{\Omega} u$.

From Lemma 2.11, we have

$$\begin{aligned}\limsup_{n\to\infty}\langle x_n - x^*, \nabla^{g_1}_{E_1}(u) - \nabla^{g_1}_{E_1}(x^*)\rangle &= \lim_{j\to\infty}\langle x_{n_j} - x^*, \nabla^{g_1}_{E_1}(u) - \nabla^{g_1}_{E_1}(x^*)\rangle \\ &= \langle z - x^*, \nabla^{g_1}_{E_1}(u) - \nabla^{g_1}_{E_1}(x^*)\rangle \\ &\le 0, \end{aligned} \tag{3.37}$$

and hence from (3.31), we obtain

$$\limsup_{n\to\infty}\langle x_{n+1} - x^*, \nabla^{g_1}_{E_1}(u) - \nabla^{g_1}_{E_1}(x^*)\rangle \le 0. \tag{3.38}$$

Using Lemma 2.4, (3.10) and (3.12), we obtain

$$\begin{aligned} D_{g_1}(z, x_{n+1}) &\le D_{g_1}\Big(z, (\nabla^{g_1}_{E_1})^{-1}(\beta_n\nabla^{g_1}_{E_1}(x_n) + \delta_n\nabla^{g_1}_{E_1}(u_n) + \alpha_n\nabla^{g_1}_{E_1}(u))\Big) \\ &= V_{g_1}(z, \beta_n\nabla^{g_1}_{E_1}(x_n) + \delta_n\nabla^{g_1}_{E_1} + \alpha_n\nabla^{g_1}_{E_1}(u)) - \alpha_n(\nabla^{g_1}_{E_1}(u) - \nabla^{g_1}_{E_1}(z)) \\ &+ \alpha_n\langle x_{n+1} - z, \nabla^{g_1}_{E_1}(u) - \nabla^{g_1}_{E_1}(z)\rangle \\ &= \beta_n D_{g_1}(z, x_n) + \delta_n D_{g_1}(z, u_n) + \alpha_n\langle x_{n+1} - z, \nabla^{g_1}_{E_1}(u) - \nabla^{g_1}_{E_1}(z)\rangle \\ &\le \beta_n D_{g_1}(z, x_n) + \delta_n D_{g_1}(z, w_n) + \alpha_n\langle x_{n+1} - z, \nabla^{g_1}_{E_1}(u) - \nabla^{g_1}_{E_1}(z)\rangle \\ &\le \beta_n D_{g_1}(z, x_n) + \delta_n((1-\theta_n)D_{g_1}(z, x_n) + \theta_n D_{g_1}(z, x_{n-1})) \\ &+ \alpha_n\langle x_{n+1} - z, \nabla^{g_1}_{E_1}(u) - \nabla^{g_1}_{E_1}(z)\rangle \\ &\le (1 - \alpha_n - \delta_n\theta_n)D_{g_1}(z, x_n) + \delta_n\theta_n D_{g_1}(z, x_{n-1}) \\ &+ \alpha_n\langle x_{n+1} - z, \nabla^{g_1}_{E_1}(u) - \nabla^{g_1}_{E_1}(z)\rangle. \end{aligned} \tag{3.39}$$

By applying (3.39) and Lemma 2.12, we have that $x_n \to z$.

Case B: Assume $\{D_{g_1}(z, x_n)\}$ is non-decreasing. Set Γ_n of Lemma 2.13, as $\Gamma_n := D_{g_1}(z, x_n)$ and let $\tau : \mathbb{N} \to \mathbb{N}$ be a mapping for all $n \ge n_0$ (for some n_0 large enough), defined by

$$\tau(n) := \max\{k \in \mathbb{N} : k \le n, \Gamma_k \le \Gamma_{k+1}\}.$$

Then τ is non-decreasing sequence such that $\tau(n) \to \infty$ as $n \to \infty$. Thus

$$0 < \Gamma_{\tau(n)} \le \Gamma_{\tau(n)+1},\ \forall\, n \ge n_0,$$

this implies that

$$D_{g_1}(z, x_{\tau(n)}) \le D_{g_1}(z, x_{\tau(n)+1}),\ n > n_0.$$

Since $\{D_{g_1}(z, x_{\tau(n)})\}$ is bounded, therefore $\lim_{n\to\infty} D_{g_1}(z, x_{\tau(n)})$ exists. Then the following estimates can be obtained, using same argument as in case A above.

