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Some operators of fractional calculus and their applications regarding various complex functions analytic in certain domains

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Abstract. In this academic research note, some familiar operators prearranged by fractional-order calculus will first be introduced and various characteristic properties of those operators will next be propounded. Through the instrumentality of various earlier results associating with both those operators and some complex-exponential forms, and also in the light of certain special information in [1], [20], [17] and [38], an extensive result together with a variety of its implications consisting of several exponential type inequalities will then be determined. A number of its possible implications will extra be pointed out.

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1. Introduction and rudiments

In the literature consisted of mathematically academic studies, particularly, fractional-order calculations have been continually encountering either as fractional-order integral(s) *or* as fractional-order derivative(s) in metamathematics. The mentioned-specially calculations, *which* are closely related to each other, are extensive calculations that are frequently applied for both the functions with real variable and the functions with complex variable. There are a wide range of both theoretical and applied research in relation with those. In this respect, in particular, a great variety of

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scientific articles are also presented as extensive works in the section of the references of this research. For instance, one may refer to certain main works in [4], [6], [7], [11], [13], [24], [34], [39] and [40].

As indicated in the abstract, in this scientific note, various fundamental operators associating with fractional calculus, which are specially fractional type derivative operators, will be firstly considered for certain complex functions which are regular in certain domains of the complex plane. In special, in consideration of the fractional derivative(s) operator, a fractional type operator, *which* is encountered as the Tremblay operator in the academic literature, will be then introduced here. Especially, comprehensive studies are still ongoing regarding both this operator and the other specified operators. Nevertheless, for related researchers, we can also offer the results in the earlier papers given in [2], [9], [14], [20], [19] and [31] as a variety of examples.

Furthermore, as several applications of fractional (order) calculus used in various different fields of sciences, numerous papers are also presented in the papers in [3], [10], [12], [15], [16], [22], [25]-[27], [30], [32], [33] and [37]-[40] as examples.

We have given some literature information above. We can now begin to introduce various special information, definitions and several important relationships between those operators that will be necessary for our investigations.

Firstly, let the familiar notations:

$$\mathbb{U} \quad , \quad \mathbb{C} \quad , \quad \mathbb{R} \quad \text{and} \quad \mathbb{N}$$

represent, respectively, the *open unit* disk, the *complex* numbers' set, the *real* numbers' set and the *natural* numbers' set.

Next, for the following numbers:

$$\mathrm{s} \in \mathbb{N} \quad , \quad \tilde{\alpha} \in \mathbb{C} - \{0\} \quad \text{and} \quad \tilde{\alpha}_{\mathrm{S}} \in \mathbb{C} \, ,$$

the notation $\mathbf{H}_{\tilde{\alpha}}(\mathrm{s})$ represents the family of the functions $\varrho := \varrho(z)$ being of the forms given by the complex-series expansion:

$$\varrho(z) = \tilde{\alpha} z^{\mathrm{S}} + \tilde{\alpha}_{\mathrm{S}+1} z^{\mathrm{S}+1} + \tilde{\alpha}_{\mathrm{S}+2} z^{\mathrm{S}+2} + \tilde{\alpha}_{\mathrm{S}+3} z^{\mathrm{S}+3} + \cdots \;, \tag{1.1}$$

which are also regular in $\mathbb{U}$.

Most especially, we indicate here that, as simpler expression and more convenient, the following special classes of the regular-functions in the class $\mathbf{H}_{\tilde{\alpha}}(\mathrm{s})$:

$$\mathbf{H}(\mathrm{s}) := \mathbf{H}_1(\mathrm{s}) \quad \text{and} \quad \mathbf{H} := \mathbf{H}(1)$$

can be pointed out as examples and they will also be played important roles for investigations. For this reason, those (more) special classes will taken consideration as revealing various applications of our basic result for researchers. We specially note that, in the mathematical literature, the functions in the class $\mathbf{H}_{\tilde{\alpha}}(\mathrm{s})$ are called as multivalently (*or s*-valently) regular functions (in $\mathbb{U}$) and the functions in the class $\mathbf{H}$ are also called the normalized regular functions in the open set $\mathbb{U}$. For their details and some examples, see [3], [5], [8], [9], [16], [21] and [30].

