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Some variants of contraction principle, generalizations and applications

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Dedicated to Professor Gheorghe Coman on the occasion of his 80th anniversary

Abstract. In this paper we present the following variant of contraction principle: *Saturated principle of contraction.* Let (X,d) be a complete metric space and $f : X \to X$ be an l-contraction. Then we have:

(i) $F_{f^n} = \{x^*\}$, $\forall\ n \in \mathbb{N}^*$.

(ii) $f^n(x) \to x^*$ as $n \to \infty$, $\forall\ x \in X$.

(iii) $d(x,x^*) \leq \psi(d(x,f(x)))$, $\forall\ x \in X$ where $\psi(t) = \frac{t}{1-l}$, $t \geq 0$.

(iv) $y_n \in X$, $d(y_n, f(y_n)) \to 0$ as $n \to \infty \Rightarrow y_n \to x^*$ as $n \to \infty$.

(v) $y_n \in X$, $d(y_{n+1}, f(y_n)) \to 0$ as $n \to \infty \Rightarrow y_n \to x^*$ as $n \to \infty$.

(vi) If $Y \subset X$ is a nonempty bounded and closed subset with $f(Y) \subset Y$, then $x^* \in Y$ and $\bigcap\limits_{n\in\mathbb{N}} f^n(Y) = \{x^*\}$.

The basic problem is: which other metric conditions imply the conclusions of this variant ? We give some answers for this problem. Some applications and open problems are also presented.

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1. Introduction and preliminaries

The number of papers on fixed point theory in which appear metric conditions is a large one (see: [20], [49], [32], [48], [4], [9], [16], [17], [24], [25], [26], [34], [36], [52], [64], [68],. . .). In these papers two fixed point theorems appear under the same name, *contraction principle*:

(1) Let (X, d) be a complete metric space and $f : X \to X$ be a contraction. Then f has a unique fixed point (i.e., $F_f = \{x^*\}$).
(2) Let (X, d) be a complete metric space and $f : X \to X$ be an l-contraction. Then we have:
 (i) $F_f = \{x^*\}$.
 (ii) $f^n(x) \to x^*$ as $n \to \infty$, $\forall\, x \in X$.

By Contraction Principle (CP) we understand this (2) variant.

On the other hand, in many papers appear some properties of fixed point equations, where the corresponding operator is a contraction (see [56], [65], [40], [4], [6], [32], [25], [47], [49], [50], [51], [52], [53], [54], [56], [57], [64], [65], [72], [73],...). So, in this paper we present a new variant of contraction principle, a variant with generous conclusions. This variant is the following:

Theorem 1.1 (Saturated principle of contraction (SPC)). *Let (X, d) be a complete metric space and $f : X \to X$ be an l-contraction. Then we have:*

(i) *There exists $x^* \in X$ such that,*

$$F_{f^n} = \{x^*\},\ \forall\, n \in \mathbb{N}.$$

(ii) *For all $x \in X$, $f^n(x) \to x^*$ as $n \to \infty$.*
(iii) $d(x, x^*) \le \psi(d(x, f(x)))$, $\forall\, x \in X$, *where* $\psi(t) = \frac{t}{1-l}$, $t \ge 0$.
(iv) *If $\{y_n\}_{n\in\mathbb{N}}$ is a sequence in X such that*

$$d(y_n, f(y_n)) \to 0 \text{ as } n \to \infty,$$

then, $y_n \to x^*$ as $n \to \infty$.
(v) *If $\{y_n\}_{n\in\mathbb{N}}$ is a sequence in X such that*

$$d(y_{n+1}, f(y_n)) \to 0 \text{ as } n \to \infty,$$

then, $y_n \to x^*$ as $n \to \infty$.
(vi) *If $Y \subset X$ is a closed subset such that $f(Y) \subset Y$, then $x^* \in Y$. Moreover, if in addition Y is bounded, then*

$$\bigcap_{n\in\mathbb{N}} f^n(Y) = \{x^*\}.$$

It is clear that, all conclusions in this theorem are well known. For a better understanding of this variant of contraction principle, some remarks and commentaries are necessary.