$$\begin{cases} \lim_{n\to\infty}\|y_{\tau(n)} - z_{\tau(n)}\| = 0, \\ \lim_{n\to\infty}\|Kw_{\tau(n)} - J^{g_2}_{\lambda B_2}Kw_{\tau(n)}\| = 0, \\ \lim_{n\to\infty}\|z_{\tau(n)} - x_{\tau(n)}\| = 0, \\ \lim_{n\to\infty}\|w_{\tau(n)} - x_{\tau(n)}\| = 0, \\ \limsup_{n\to\infty}\langle x_{\tau(n)} - z, \nabla^{g_1}_{E_1}(u) - \nabla^{g_1}_{E_1}(z)\rangle \le 0. \end{cases} \tag{3.40}$$

From (3.39) and $\Gamma_{\tau(n)} \leq \Gamma_{\tau(n)+1}$, we have

$$D_{g_1}(z, x_{\tau(n)}) \leq (1-\alpha_{\tau(n)})D_{g_1}(z, x_{\tau(n)}) + \delta_{\tau(n)}\theta_{\tau(n)}\big(D_{g_1}(z, x_{\tau(n)-1} - D_{g_1}(z, x_{\tau(n)})\big)$$
$$+ \alpha_{\tau(n)}\langle x_{\tau(n)+1} - z, \nabla^{g_1}_{E_1}(u) - \nabla^{g_1}_{E_1}(z)\rangle.$$

and hence

$$\lim_{n\to\infty} \Gamma_{\tau(n)} = \lim_{n\to\infty} \Gamma_{\tau(n)+1} = 0,$$

for all $n \geq n_0$, we have $\Gamma_{\tau(n)} \leq \Gamma_{\tau(n)+1}$, if $n \neq \tau(n)$ (that is, $\tau(n) < n$), because $\Gamma_{k+1} \leq \Gamma_k$, for $\tau(n) \leq k \leq n$. This gives for all $n \geq n_0$

$$0 < \Gamma_n \leq \max\{\Gamma_{\tau(n)}, \Gamma_{\tau(n)+1}\} = \Gamma_{\tau(n)+1}.$$

This implies that $\lim_{n\to\infty} \Gamma_n = 0$ which yields that $\lim_{n\to\infty} D_{g_1}(z, x_n) = 0$. Hence, $x_n \to z = P^{g_1}_{\Omega} u$ as $n \to \infty$. □

Remark 3.3. Our main result improve and generalize the main results of [22, 23, 33, 40, 45] in the following ways:

- (i) We extend Theorem 3.1 of [40] from 2-uniformly Banach spaces which are uniformly smooth to a reflexive Banach space and also extend the results of [22, 23, 45] from real Hilbert spaces to reflexive Banach spaces.
- (ii) We relax the strict assumption of the mapping A in [22, 23, 33] with the weaker assumption that A is a monotone and L-Lipschitz continuous mapping.

4. Numerical examples

In this section, we give a couple of examples to implement our main result.

Example 4.1. This is an implementation of our result in infinite dimensional Hilbert space with our application to split feasibility problem. Let C and Q be nonempty, closed and convex subsets of real Hilbert spaces H_1 and H_2, respectively. Let $K : H_1 \to H_2$ be a bounded linear operator with its adjoint K^* and Θ denote the solution set of (1.4). Let $H_1 = H_2 = L_2([0,1])$ with norm

$$\|x\|_2 = \left(\int_0^1 |x(t)|^2 dt\right)^{\frac{1}{2}},$$

and inner product

$$\langle x, y\rangle = \int_0^1 x(t)y(t)dt,$$

for all $x, y \in L_2([0,1])$.
Now, let

$$C = \{x \in L_2([0,1]) : \|x\| \leq 1\},$$

and

$$Q = \{x \in L_2([0,1]) : \langle \frac{t}{2}, x\rangle = 0\}.$$

Let $K : L_2([0,1]) \to L_2([0,1])$ be a mapping defined by $(Kx)(t) = \frac{x(t)}{3}$ for all $x \in L_2([0,1])$. Then, we have $(K^*x)(t) = \frac{x(t)}{3}$ and $\|K\| = \frac{1}{3}$. We see that the $\Theta \neq \emptyset$ because $x^*(t) = 0$ is a solution. We define

$$A_1(x) = \nabla\left(\frac{1}{2}\|Kx - P_Q Kx\|^2\right) = K^*(I - P_Q)Kx,\ B_1(x) = N_C(x)$$

and

$$B_2(x) = N_Q(x) \text{ for all } x \in L_2([0,1]).$$

For our algorithm, we take

$$\alpha_n = \frac{1}{12n+3},\ \beta_n = \frac{8n+1}{12n+3},\ \delta_n = \frac{4n+1}{12n+3},$$

$$\gamma = 0.002,\ l = 0.0001,\ \mu = 0.03 \text{ and } \theta_n = \frac{1}{4}.$$