Secondly, for a function $\varrho := \varrho(z) \in \mathbf{H}_{\tilde{\alpha}}(\mathrm{s})$, we also denote the notation of the Tremblay operator, *which* is specified by fractional derivative (of order λ ($\lambda :=$

$\alpha - \beta; 0 \leq \lambda < 1)$), by any one of the equivalent notations:

$$\mathbf{T}_z^{\alpha,\beta}[\varrho] \quad \Big(or, \quad \mathbf{T}_z^{\alpha,\beta}[\varrho(z)] \Big).$$

At that time, it is generally defined by

$$\mathbf{T}_z^{\alpha,\beta}[\varrho] = \frac{\Gamma(\beta)}{\Gamma(\alpha)} z^{1-\beta}\, \mathbf{D}_z^{\alpha-\beta}\left[z^{\alpha-1}\varrho(z)\right], \tag{1.2}$$

where

$$\beta \in \big(0,1\big] \quad , \quad \alpha \in \big(0,1\big] \quad , \quad \alpha - \beta \in \big[0,1\big) \quad \text{and} \quad z \in \mathbf{U}, \tag{1.3}$$

and, for a function $\zeta := \zeta(z)$, any one of the equivalent notations:

$$\mathbf{D}_z^{\delta}[\zeta] \quad \Big(or, \quad \mathbf{D}_z^{\delta}\big[\zeta(z)\big] \Big)$$

denotes the Fractional Derivative Operator (of order δ) and it also identified as in the form given by

$$\mathbf{D}_z^{\delta}[\zeta] = \frac{1}{\Gamma(1-\delta)} \frac{d}{dz} \int_0^z \frac{\zeta(q)}{(z-q)^{\delta}}\, dq \quad \big(0 \leq \delta < 1\big), \tag{1.4}$$

where ζ is a regular function in a simply connected region of the complex plane comprising its origin, and the multiplicity of $(z-q)^{-\delta}$ is raised by behoving $log(z-q)$ to be real *when* $z - q > 0$.

By taking notice of the restricted conditions in (1.3), as a fairly simple implementation of the respective operators designated by (1.2) and (1.4), for a simple-complex power function (just below), the following-special calculations can be easily propounded:

$$\mathbf{D}_z^{\delta}\left[z^{\mathrm{s}}\right] = \frac{\Gamma(\mathrm{s}+1)}{\Gamma(\mathrm{s}-\delta+1)} z^{\mathrm{s}-\delta} \tag{1.5}$$

and

$$\mathbf{T}_z^{\alpha,\beta}\left[z^{\mathrm{s}}\right] = \frac{\Gamma(\beta)\Gamma(\mathrm{s}+\alpha)}{\Gamma(\alpha)\Gamma(\mathrm{s}+\beta)} z^{\mathrm{s}}, \tag{1.6}$$

where

$$\delta \in \big[0,1\big) \quad , \quad \alpha \in \big(0,1\big] \quad , \quad \beta \in \big(0,1\big] \quad , \quad \alpha - \beta \in \big[0,1\big) \quad \text{and} \quad \mathrm{s} \in \mathbb{N}. \tag{1.7}$$

In the same time, in the light of the conditions created by (1.7) and also with the help of the results (1.5) and (1.6), respectively, the following-extra-special results can be also given by

$$z\frac{d}{dz}\Big(\mathbf{D}_z^{\delta}\left[z^{\mathrm{s}}\right]\Big) \equiv z\mathbf{D}_z^{1+\delta}\left[z^{\mathrm{s}}\right] = \frac{\Gamma(\mathrm{s}+1)}{\Gamma(\mathrm{s}-\delta)} z^{\mathrm{s}-\delta} \tag{1.8}$$

and

$$z\frac{d}{dz}\Big(\mathbf{T}_z^{\alpha,\beta}\left[z^{\mathrm{s}}\right]\Big) = \frac{\mathrm{s}\Gamma(\beta)\Gamma(\mathrm{s}+\alpha)}{\Gamma(\alpha)\Gamma(\mathrm{s}+\beta)} z^{\mathrm{s}} \tag{1.9}$$

for all $\mathrm{s} \in \mathbb{N}$. For these determinations in (1.5)-(1.6) and (1.8)-(1.9) and also some of their applications, one can see the recent works in [21] and [20].