Conclusion (i) is a set-theoretical one. If X is a nonempty set and $f : X \to X$ is an operator such that, $F_{f^n} = \{x^*\}$, for all $n \in \mathbb{N}^*$, then by definition we call f a Bessaga operator.

Conclusion (ii) is a topological one. All Picard iterations converge to the unique fixed point of the operator. If $(X, \to)$ is an L-space and $f : X \to X$ is an operator such that we have (i) and (ii), then by definition f is a Picard operator.

Conclusion (iii) is a metrical one and is very important in the theory of fixed point equations. We obtain from this estimate, for example, a data dependence of the fixed point under operator perturbation.

If in a metric space an operator f satisfies (i), (ii) and (iii), then by definition the operator f is a ψ-Picard operator and the estimation in (iii) is called retraction-displacement estimation. In this definition, ψ is a function, $\psi : \mathbb{R}_+ \to \mathbb{R}_+$, increasing and continuous in 0 with $\psi(0) = 0$.

If in a metric space (X, d) an operator $f : X \to X$ satisfies (i) and (iv) then by definition the fixed point problem for f is well posed. We remark that we can present this notion in a linear L-space. Let $(X, +, \mathbb{R}, \to)$ be a linear L-space and $f : X \to X$ be an operator. By definition the fixed point problem for f is well posed if:

(i) $F_f = \{x^*\}$.

(ii) If $\{y_n\}_{n\in\mathbb{N}}$ is a sequence in X such that $y_n - f(y_n) \to 0$ as $n \to \infty$, then $y_n \to x^*$ as $n \to \infty$.

If in a metric space (X, d) an operator satisfies (i) and (v), then by definition the operator f has the Ostrowski property. We remark that we can present this notion in a linear L-space.

Let $(X, +, \mathbb{R}, \to)$ be a linear L-space and $f : X \to X$ be an operator. By definition the operator f has the Ostrowski property if:

(i) $F_f = \{x^*\}$.

(ii) If $\{y_n\}_{n\in\mathbb{N}}$ is a sequence in X such that $y_{n+1} - f(y_n) \to 0$ as $n \to \infty$, then $y_n \to x^*$ as $n \to \infty$.

First part of conclusion (vi) is useful for the localization of the fixed point. Second part is a set-theoretical one, under metrical conditions. If X is a nonempty set and $f : X \to X$ is an operator such that

$$\bigcap_{n\in\mathbb{N}} f^n(X) = \{x^*\},$$

then by definition f is a Janos operator. On the other hand, from (vi) we have the following property of a contraction:

If (X, d) is a complete metric space and $f : X \to X$ is a contraction with $F_f = \{x^*\}$ and $\{y_n\}_{n\in\mathbb{N}}$ is a bounded sequence in X, then $f^n(y_n) \to x^*$ as $n \to \infty$.

It is well known that there is an extensive bibliography of the generalized contractions (see Ortega and Rheinboldt [32], Istrăţescu [20], Rhoades [48], Krasnoselskii and Zabrejko [26], Kirk and Sims [25], Granas and Dugundji [17], Goebel [16], Berinde [4], Rus [49], Rus [52], Rus, Petruşel and Petruşel [64], Rus and Şerban [65], Petruşel, Rus and Şerban [41], Rus and Şerban [65], Kirk and Shahzad [24],...). The problem is which metrical conditions which appear in the metrical fixed point theorems imply conclusions in the SPC ? We shall consider the problem in this paper. Some applications are given and open problems are presented.

Throughout this paper the notations and terminologies in [56], [65] and [40] are used. Moreover we consider these references as starting papers for our study.

The structure of the paper is the following:

2. Some variants of SPC
3. Examples of relevant metrical conditions
4. The case of generalized metric spaces
5. Applications

6. Other research directions

2. Some variants of SPC

We start our considerations with the following useful variant.