We present the result of this experiment in Figure 1 with $\|x_{n+1} - x_n\|_2 = 10^{-4}$ and varying initial values of x_0 and x_1 as follows:

(I) $x_0 = t^{\frac{2}{3}} + 11t$ and $x_1 = t$;
(II) $x_0 = 2t$ and $x_1 = \cos t$;
(III) $x_0 = -2t + 5$ and $x_1 = t + 1$;
(IV) $x_0 = 2t$ and $x_1 = \frac{7t^2}{11}$;

Example 4.2. Let $E_1 = E_2 = E = \mathbb{R}^2$ be the two-dimensional Euclidean space of the real number with an inner product $\langle \cdot, \cdot \rangle : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$\langle x, y \rangle = x \cdot y = x_1 y_1 + x_2 y_2$$

where $x = (x_1, x_2) \in \mathbb{R}^2$ and $y = (y_1, y_2) \in \mathbb{R}^2$ and a usual norm $\|\cdot\| : \mathbb{R}^2 \to \mathbb{R}$ be defined by $\|x\| = (x_1^2 + x_1^2)^{\frac{1}{2}}$ where $x = (x_1, x_2) \in \mathbb{R}^2$. Let $B_1 : \mathbb{R}^2 \to \mathbb{R}^2$ and $B_2 : \mathbb{R}^2 \to \mathbb{R}^2$ be defined respectively by

$$B_1 = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, B_2 = \begin{pmatrix} 1 & 2 \\ 2 & 5 \end{pmatrix}.$$

Since B_1 and B_2 are positive definite, they are maximal monotone operators. Also, let $A_1 : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by

$$A_1(x) = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

Now, define $h_i : \mathbb{R} \to (-\infty, +\infty]$ by $h_i(x) = \frac{x^2}{2}$ for $i = 1, 2$, then $\nabla h_i(x) = x$. We also define $g_1 = g_2 = g$ by

$$g : \mathbb{R}^2 \to (-\infty, +\infty], \quad g(x) = h_1(x_1) + h_2(x_2) = \frac{x_1^2}{2} + \frac{x_2^2}{2}, \quad x = (x_1, x_2).$$

Therefore, we have

$$\nabla g(x) = (\nabla h_1(x_1), \nabla h_2(x_2)) = (x_1, x_2) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