In terms of this academic study, we specially note here that, for convenience, both the indicated functions belonging to the general class $\mathbf{H}_{\tilde{\alpha}}(n)$ (defined by any forms of the complex-series expansions like (1.1)) will be considered for our major results and the fundamental definitions (given by (1.2) and (1.4) together with the related special conclusions (determined by (1.5)-(1.9)) will be a quite basic-necessary information for main results of this scientific investigations.

As both a final reminder of this introductory section and one of various properties of the special operator advertised by (1.2), especially, for any regular function $\varrho(z)$ having the form (1.1), the following two-important relationships consisting of a form of the Srivastava-Owa operator and an identity transformation of the Tremblay operator:

$$\mathbf{T}_z^{1,\beta}[\varrho(z)] \equiv \Gamma(\beta) z^{1-\beta}\, \mathbf{D}_z^{1-\beta}\big[\varrho(z)\big] \quad \big(0 < \beta \leq 1\big) \tag{1.10}$$

and

$$\mathbf{T}_z^{\gamma,\gamma}[\varrho(z)] \equiv \varrho(z) \quad \big(0 \leq \gamma < 1\big) \tag{1.11}$$

can be easily ascertained in terms of the character of that operator as its implications *when* the concerned parameters are then selected by letting

$$\alpha := 1$$

and

$$\alpha := \gamma \quad \text{and} \quad \gamma =: \beta\,,$$

respectively.

Specially, for pertinent researchers, recently, by taking advantage of the mentioned fractional derivative operator, the main works (in relation with the Tremblay operator) can be firstly presented and certain relations and also several elementary results for normalized analytic functions (with negative coefficient) are also determined. (cf., e.g., [35]; and, see also [14].)

For various operators specified by fractional-order calculus, it can be looked over the results in the papers in [8], [16] and [17]. By considering certain different methods (*or* ideas), numerous interesting applications of related operators to certain functions analytic in $\mathbb{U}$ can be given in [17], as examples.

Additionally, we also indicate that the fractional derivative(s) operator, identified by (1.4), has comprehensive implications of the well-recognized operator for the literature, *which* also is the Srivastava-Owa fractional derivative operator being of similar form like (1.10). For those and their special forms, one check the work in [1]. See also the results in [15], [16] and [17].

2. Lemmas and results

In this section, in order to get a line on our essential objective, we need some fundamental lemmas with some of applications of fractional calculus (derivatives). Those are only three lemmas, *which* will be taken advantage of starting and then proving for principal results of this investigations.

Firstly, in the light of the conditions given in (1.7), the first assertion, *which* is Lemma 2.1 just below, can be easily demonstrated by applying the elementary results stated in (1.9) and (1.8) (a long with the results in (1.5) and (1.6)) to any function belonging to the class $\mathbf{H}_{\tilde{\alpha}}(n)$. Accordingly, its detail is omitted here. For similar results, one may also center upon the recent papers in [8, 21, 17, 19].

Lemma 2.1. *Let a regular function $\varrho(z)$ have the series form given as in the family $\mathbf{H}_{\tilde{\alpha}}(n)$. For $z \in \mathbb{U}$ and any function $\varrho := \varrho(z)$, the following basic result then holds:*

$$\begin{aligned}\mathbf{T}_z^{\alpha,\beta}[\varrho] &\equiv \mathbf{T}_z^{\alpha,\beta}[\varrho(z)]\\ &= \tilde{\alpha}\,\frac{\Gamma(\beta)\Gamma(n+\alpha)}{\Gamma(\alpha)\Gamma(n+\beta)}z^n + \sum_{\ell=n+1}^{\infty}\tilde{\alpha}_n\,\frac{\Gamma(\beta)\Gamma(\ell+\alpha)}{\Gamma(\alpha)\Gamma(\ell+\beta)}z^\ell. \end{aligned} \tag{2.1}$$

The second assertion is just below and it is a special form of the well-known-elementary results of complex exponential. For both it and some of its implications, one may check the paper given in [18].