Theorem 2.1 (SPC with respect to a strongly equivalent metric). *Let X be a nonempty set, d and ρ be two metrics on X and $f : X \to X$ be an operator. We suppose that:*

(a) *(X, ρ) is a complete metric space.*

(b) *There exist $c_1, c_2 > 0$ such that*

$$c_1 d(x, y) \leq \rho(x, y) \leq c_2 d(x, y), \ \forall \ x, y \in X.$$

(c) *f is an l-contraction with respect to the metric ρ.*

Then we have:

(i) $F_{f^n} = \{x^*\}$, $\forall \ n \in \mathbb{N}^*$.

(ii) $f^n(x) \xrightarrow{d} x^*$ *as* $n \to \infty$.

(iii) $d(x, x^*) \leq \psi(d(x, f(x)))$, $\forall \ x \in X$, *where*

$$\psi(t) = \frac{c_2 t}{c_1(1 - l)}, \ t \geq 0.$$

(iv) *The fixed point problem for f is well posed with respect to the metric d.*

(v) *The operator f has the Ostrowski property with respect to the metric d.*

(vi) *If $Y \subset X$ is a bounded and close subset in (X, d) with $f(Y) \subset Y$, then $x^* \in Y$ and*

$$\bigcap_{n \in \mathbb{N}} f^n(Y) = \{x^*\}.$$

Proof. The proof follows from SPC in (X, ρ) and the condition (b) of strongly equivalence of the metrics d and ρ (see [40]). □

An other variant is the following.

Theorem 2.2 (Saturated Principle of Quasicontraction (SPQC)). *Let (X, d) be a metric space and $f : X \to X$ be an operator. We suppose that there exists a fixed point x^* of f and $0 < l < 1$ such that:*

$$d(f(x), x^*) \leq l d(x, x^*), \ \forall \ x \in X.$$

Then, we have (i)-(vi) in Theorem 1.1.

Proof. For (i)-(v) the proofs are similar with the proofs in Theorem 1.1.

(vi) Let $x \in Y$. Then $f^n(x) \in Y$, $\forall \ n \in \mathbb{N}$. But, $f^n(x) \to x^*$ as $n \to \infty$. Since Y is closed, it follows that $x^* \in Y$. In fact, we have (i)-(vi) in Theorem 1.1, with respect to Y. Indeed, for the second part of (vi), we have $\delta(f(Y), \{x^*\}) \leq \delta(Y, \{x^*\})$, where δ is the diameter functional with respect to d. Moreover, $\delta(f^n(Y), \{x^*\}) \leq l^n \delta(Y, \{x^*\}) \to 0$ as $n \to \infty$. So, $\bigcap_{n \in \mathbb{N}} f^n(Y) = \{x^*\}$. □

We also have the following result.

Theorem 2.3 (SPQC with respect to a strongly equivalent metric). *Let X be a nonempty set, d and ρ be two metrics on X and $f : X \to X$ be an operator. We suppose that:*

(a) *There exist $c_1, c_2 > 0$ such that*

$$c_1 d(x,y) \leq \rho(x,y) \leq c_2 d(x,y), \ \forall \ x,y \in X.$$

(b) *There exists a fixed point x^* of f and $0 < l < 1$ such that*

$$\rho(f(x), x^*) \leq l\rho(x,x^*), \ \forall \ x \in X.$$

Then we have (i)-(vi) in Theorem 2.1

Proof. The proof follows from Theorem 2.2 in (X, ρ) and condition (a). □

Now, we finish this section with the following definition.

Definition 2.4. *Let (X,d) be a complete metric space and $f : X \to X$ be an operator. We call relevant a metric condition on f which implies the uniqueness of fixed point if it implies also, conclusions such as in SPC.*

3. Examples of relevant metrical conditions

We start with the following remark.

Lemma 3.1. *Let (X,d) be a complete metric space and $f : X \to X$ be an operator. If a metrical condition on f implies the conclusions in CP and if in addition f is an l-quasicontraction, then we have for f the conclusions in SPC with*

(iii) $d(x,x^*) \leq \frac{1}{1-l} d(x, f(x)), \ \forall \ x \in X.$

Proof. The proof follows from SPQC. □

From this Lemma the following question rises.

Problem 3.2. Which metric conditions on f imply that f is a quasicontraction ?

From Lemma 3.1, we have, as examples, the following results.