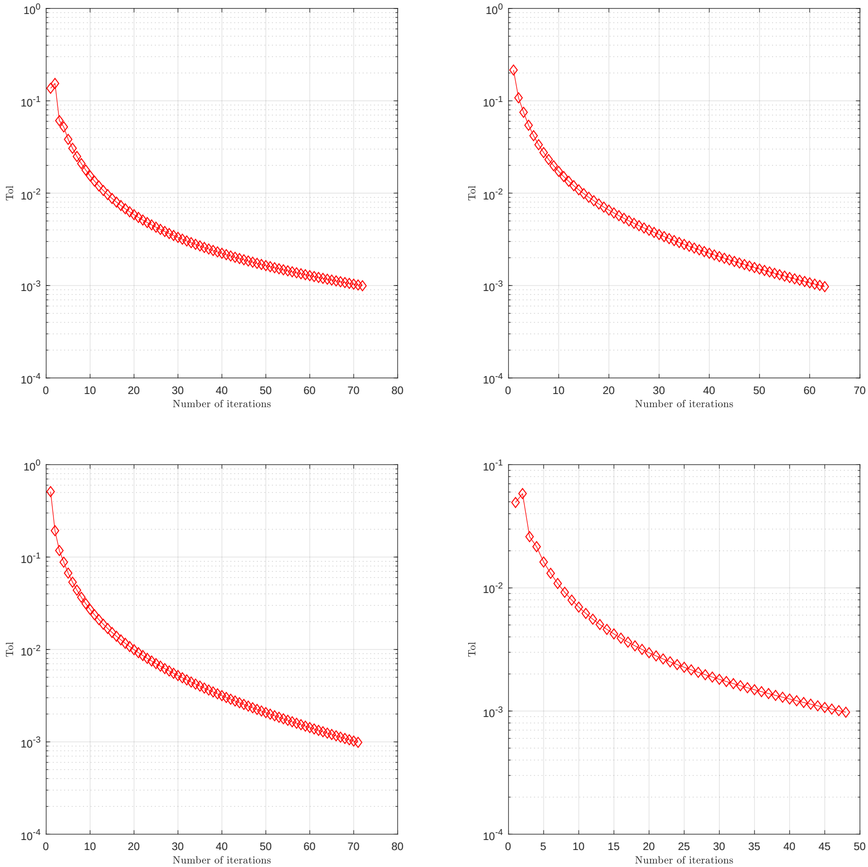

FIGURE 1. Example 4.1. Top left: Case I, Top right: II, Bottom left: III, Bottom right: IV.

For $\lambda > 0$, we compute the resolvents of B_1 and B_2 as follows:

$$J^{g_1}_{\lambda B_1} = \nabla g_1 + rB_1 = \begin{pmatrix} 1+\lambda & 2\lambda \\ 0 & 1+\lambda \end{pmatrix}, \quad (\nabla g_1 + rB_1)^{-1} = \frac{1}{(1+\lambda)^2} \begin{pmatrix} 1+\lambda & -2\lambda \\ 0 & 1+\lambda \end{pmatrix}$$

and

$$J^{g_2}_{\lambda B_2} = \nabla g_1 + rB_2 = \begin{pmatrix} 1+\lambda & 2\lambda \\ 2\lambda & 1+\lambda \end{pmatrix},$$

$$(\nabla g_1 + rB_2)^{-1} = \frac{1}{1+6\lambda+\lambda^2} \begin{pmatrix} 1+5\lambda & -2\lambda \\ -2\lambda & 1+\lambda \end{pmatrix}.$$

Let the operator $K : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by

$$K(x) = (2x_1 - x_2, x_1 + 2x_2) \text{ for all } x = (x_1, x_2) \in \mathbb{R}^2$$

and $K^* : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by

$$K^*(y) = (2y_1 - y_2, y_1 + 2y_2) \text{ for all } y = (y_1, y_2) \in \mathbb{R}^2.$$

For this experiment, we choose the parameters

$$\alpha_n = \frac{3n}{4n^2 + 5n + 3}, \ \beta_n = \frac{n^2 + 3}{4n^2 + 5n + 3}, \ \delta_n = \frac{3n^2 + 2n}{4n^2 + 5n + 3},$$

$$\gamma = 0.002, \ l = 0.0001, \ \mu = 0.03 \text{ and } \theta_n = \frac{1}{4}.$$

For $u = 0.1$ and initial values of x_0 and x_1, we report our test for the following cases in Figure 2 with $\|x_{n+1} - x_n\| = 10^{-5}$.