Lemma 2.2. *Let $\omega \in \mathbb{R}$ and also let $z \in \mathbb{C} - \{0\}$. Then, the following-complex exponentiation is true.*

$$z^\omega = |z|^\omega\big[\mathit{Cos}\big(\omega\, arg(z)\big) + i\,\mathit{Sin}\big(\omega\, arg(z)\big)\big]. \tag{2.2}$$

The last assertion, *which* is Lemma 2.3 just below, is a well-known important tool and very useful auxiliary theorem proven in [28]. For some of its applications, one can easily arrive at various works in the literature. For its detail, one may also refer to the paper given by [23].

Lemma 2.3. *Let $\xi := \xi(z)$ be a regular function in the domain $\mathbb{U}$ and also be of the form given as in (1.1). For $z \in \mathbb{U}$ and for any $z_0 \in \mathbb{U}$, if*

$$\big|\xi(z_0)\big| = \max\Big\{\,\big|\xi(z)\big| \;:\; |z| \le |z_0|\,\Big\}, \tag{2.3}$$

then there exists any positive number λ such that

$$z_0\xi'(z_0) = \lambda\xi(z_0), \tag{2.4}$$

where $\lambda \in \mathbb{R}$ with $\lambda \ge n$ $(n \in \mathbb{N})$.

In accordance with principal assertions, namely, Lemmas 2.1-2.3 just above, we can then compose our comprehensive result appertaining to the functions belonging to in the class $\mathbf{H}_{\tilde{\alpha}}(n)$, *which* will be specified by the special operator (1.2).

Theorem 2.4. *Under the mentioned conditions of both the parameters in (1.3) and the definitions in (1.2) and (1.4), let the parameters Υ, Λ, ∇ and Θ have the conditions determined as follows:*

$$\Upsilon \in \mathbb{R} - \{0\} \quad , \quad \Lambda \ge m \quad , \quad \nabla \in \mathbb{C} \quad \text{and} \quad 0 \le \Theta < 2\pi\,, \tag{2.5}$$

where $m \in \mathbb{N}$ and $0 < |\nabla| < 1$. Then, for some $z \in \mathbb{U}$ and for any function $\varrho(z) \in \mathbf{H}_{\tilde{\alpha}}(n)$, if any one of the statements given by

$$\Re e\left\{\left[z\Big(\mathbf{T}_z^{\alpha,\beta}[\varrho(z)]\Big)^{(n+1)}\right]^{\Upsilon}\right\} \ne \Lambda^{\Upsilon}|\nabla|^{\Upsilon}\,\mathit{Cos}\Big(\Upsilon\big[\Theta + \mathit{Arg}(\nabla)\big]\Big) \tag{2.6}$$

and

$$\Im m\left\{\left[z\left(\mathbf{T}_z^{\alpha,\beta}[\varrho(z)]\right)^{(n+1)}\right]^{\Upsilon}\right\} \neq \lambda^{\Upsilon}|\nabla|^{\Upsilon}\,Sin\Big(\Upsilon\left[\Theta + Arg(\nabla)\right]\Big) \tag{2.7}$$

is satisfied, then the statement given by

$$\left|\left(\mathbf{T}_z^{\alpha,\beta}[\varrho(z)]\right)^{(n)} - \tilde{\alpha}\,\mathbb{I}_n^n(\alpha,\beta)\right| < |\nabla| \tag{2.8}$$

is satisfied, which also is quite clear that

$$\left|\Re e\left[\left(\mathbf{T}_z^{\alpha,\beta}[\varrho(z)]\right)^{(n)}\right] - \Re e\big(\tilde{\alpha}\big)\,\mathbb{I}_n^n(\alpha,\beta)\right| \leq |\nabla| \tag{2.9}$$

and

$$\left|\Im m\left[\left(\mathbf{T}_z^{\alpha,\beta}[\varrho(z)]\right)^{(n)}\right] - \Im m\big(\tilde{\alpha}\big)\,\mathbb{I}_n^n(\alpha,\beta)\right| \leq |\nabla| \tag{2.10}$$