Theorem 3.3. *Let (X,d) be a complete metric space and $f : X \to X$ be such that there exists $0 < l < \frac{1}{2}$, with*

$$d(f(x), f(y)) \leq l[d(x,f(x)) + d(y,f(y))], \ \forall \ x,y \in X.$$

Then we have the conclusions in SPC, with

(iii) $d(x,x^*) \leq \frac{1}{1-2l} d(x,f(x)), \ \forall \ x \in X.$

Proof. (i)-(ii). This is Kannan's theorem. Kannan's theorem is not a generalization of CP, but implies conclusions in CP.

(iii)-(vi). From Kannan's metrical condition it follows that f is a $2l$-quasicontraction. From SPQC we have (iii)-(vi). □

Theorem 3.4. (see [4], [52]) *Let (X,d) be a complete metric space and $f : X \to X$ be an operator. We suppose that there exist $a,b,c \in \mathbb{R}_+$, $a < 1$, b and $c < \frac{1}{2}$, such that for each $x,y \in X$ at least one of the following conditions is true:*

(1) $d(f(x), f(y)) \leq ad(x, y)$,
(2) $d(f(x), f(y)) \leq b[d(x, f(x)) + d(y, f(y))]$,
(3) $d(f(x), f(y)) \leq c[d(x, f(y)) + d(y, f(x))]$.

Then we have the conclusions in SPC with

(iii) $d(x, x^*) \leq \frac{1}{1-l} d(x, f(x))$, $\forall\ x \in X$,

where $l = \max\{a, 2b, 2c\}$.

Proof. (i)-(ii). This is Zamfirescu's theorem. It is a generalization of CP.

(iii)-(vi). From Zamfirescu's metrical conditions it follows that f is an l-quasicontraction with $l = \max\{a, 2b, 2c\}$. The proof follows from SPQC. □

Theorem 3.5. (see [35]) *Let* (X, d) *be a complete metric space,* m *be a positive integer,* $A_1, \ldots, A_m \in P_{cl}(X)$, $Y := \bigcup_{i=1}^{m} A_i$, *and* $f : Y \to Y$ *be an operator. We suppose that:*

(a) $\bigcup_{i=1}^{m} A_i$ *is a cyclic representation of* Y *with respect to* f.
(b) f *is a cyclic* l*-contraction.*

Then we have the conclusions in SPC with

(iii) $d(x, x^*) \leq \frac{1}{1-l} d(x, f(x))$, $\forall\ x \in Y$.

Proof. (i)-(ii). This is Kirk-Srinivasan-Veeramany's theorem. It is a generalization of CP.

(iii)-(vi). From the definition of cyclic representation it follows that $x^* \in \bigcap_{i=1}^{m} A_i$. From the definition of cyclic l-contraction it follows that $f : Y \to Y$ is an l-quasicontraction. The proof follows from SPQC. □

Theorem 3.6. (see [71]) *Let* (X, d) *be a complete metric space and* $f : X \to X$ *be an operator. Let* $\theta : [0, 1[\to]\frac{1}{2}, 1]$ *be defined by*

$$\theta(t) := \begin{cases} 1 & \text{if } 0 \leq t \leq (\sqrt{5} - 1)/2, \\ (1 - t)t^{-2} & \text{if } (\sqrt{5} - 1)/2 \leq t \leq 2^{-\frac{1}{2}}, \\ (1 + t)^{-1} & \text{if } 2^{-\frac{1}{2}} \leq t < 1. \end{cases}$$

We suppose that there exists $l \in [0, 1[$ *such that*

$$x, y \in X,\ \theta(l)d(x, f(x)) \leq d(x, y) \Rightarrow d(f(x), f(y)) \leq ld(x, y).$$

Then, we have the conclusions in SPC with

(iii) $d(x, x^*) \leq \frac{1}{1-l} d(x, f(x))$, $\forall\ x \in X$.

Proof. (i)-(ii). This is Suzuki's theorem. It is a generalization of CP.

(iii)-(vi). From the Suzuki's metrical condition we have that f is an l-quasicontraction. The proof follows from SPQC. □

Now, we give an example in a set with two metrics.