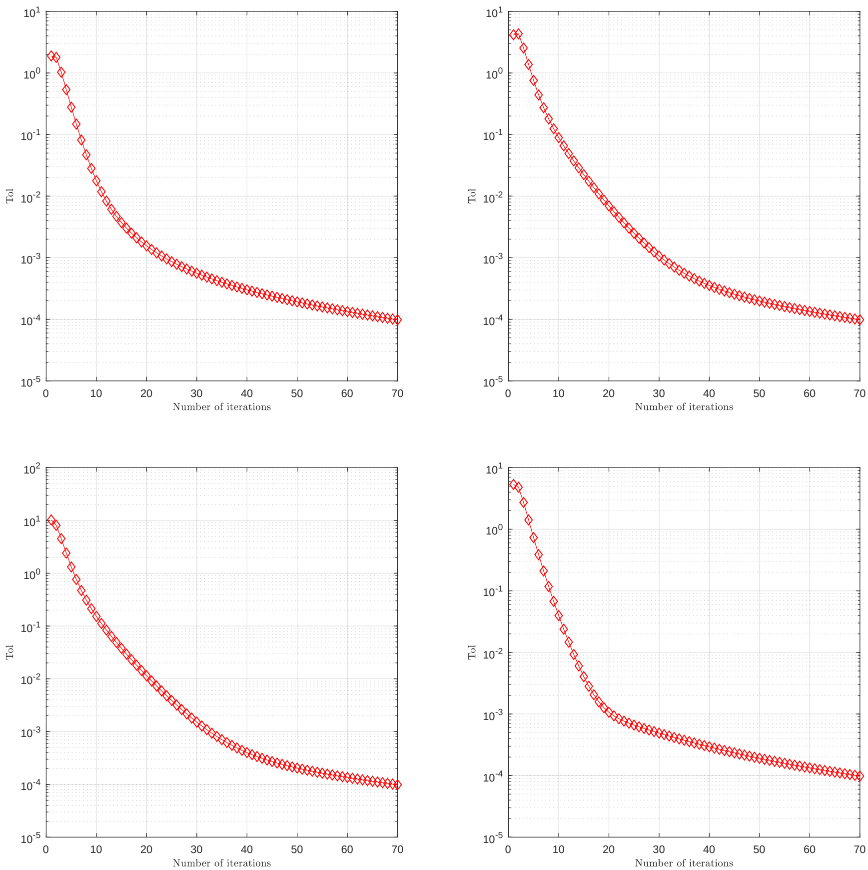

FIGURE 2. Example 4.2. Top left: Case 1, Top right: Case 2, Bottom left: Case 3, Bottom right: Case 4.

Case 1. $x_0 = [5, -5]$ and $x_1 = [3, 5]$;
Case 2. $x_0 = [-5, -5]$ and $x_1 = [10, 10]$;
Case 3. $x_0 = [10, 10]$ and $x_1 = [20, 20]$;
Case 4. $x_0 = [10, -5]$ and $x_1 = [5, 15]$.

Acknowledgement. The first author acknowledge with thanks the bursary and financial support from Department of Science and Technology and National Research Foundation, Republic of South Africa Center of Excellence in Mathematical and Statistical Sciences (DSI-NRF COE-MaSS) Post-Doctoral Fellowship. Opinions expressed and conclusions arrived are those of the authors and are not necessarily to be attributed to the CoE-MaSS. We would like to thank Professor Vasile Berinde for his suggestions to improve our article. We would also like to appreciate Dr. Oyewole K. O. for his contributions on our graphical representations.

References

[1] Abass, H.A., Aremu, K.O., Jolaoso, L.O., Mewomo, O.T., *An inertial forward-backward splitting method for approximating solutions of certain optimization problem*, J. Nonlinear Funct. Anal., 2020, (2020), Article ID 6.

[2] Abass, H.A., Godwin, G.C., Narain, O.K., Darvish, V., *Inertial extragradient method for solving variational inequality and fixed point problems of a Bregman demigeneralized mapping in a reflexive Banach spaces*, Numerical Functional Analysis and Optimization, (2022), 1-28.

[3] Abass, H.A., Izuchukwu, C., Mewomo, O.T., Dong, Q.L., *Strong convergence of an inertial forward-backward splitting method for accretive operators in real Banach space*, Fixed Point Theory, **20**(2020), no. 2, 397-412.

[4] Abass, H.A., Mebawondu, A.A., Narain, O.K., Kim, J.K., *Outer approximation method for zeros of sum of monotone operators and fixed point problems in Banach spaces*, Nonlinear Funct. Anal. and Appl., **26**(2021), no. 3, 451-474.

[5] Afassinou, K., Narain, O.K., Otunuga, O.E., *Iterative algorithm for approximating solutions of split monotone variational inclusion, variational inequality and fixed point problems in real Hilbert spaces*, Nonlinear Funct. Anal. and Appl., **25**(2020), no. 3, 491-510.

[6] Ansari, Q.H., Rehan, A., *Iterative methods for generalized split feasibility problems in Banach spaces*, Carapathian J. Math., **33**(2017), no. 1, 9-26.

[7] Barbu, V., *Nonlinear Differential Equations of Monotone Types Nonlinear Differential in Banach Spaces*, Springer, New York, 2010.

[8] Bauschke, H.H., Borwein, J.M., *Legendre functions and method of random Bregman functions*, J. Convex Anal., **4**(1997), 27-67.

[9] Bauschke, H.H., Borwein, J.M., Combettes, P.L., *Essentially smoothness, essentially strict convexity and Legendre functions in Banach spaces*, Commun. Contemp. Math., **3**(2001), 615-647.