where

$$\left(\mathbf{T}_z^{\alpha,\beta}[\varrho(z)]\right)^{(n)} := \frac{d^n}{dz^n}\left(\mathbf{T}_z^{\alpha,\beta}[\varrho(z)]\right) \quad \big(n \in \mathbb{N} \cup \{0\}\big) \tag{2.11}$$

and

$$\mathbb{I}_v^u(\alpha,\beta) = \frac{u!}{(u-v)!}\,\frac{\Gamma(\beta)\Gamma(u+\alpha)}{\Gamma(\alpha)\Gamma(u+\beta)} \quad \big(v < u; u \in \mathbb{N}; v \in \mathbb{N}\big), \tag{2.12}$$

and, also, here and throughout this research note, the values of the complex powers in (2.6) and (2.7) are considered as their principal values.

Proof. Let the interested function $\varrho := \varrho(z)$ be of the form in the class $\mathbf{H}_{\tilde{\alpha}}(n)$. When taking into account the equivalent relation in (2.11) and the determined result in (1.9) (of Lemma 2.1), its nth derivative:

$$\begin{aligned}\left(\mathbf{T}_z^{\alpha,\beta}[\varrho]\right)^{(n)} &= \frac{d^n}{dz^n}\left(\tilde{\alpha}\,\frac{\Gamma(\beta)\Gamma(n+\alpha)}{\Gamma(\alpha)\Gamma(n+\beta)}\,z^{\mathrm{n}}\right.\\ &\qquad\left. + \tilde{\alpha}_{\mathrm{n+1}}\frac{\Gamma(\beta)\Gamma(n+1+\alpha)}{\Gamma(\alpha)\Gamma(n+1+\beta)}\,z^{\mathrm{n+1}} + \cdots\right) \\ &= \tilde{\alpha}\,\mathbb{I}_n^n(\alpha,\beta) + \tilde{\alpha}_{\mathrm{n+1}}\,\mathbb{I}_n^{n+1}(\alpha,\beta)z^1 + \tilde{\alpha}_{\mathrm{n+2}}\,\mathbb{I}_n^{n+2}(\alpha,\beta)z^2 + \cdots\end{aligned} \tag{2.13}$$

can be easily calculated, *where* the notation $\mathbb{I}_r^s(\alpha,\beta)$ above is defined by (2.12).

For the proof of Theorem 1, in the light of such information (2.12) and (2), for a n-valently regular function like any form $\varrho := \varrho(z)$ in $\mathbf{H}_{\tilde{\alpha}}(n)$, there is a need to consider a function $\Omega(z)$ in the form given by

$$\left(\mathbf{T}_z^{\alpha,\beta}[\varrho]\right)^{(n)} = \tilde{\alpha}\,\mathbb{I}_n^n(\alpha,\beta) + \Phi\,\Omega(z) \quad \big(0 < |\Phi| < 1; z \in \mathbb{U}\big). \tag{2.14}$$

In that case, as a result of simple elementary operations, one can easily distinguish that the described function $\Omega(z)$ belongs to the class $\mathbf{H}_{\tilde{\alpha}}(m)$ $(m \in \mathbb{N})$. Thereby, both the related function $\Omega(z)$ both is regular in the set $\mathbb{U}$ and it can be considered for the proof of Theorem 2.4. By differentiating of both sides of (2.14) with respect to the complex variable z, we then get that

$$\frac{d}{dz}\left\{\left(\mathbf{T}_z^{\alpha,\beta}[\varrho(z)]\right)^{(n)}\right\} \equiv \left(\mathbf{T}_z^{\alpha,\beta}[\varrho]\right)^{(n+1)} = \Phi\,\Omega'(z) \quad \big(0 < |\Phi| < 1; z \in \mathbb{U}\big). \tag{2.15}$$

We now assert that $|\Omega(z)| < 1$ in $\mathbb{U}$. In fact, if not, then, according to (2.3) (of Lemma 2.3), there exists a point z_0 belonging to $\mathbb{U}$ such that

$$\max\left\{ \left|\Omega(z)\right| \,:\, |z| \leq |z_0| \;\; \left(z, z_0 \in \mathbb{U}\right)\right\} = \left|\Omega(z_0)\right| = 1\,,$$

which readily yields that

$$\Omega(z_0) = e^{i\Delta} \quad \left(0 \leq \Delta < 2\pi; z_0 \in \mathbb{U}\right).$$