Theorem 3.7. (see [64], p. 40; see also [15], [43], [51]) *Let X be a nonempty set, d and ρ be two metrics on X and $f : X \to X$ be an operator. We suppose that:*

(a) $d(x,y) \leq \rho(x,y)$, $\forall\ x,y \in X$.
(b) (X,d) *is a complete metric space.*
(c) f *is an l-contraction with respect to ρ.*
(d) f *is continuous with respect to d.*

Then we have:

(i) $F_{f^n} = \{x^*\}$, $\forall\ n \in \mathbb{N}^*$.
(ii) $f^n(x) \xrightarrow{d} x^*$ *as* $n \to \infty$, *and* $f^n(x) \xrightarrow{\rho} x^*$ *as* $n \to \infty$.
(iii) $\rho(x,x^*) \leq \frac{1}{1-l}\rho(x,f(x))$, $\forall\ x \in X$.
(iv) *The fixed point problem for f is well posed with respect to ρ.*
(v) *The operator f has the Ostrowski property with respect to the metric ρ.*
(vi) *If $Y \subset X$ is a bounded and closed subset in (X,ρ) with $f(Y) \subset Y$, then $x^* \in Y$ and*

$$\bigcap_{n\in\mathbb{N}} f^n(Y) = \{x^*\}.$$

Proof. (i)-(ii). This is Maia's fixed point theorem. It is a generalization of CP.

(iii)-(vi). We remark that f is an l-quasicontraction. The proof follows from SPQC. □

In what follows we shall present an example from asymptotical fixed point theorems.

There are many asymptotic metrical fixed point results. We mention the contributions made by R. Caccioppoli (1930), J. Weisinger (1952), A.N. Kolmogorov and S.V. Fomin (1957), I.I. Kolodner (1964), S.C. Chy and J.B. Diaz (1965), V.W. Bryant (1968), V.M. Sehgal (1969), L.F. Guseman (1970), V.I. Istrăţescu (1973), W. Walter (1970, 1981), F. Browder (1979), I.A. Rus (1980), J.D. Stein (1998 (2000)), J. Jachymski and J.D. Stein (1999), K. Goebel (2002), W.A. Kirk (2003), S. Andras (2003), A.D. Arvanitakis (2003), A.S. Mureşan (2014) (see [13], [20], [72], [50], [73], [66], [70](this paper of Stein has no references on asymptotic conditions!), [23], [1], [2], [16],...).

Our example in this direction is the following.

Theorem 3.8. (see [72]) *Let (X,d) be a complete metric space and $f : X \to X$ be such that there exists $k \in \mathbb{N}^*$ for which f^k is an l-contraction. Then we have:*

(i) $F_{f^n} = \{x^*\}$, $\forall\ n \in \mathbb{N}$.
(ii) $f^n(x) \to x^*$ *as* $n \to \infty$, $\forall\ x \in X$.

If in addition, f^s is l_s-Lipschitz, $s \in \mathbb{N}^*$, then:

(iii) $d(x,x^*) \leq \frac{c_2}{1-l^{\frac{1}{k}}} d(x,f(x))$, $\forall\ x \in X$,
where $c_2 = 1 + l_1 l^{-\frac{1}{k}} + \ldots + l_{k-1} l^{\frac{1-k}{k}}$.
(iv) *The fixed point problem for f is well posed.*
(v) *The operator f has the Ostrowski property.*

(*vi*) *If $Y \subset X$ is a bounded and closed subset with $f(Y) \subset Y$, then $x^* \in Y$ and*

$$\bigcap_{n\in\mathbb{N}} f^n(Y) = \{x^*\}.$$

Proof. (i)-(ii). It follows from the following remark:

If (X, τ) is a Hausdorff topological space and $f : X \to X$ is an operator, then the following statements are equivalent:

(1) f is a Picard operator
(2) There exists $k \in \mathbb{N}^*$ such that f^k is a Picard operator.