[10] Bello, J.Y., Shehu, Y., *An iterative method for split inclusion problem without prior knowledge of operator norm*, J. Fixed Point Theory Appl., **19**(2017), no. 3.

[11] Bregman, L.M., *The relaxation method for finding the common point of convex sets and its application to solution of problems in convex programming*, U.S.S.R Comput. Math. Phys., **7**(1967), 200-217.

[12] Bryne, C., *Iterative oblique projection onto convex subsets and the split feasibility problems*, Inverse Probl., **18**(2002), 441-453.

[13] Bryne, C., Censor, Y., Gibali, A., *The split common null point problem*, J. Nonlinear Convex Anal., **13**(2012), 759-775.

[14] Burwein, M., Reich, S., Sabach, S., *A characterization of Bregman firmly nonexpansive operators using a new monotonicity concept*, J. Nonlinear Convex Anal., **12**(2011), 161-184.

[15] Butnariu, D., Iusem, A.N., *Totally Convex Functions for Fixed Points Computation and Infinite Dimensional Optimization*, Kluwer Academic Publishers, Dordrecht, 2000.

[16] Butnariu, D., Resmerita, E., *Bregman distances, totally convex functions and a method for solving operator equations in Banach spaces*, Abstract and Applied Analysis, **2006**(2006), Art. ID 84919, 1-39.

[17] Censor, Y., Elfving, T., *A multiprojection algorithms using Bregman projections in a product space*, Numer. Algor., **8**(1994), 221-239.

[18] Censor, Y., Segal, A., *The split common fixed point problem for directed operators*, J. Convex Anal., **16**(2009), no. 2, 587-600.

[19] Cholamjiak, P., Sunthrayuth, P., *A Halpern-type iteration for solving the split feasibility problem and fixed point problem of Bregman relatively nonexpansive semigroup in Banach spaces*, Filomat, **32**(2018), no. 9, 3211-3227.

[20] Gazmeh, H., Naraghirad, E., *The split common null point problem for Bregman generalized resolvents in two Banach spaces*, Optimization, (2020), DOI:10.1080/02331934.2020.1751157.

[21] Iiduka, H., *Acceleration method for convex optimization over the fixed point set of a nonexpansive mappings*, Math. Prog. Series A, **149**(2015), 131-165.

[22] Izuchukwu, C., Okeke, C.C., Isiogugu, F.O., *A viscosity iterative technique for split variational inclusion and fixed point problems between a Hilbert and a Banach space*, J. Fixed Point Theory Appl., **20**(157)(2018).

[23] Jailoka, P., Suantai, S., *Split null point problems for demicontractive multivalued mappings*, Mediterr. J. Math., **15**(2018), 1-19.

[24] Kazmi, K.R., Ali, R., Yousuf, S., *Generalized equilibrium and fixed point problems for Bregman relatively nonexpansive mappings in Banach spaces*, J. Fixed Point Theory Appl., (2018), 20:151.

[25] Kinderlehrer, D., Stampacchia, G., *An Introduction to Variational Inequalities and Their Applications*, Society for Industrial and Applied Mathematics, Philadelphia, 2000.

[26] Lions, P.L., Mercier, B., *Splitting algorithms for the sum of two nonlinear operators*, SIAM J. Numer. Anal., **16**(1979), 964-979.

[27] Mainge, P.E,, *Viscosity approximation process for quasi nonexpansive mappings in Hilbert space*, Comput. Math. Appl., **59**(2010), 74-79.

[28] Martin Marquez, V., Reich, S., Sabach, S., *Bregman strongly nonexpansive operators in reflexive Banach spaces*, J. Math. Anal. Appl., **400**(2013), 597-614.

[29] Moudafi, A., *A note on the split common fixed point problem for quasi-nonexpansive operator*, Nonlinear Anal., **74**(2011), 4083-4087.

[30] Ogbuisi, F.U., Izuchukwu, C., *Approximating a zero of sum of two monotone operators which solves a fixed point problem in reflexive Banach spaces*, Numer. Funct. Anal., **40**(13)(2019), DOI:10.1080/01630563.2019.1628050.

[31] Ogbuisi, F.U., Mewomo, O.T., *Iterative solution of split variational inclusion problem in real Banach spaces*, Afr. Mat., **28**(2017), 295-309.

[32] Okeke, C.C., Izuchukwu, C., *Strong convergence theorem for split feasibility problems and variational inclusion problems in real Banach spaces*, Rendiconti de Circolo Matematico di Palermo, Series 2, doi.10.1007/s12215-020-00508-3.