In the present case, the expression (2.4) (of Lemma 2.3) also gives rise to

$$z_0\,\Omega'(z_0) = \lambda\Omega(z_0) = \lambda e^{i\Delta} \quad \left(\lambda \geq m; m \in \mathbb{N}\right).$$

Therefore, of course, for all $\lambda \geq m \geq n$ $(n, m \in \mathbb{N})$, by setting $z := z_0$ and also by means of the main relations (2.2) (of Lemma 2.2) and (2.4) (of Lemma 2.3), the expression (2.15) lightly follows that

$$\begin{aligned}
\Re e &\left\{ \left(z\left(\mathbf{T}_z^{\alpha,\beta}[\varrho(z)]\right)^{(n+1)}\right)^{\mathrm{r}}\Bigg|_{z:=z_0} \right\} \\
&= \Re e\left\{\left(\Phi\, z_0\, \Omega'\left(z_0\right)\right)^{\mathrm{r}}\right\} \\
&= \Re e\left\{\left(\Phi\lambda\Omega\left(z_0\right)\right)^{\mathrm{r}}\right\} \\
&= \Re e\left\{\left[\lambda\Phi e^{i\Delta}\right]^{\mathrm{r}}\right\} \\
&= \Re e\left\{\left|\lambda\Phi e^{i\Delta}\right|^{\mathrm{r}} e^{i\mathrm{r}\mathrm{Arg}\left(\lambda\Phi e^{i\Delta}\right)}\right\} \\
&= \Re e\left\{\left|\lambda\Phi\right|^{\mathrm{r}} e^{i\mathrm{r}\mathrm{Arg}\left(\Phi e^{i\Delta}\right)}\right\} \quad (\textit{since } \lambda \geq m \geq 1) \\
&= \lambda^{\mathrm{r}}|\Phi|^{\mathrm{r}}\mathrm{Cos}\left[\mathrm{r}\mathrm{Arg}\left(\Phi e^{i\Delta}\right)\right] \\
&= \lambda^{\mathrm{r}}|\Phi|^{\mathrm{r}}\mathrm{Cos}\left[\mathrm{r}\left(\mathrm{Arg}(\Phi) + \mathrm{Arg}\left(e^{i\Delta}\right)\right]\right. \\
&= \lambda^{\mathrm{r}}|\Phi|^{\mathrm{r}}\mathrm{Cos}\left[\mathrm{r}\left(\Delta + \mathrm{Arg}(\Phi)\right)\right]
\end{aligned} \tag{2.16}$$

and

$$\begin{aligned}
\Im m &\left\{ \left(z\left(\mathbf{T}_z^{\alpha,\beta}[\varrho(z)]\right)^{(n+1)}\right)^{\mathrm{r}}\Bigg|_{z:=z_0} \right\} \\
&= \Im m\left\{\left(\Phi\, z_0\, \Omega'\left(z_0\right)\right)^{\mathrm{r}}\right\} \\
&= \Im m\left\{\left(\Phi\lambda\Omega\left(z_0\right)\right)^{\mathrm{r}}\right\} \\
&= \Im m\left\{\left[\lambda\Phi e^{i\Delta}\right]^{\mathrm{r}}\right\} \\
&= \Im m\left\{\left|\lambda\Phi e^{i\Delta}\right|^{\mathrm{r}} e^{i\mathrm{r}\mathrm{Arg}\left(\lambda\Phi e^{i\Delta}\right)}\right\}
\end{aligned} \tag{2.17}$$

$$= \Im m \left\{ \left|\lambda\Phi\right|^{\mathrm{r}} e^{i\mathrm{r}\mathrm{Arg}\left(\Phi e^{i\Delta}\right)} \right\} \quad \left(since\ \lambda \geq m \geq 1\right)$$
$$= \lambda^{\mathrm{r}}|\Phi|^{\mathrm{r}}\mathrm{Sin}\left[\mathrm{r}\mathrm{Arg}\left(\Phi e^{i\Delta}\right)\right]$$
$$= \lambda^{\mathrm{r}}|\Phi|^{\mathrm{r}}\mathrm{Sin}\left[\mathrm{r}\left(\Delta + \mathrm{Arg}(\Phi)\right)\right],$$