(iii)-(vi). The functional, $\rho : X \times X \to \mathbb{R}_+$, defined by

$$\rho(x, y) := d(x, y) + l^{-\frac{1}{k}} d(f(x), f(y)) + \ldots + l^{\frac{1-k}{k}} d(f^{k-1}(x), f^{k-1}(y))$$

is a metric on X which is strongly equivalent with the metric d (see, for example [72]), with $c_1 = 1$ and $c_2 = 1 + l_1 l^{-\frac{1}{k}} + \ldots + l_{k-1} l^{\frac{1-k}{k}}$. Moreover the operator f is an $l^{\frac{1}{k}}$-contraction with respect to ρ. The proof follows from Theorem 2.1. □

In order to present the next example we need some preliminaries.

Let $(X, +, \mathbb{R}, \|\cdot\|, K)$ be an ordered Banach space. By definition the cone K is normal if there exists $c_N > 0$ such that,

$$x, y \in X,\ 0 \le x \le y \Rightarrow \|x\| \le c_N \|y\|.$$

The cone K is reproducing if, $X = K - K$. So, each element $x \in X$ admits a presentation, $x = u - v$, where $u, v \in K$. Moreover each element $x \in X$ admits a presentation, $x = u - v$ such that, $\|u\|, \|v\| \le c_r \|x\|$, where c_r does not depend of x.

In an ordered Banach space with reproducing cone, the functional, $\|\cdot\|_r : X \to \mathbb{R}_+$, defined by, $\|x\|_r := \inf\{\|y\| \mid -y \le x \le y\}$, is a norm on X. For this norm we have (see [26], p. 320),

$$(2c_N + 1)^{-1}\|x\| \le \|x\|_r \le 2c_g \|x\|,\ \forall\ x \in X.$$

Our example in an ordered Banach space is the following.

Theorem 3.9. (see [26]) *Let X be an ordered Banach space with a reproducing and normal cone K and $g : X \to X$ be a positive linear operator with, $\|g\| < 1$. If an operator $f : X \to X$ satisfies the condition*

$$-g(x - y) \le f(x) - f(y) \le g(x - y),\ \forall\ x, y \in X,\ x \ge y$$

then f satisfies the conclusions in SPC, with

(*iii*) $\|x - x^*\| \le \frac{2c_g(2c_N+1)}{1-\|g\|}\|x - f(x)\|,\ \forall\ x \in X.$

Proof. (i)-(ii). It is the Krasnoselskii's theorem. From the Krasnoselskii's proof it follows that the operator f is a $\|g\|$-contraction with respect to the strongly equivalent norm, $\|\cdot\|_r$. So, the conclusions (iii)-(vi), follows from Theorem 2.1. □

4. The case of generalized metric spaces

The universe of generalized metric spaces is a very large one (see, for example, [5], [15], [22], [24], [43], [44], [46], [55], [64],...). In what follows we shall present only some examples for our problem in some generalized metric spaces.

Theorem 4.1 (Saturated principle of contraction in a partial metric space (see [55]**).** *Let (X,p) be a complete partial metric space and $f : X \to X$ be an l-contraction. Then we have:*

(i) $F_{f^n} = \{x^*\}$, $\forall\ n \in \mathbb{N}^*$.
(ii) $p(f^n(x), x^*) \to 0$ *as* $n \to \infty$, $\forall\ x \in X$.
(iii) $p(x, x^*) \leq \frac{1}{1-l} p(x, f(x))$, $\forall\ x \in X$.
(iv) $\{y_n\}_{n\in\mathbb{N}} \subset X$, $p(y_n, f(y_n)) \to 0$ *as* $n \to \infty \Rightarrow p(y_n, x^*) \to 0$ *as* $n \to \infty$.
(v) $\{y_n\}_{n\in\mathbb{N}} \subset X$, $p(y_{n+1}, f(y_n)) \to 0$ *as* $n \to \infty \Rightarrow p(y_n, x^*) \to 0$ *as* $n \to \infty$.
(vi) *Let* $Y \subset X$ *be a nonempty subset such that* $f(Y) \subset Y$, $x^* \in Y$ *and* $\sup\{p(x,y) \mid x, y \in Y\} < +\infty$. *Then,*