[33] Oyewole, O.K., Abass, H.A., Mewomo, O.T., *A strong convergence algorithm for a fixed point constraint split null point problem*, Rendiconti de Circolo Matematico di Palermo, Series 2, (2020), 1-20.

[34] Reich, S., Sabach, S., *A strong convergence theorem for a proximal-type algorithm in reflexive Banach spaces*, J. Nonlinear Convex Anal., **10**(2009), 471-485.

[35] da Reich, S., Sabach, G., *Iterative methods for solving systems of variational inequalities in reflexive Banach spaces*, J. Nonlinear Convex Anal., **10**(2009), 471-485.

[36] Reich, S., Sabach, S., *Two strong convergence theorems for a proximal method in reflexive Banach spaces*, Numer. Funct. Anal. Optim., **31**(2010), 24-44.

[37] Rockafellar, R.T., *Characterization of the subdifferentials of convex functions*, Pac. J. Math., **17**(1966), 497-510.

[38] Rockafellar, R.T., *On the maximality of sums of nonlinear monotone operators*, Trans. Amer. Math. Soc., **149**(1970), 75-88.

[39] Schopfer, F., Schuster, T., Louis, A.K., *An iterative regularization method for the solution of the split feasibilty problem in Banach spaces*, Inverse Probl., **24**(5)(2008), 055008.

[40] Shehu, Y., *Convergence results of forward-backward algorithms for sum of monotone operators in Banach spaces*, Results Math., **74**(2019), 138.

[41] Shehu, Y., Ogbuisi, F.U., *Approximation of common fixed points of left Bregman strongly nonexpansive mappings and solutions of equilibrium problems*, J. Appl. Anal., **21**(2)(2015), 63-77, DOI: 10.1515/jaa-2015-0007.

[42] Shehu, Y., Ogbuisi, F.U., *An iterative method for solving split monotone variational inclusion and fixed point problem*, RACSAM, **110**(2016), 503-518.

[43] Shehu, Y., Ogbuisi, F.U., Iyiola, O.S., *Convergence analysis of an iterative algorithm for fixed point problems and split feasibility problems in certain Banach spaces*, Optimization, **65**(2016), 299-323.

[44] Sunthrayuth, P., Pholasa, N., Cholamjiak, P., *Mann-type algorithms for solving the monotone inclusion problem and the fixed point problem in reflexive Banach spaces*, Ricerche di Matematica, (2021), 1-28.

[45] Takahashi, S., Takahashi, W., *The split common null point problem and the shrinking projection method in Banach spaces*, Optimization, **65**(2016), no. 2, 281-287.

[46] Tie, J.V., *Convex Analysis: An Introductory Text*, Wiley, New York, 1984.

[47] Timnak, S., Naraghirad, E., Hussain, N., *Strong convergence of Halpern iteration for products of finitely many resolvents of maximal monotone operators in Banach spaces*, Filomat, **31**(15)(2017), 4673-4693.

[48] Tseng, P., *A modified forward-backward splitting method for maximal monotone mappings*, SIAM J. Control Optim., **38**(2000), 431-446.

[49] Xia, F.Q., Huang, N.J., *Variational inclusions with a general H-monotone operators in Banach spaces*, Comput. Math. Appl., **54**(2010), no. 1, 24-30.

[50] Xu, H.K., *Iterative algorithms for nonlinear operators*, J. London Math. Soc., **66**(2)(2002), no. 1, 240-256.

[51] Zalinescu, C., *Convex Analysis in General Vector Spaces*, World Scientific Publishing Co. Inc., River Edge NJ, 2002.

Hammed Anuoluwapo Abass
Department of Mathematics and Applied Mathematics,
Sefako Makgatho Health Science University,
P.O. Box 94, Pretoria 0204, South Africa
e-mail: hammedabass548@gmail.com, hammed.abass@smu.ac.za

Godwin Ugwunnadi
Department of Mathematics,
University of Eswatini, Kwaluseni
and
Department of Mathematics and Applied Mathematics,
Sefako Makgatho Health Science University,
P.O. Box 94, Pretoria 0204, South Africa
e-mail: Ugwunnadi4u@yahoo.com

Ojen Narain
School of Mathematics, Statistics and Computer Science,
University of KwaZulu-Natal, Durban, South Africa
e-mail: Naraino@ukzn.ac.za