where $\lambda \geq m$ $(m \in \mathbb{N})$, $0 \leq \Delta < 2\pi$, $\mathrm{r} \in \mathbb{R}$ and $\Phi \in \mathbb{C}$ $(0 < |\Phi| < 1)$. But, unfortunately, the results determined as in (2.16) and (2.17) are, respectively, contradictions with the hypotheses of Theorem 2.4, *which* are the mentioned results presented by (2.16) and (2.17) *when* setting

$$\lambda := \Lambda \quad , \quad \Phi := \nabla \quad , \quad \mathrm{r} := \Upsilon \quad \text{and} \quad \Delta := \Theta.$$

This means that there is no $z_0 \in \mathbb{U}$ satisfying the condition $\left|\Omega(z_0)\right| = 1$. Therefore, we decide upon that it has to be in the form $\left|\Omega(z)\right| < 1$ for all $z \in \mathbb{U}$. Consequently, for all functions $\varrho(z) \in \mathbf{H}_{\tilde{\alpha}}(n)$, the expression (2.14) follows that

$$\left| \left(\mathbf{T}_z^{\alpha,\beta}[\varrho(z)] \right)^{(n)} - \tilde{\alpha}\, \mathbb{I}_n^n(\alpha,\beta) \right| = \left|\Phi\, \Omega(z)\right| < \left|\Phi\right| \quad \left(0 < |\Phi| < 1; z \in \mathbb{U}\right),$$

which is equivalent to the provision of Theorem 2.4, namely, the statement in (2.8) *when* $\Phi := \nabla$. Finally, the basic relationships, *which* are both between (2.8) and(2.9) and between (2.8) and (2.10), can be easily seen propositions. Thus, this ends the desired proof. □

3. Conclusion and recommendations

In this part, *which* is the last part of this comprehensive-research note, we would like to mention various special implications and suggestions relating to our investigations for our readers.

Here we want to bring forward certain conclusions and also to give implicit recommendations concerning our main results. As emphasized in the abstract of this study, the main purpose of this comprehensive study was to present both the basic concepts about some operators of fractional derivatives and to introduce a special operator defined with the help of those, *which* is expressed as various works relating with the Tremblay operator (cf., e.g., [1], [38], [15] and [8]), as indicated before. In any case, these were also carried out in the first chapter. In fact, some important relationships between the respective operators as in (1.2) and (1.4), and also, for certain regular functions like (1.1), a number of their basic applications were presented as in (1.5), (1.6) and (1.8)-(1.11). Clearly, those relevant relations and special implications play a big role both for our essential result given in this section above and for all of its possible special consequences. By focusing especially on our main comprehensive result, namely Theorem 2.4, and its proof, that is, with the help of the relevant theorem and its proof, different analytical and geometrical new results specified by the mentioned operators and naturally the mentioned functions (in the classes $\mathbf{H}_{\tilde{\alpha}}(n)$ *or* its special subclasses $\mathbf{H}(s)$ and $\mathbf{H}$) can also be determined (*or* calculated) (cf., e.g., [5], [10], [13] and [28]). Lastly, most particularly, the parameter Υ, considered in Theorem 2.4, can be chosen as complex number. For possible details of both this and other suggestions, one can check the works given in [8], [15], [21] and [17]-[22] as some of different investigations.

We choose to leave the special details of the relationships between those operators of fractional calculus (that is, that derivatives) and a large number of possible-logic implications of our principal result as an exercise for the interested researchers (*or* readers). Nevertheless, we also want to find out only one private result together with one of its special forms, *which* both associates with Theorem 2.4 and has wide range of (more) special results according to the facts of suitable values of the related parameters.