$$\bigcap_{n\in\mathbb{N}} f^n(Y) = \{x^*\}.$$

Proof. (i)-(ii). This is Matthews' theorem.
(iii)-(v). See [55]. See also [64], pp. 53-58.
(vi) It is clear that $x^* \in \bigcap_{n\in\mathbb{N}} f^n(Y)$. Let $u \in \bigcap_{n\in\mathbb{N}} f^n(Y)$. Then there exists $x_n \in Y$ such that $u = f^n(x_n)$.
We have, $p(u, x^*) = p(f^n(x_n), x^*) = p(f^n(x_n), f^n(x^*)) \leq l^n p(x_n, x^*) \leq l^n \delta_p(Y) \to 0$ as $n \to \infty$. So, $u = x^*$. □

In a similar way we have

Theorem 4.2 (Saturated principle of quasicontraction in a partial metric space). *Let (X,p) be a partial metric space and $f : X \to X$ be an operator. We suppose that:*

(a) *There exists an $x^* \in X$, fixed point of f.*
(b) *f is an l-quasicontraction.*

Then we have the conclusions in Theorem 4.1.

There are examples of saturate principle of generalized contractions in $\mathbb{R}^m_+$-metric spaces. For the example corresponding to Perov's fixed point principle, see [64], pp. 82-85.

Now we give an example in a gauge space. Let (X,d) be a generalized metric space with $d(x,y) \in s(\mathbb{R}_+)$. So, $d(x,y) = \{d_k(x,y)\}_{k\in\mathbb{N}^*}$ where d_k is a pseudometric, for all $k \in \mathbb{N}^*$ and for each $(x,y) \in X \times X$ there exists $k \in \mathbb{N}^*$ such that $d_k(x,y) \neq 0$. Let $l = (l_1, \ldots, l_n, \ldots)$ be such that $0 \leq l_k < 1$, $\forall\ k \in \mathbb{N}^*$. By definition, an operator $f : X \to X$ is an l-contraction if

$$d_k(f(x), f(y)) \leq l_k d_k(x,y),\ \forall\ x, y \in X,\ \forall\ k \in \mathbb{N}^*.$$

For the basic notions in a generalized metric space with $d(x,y) \in s(\mathbb{R}_+)$, see [64], [21], [56],...

We have

Theorem 4.3 (Saturated principle of contraction in a $s(\mathbb{R}_+)$-metric space). *Let (X,d), $d(x,y)\in s(\mathbb{R}_+)$, be a complete metric space and $f:X\to X$ be an l-contraction. Then we have:*

(i) $F_{f^n}=\{x^*\}$, $\forall\ n\in\mathbb{N}^*$.
(ii) $f^n(x)\to x^*$ *as* $n\to\infty$, $\forall\ x\in X$.
(iii) $d_k(x,x^*)\leq\frac{1}{1-l_k}d(x,f(x))$, $\forall\ x\in X$.
(iv) $\{y_n\}_{n\in\mathbb{N}}\subset X$, $d(y_n,f(y_n))\to 0$ *as* $n\to\infty\Rightarrow y_n\to x^*$ *as* $n\to\infty$.
(v) $\{y_n\}_{n\in\mathbb{N}}\subset X$, $d(y_{n+1},f(y_n))\to 0$ *as* $n\to\infty\Rightarrow y_n\to x^*$ *as* $n\to\infty$.
(vi) *Let* $Y\subset X$ *be a bounded and closed subset with* $f(Y)\subset Y$. *Then* $x^*\in Y$ *and*

$$\bigcap_{n\in\mathbb{N}}f^n(Y)=\{x^*\}.$$

Proof. (i)-(ii). This is the Cain and Nashed's fixed point theorem. The Cain-Nashed's theorem is a generalization of CP.

(iii). From the definition of l-contraction we have that f is an l_k-contraction with respect to the pseudometric d_k. Now the proof is standard.

(iv). $d(y_n,f(y_n))\to 0$ as $n\to\infty$ implies that $d_k(y_n,f(y_n))\to 0$ as $n\to\infty$. From (iii) we have (iv).