Inside of the extra information in the first section, through the instrumentality of the special relation (1.11) together with taking $\tilde{\alpha} := 1$ in Theorem 2.4, the following-extensive result can be easily designated for all nth derivative of any regular function $\varrho(z)$ in the special class $\mathbf{H}(n)$ (of the general class $\mathbf{H}_{\tilde{\alpha}}(n)$), *which* also includes numerous geometric properties of n-valently regular functions in $\mathbb{U}$. Specifically, for more detailed information in relation to those analytic-geometric properties, one may center on the main works given by [3], [10] and [27].) Shortly, the desired-special result can be easily constituted as in the following Proposition (just below).

Proposition 3.1. *Under the mentioned conditions of both the parameters* Υ, Λ, ∇ *and* Θ *designated as in (2.5) and for any n-valently regular function* $\varrho(z)$ *in the class* $\mathbf{H}(n)$, *if any one of the statements given by*

$$\Re e\left\{\left[z\Big(\varrho(z)\Big)^{(n+1)}\right]^{\Upsilon}\right\} \equiv \Re e\left\{\Big(z\varrho^{(n+1)}(z)\Big)^{\Upsilon}\right\}$$
$$\neq \Lambda^{\Upsilon}|\nabla|^{\Upsilon}\, Cos\Big(\Upsilon\left[\Theta + Arg(\nabla)\right]\Big)$$

and

$$\Im m\left\{\left[z\Big(\varrho(z)\Big)^{(n+1)}\right]^{\Upsilon}\right\} \equiv \Im m\left\{\Big(z\varrho^{(n+1)}(z)\Big)^{\Upsilon}\right\}$$
$$\neq \Lambda^{\Upsilon}|\nabla|^{\Upsilon}\, Sin\Big(\Upsilon\left[\Theta + Arg(\nabla)\right]\Big)$$

is true, then the statement given by

$$\left|\varrho^{(n)}(z) - n!\right| < |\nabla|$$

is also true, which also requires to more simple inequalities given by

$$\left|\Re e\Big(\varrho^{(n)}(z)\Big) - n!\right| \leq |\nabla|$$

and

$$\left|\Im m\Big(\varrho^{(n)}(z)\Big) - n!\right| \leq |\nabla|$$

where

$$\varrho^{(n)}(z) := \frac{d^n}{dz^n}\Big(\varrho(z)\Big)$$

for all $n \in \mathbb{N} \cup \{0\}$ *and for some* $z \in \mathbb{U}$.

By putting $n := 1$, and $\Lambda := 1$ in Proposition 3.1 (*or,* equivalently, by taking $\alpha := \gamma$, $\beta := \gamma$, $\Lambda := 1$ and $n := 1$ in the concerned theorem, *i.e.,* in Theorem 2.4), one of the exclusive-special results of any normalized-regular function $\varrho(z)$ in the more special class $\mathbf{H}$ (of the general class $\mathbf{H}_{\tilde{\alpha}}(n)$) can be easily determined as in the following Proposition (below).

Proposition 3.2. *Under the mentioned conditions of both the parameters* Λ, ∇ *and* Θ *designated as in (16) and for any normalized regular function* $\varrho(z)$ *in the class* $\mathbf{H}$, *if any one of*

$$\Re e\Big(\varrho''(z)\Big) \neq \Lambda|\nabla|Cos\Big(\Theta + Arg(\nabla)\Big)$$

and

$$\Im m\Big(\varrho''(z)\Big) \neq \Lambda|\nabla|Sin\Big([\Theta + Arg(\nabla)\Big)$$

holds true, then

$$|\varrho'(z) - 1| < \big|\nabla\big|$$

also holds true, which also requires to

$$\Big|\Re e\Big(\varrho'(z)\Big) - 1\Big| \leq \big|\nabla\big| \quad and \quad \Big|\Im m\Big(\varrho'(z)\Big) - 1\Big| \leq \big|\nabla\big|$$

where $z \in \mathbb{U}$.

As a final note of this research, in the light of the two-special propositions of our extensive result above or/and by considering certain extra conditions *when* there needs any necessity, we want to present to the attention of the related researchers to describe (*or* redescribe) each one of those possible-special results can be designated by making use of various types of the normalized-regular functions (*or* the multivalently-regular functions) in certain domains of the complex plane.

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