(v). $d(y_{n+1},f(y_n))\to 0$ as $n\to\infty\Rightarrow d_k(y_{n+1},f(y_n))\to 0$ as $n\to\infty$. But,

$$\begin{aligned}d_k(y_{n+1},x^*)&\leq d_k(y_{n+1},f(y_n))+d_k(f(y_n),x^*)\leq\\&\leq d_k(y_{n+1},f(y_n))+ld(y_n,f(y_{n-1}))+\ldots+l^{n+1}d_k(y_0,x^*).\end{aligned}$$

Now the proof follows from a Cauchy lemma.

(vi). Let $y\in Y$. Then $f^n(y)\in Y$, $\forall\ n\in\mathbb{N}$, and $f^n(y)\to x^*$ as $n\to\infty$. Since Y is closed it follows that $x^*\in Y$. It is clear that $x^*\in\bigcap_{n\in\mathbb{N}}f^n(Y)$. Let $u\in\bigcap_{n\in\mathbb{N}}f^n(Y)$. Then there exists $x_n\in Y$ such that $u=f^n(x_n)$. We have that

$$d_k(u,x^*)=d_k(f^n(x_n),x^*)\leq l_k^n\delta_{d_k}(Y)\to 0\text{ as }n\to\infty,\ \forall\ k\in\mathbb{N}^*.$$

This implies, $u=x^*$. □

We also have

Theorem 4.4 (Saturated principle of quasicontraction in a $s(\mathbb{R}_+)$-metric space). *Let (X,d), $d(x,y)\in s(\mathbb{R}_+)$, be a generalized metric space and $f:X\to X$ be an operator. We suppose that:*

(a) *There exists an* $x^*\in X$, *a fixed point of* f.
(b) f *is an* l-*quasicontraction.*

Then we have the conclusions in Theorem 4.3.

From the above considerations the following questions rise:

Problem 4.5. Which metric conditions in a $s(\mathbb{R}_+)$-metric space (Colojoară, Gheorghiu,...) imply the conclusions in Theorem 4.3 ?

Problem 4.6. Let (X, d) be a generalized metric space with $d(x, y) \in s(\mathbb{R}_+)$. Let $f : X \to X$ be an operator. Let $M(\mathbb{R}_+)$ be the set of infinite matrices with elements in $\mathbb{R}_+$ and let I be the identity matrix in $M(\mathbb{R}_+)$. Our definition of l-contraction reads as follows

$$d(f(x), f(y)) \leq lId(x, y),\ \forall\ x, y \in X.$$

For a good definition for contractions in a such generalized metric space it is necessarily to put a more general matrix instead of lI (see [56]).

In the above setting, which are the contractions with the properties (i)-(vi) ?

5. Applications

5.1. More applications of SPC appear as applications of Picard operators. Let us mention abstract applications to: data dependence of fixed point under the operator perturbation ([49], [52], [64], [3], [4], [15], [12], [34], [56], [68]), Ulam stability of fixed point equations ([60], [65],...), abstract Gronwall lemmas ([57], [9], [15], [27], [53],...). For concrete applications to functional differential equations and to functional integral equations, see: [1], [3], [9], [14], [15], [19], [27], [30], [31], [33], [34], [39], [53], [58], [68], [74], [75],...

5.2. An other application is concerning iterated Picard operator systems. Let (X, d) be a complete metric space and $f_1, \ldots, f_m : X \to X$ be some Picard operators. These operators generate the following operator on $P(X)$,

$$T_f : P(X) \to P(X),\ T_f(A) := f_1(A) \cup \ldots \cup f_m(A),\ \forall\ A \in P(X).$$

The problem is to study the properties of T_f in terms of properties of $f_1, \ldots, f_m$. This problem is a particular case of the following Nadler problem:

Let (X, d) be a complete metric space and $f : X \to P(X)$ be a multivalued operator. Let $T_f : P(X) \to P(X)$ be the operator defined by, $T_f(A) = \bigcup\limits_{a \in A} f(a)$. The problem is to study the properties of T_f in terms of properties of f.

For example it is well known the following result:

Theorem 5.1 (Nadler (1969), Hutchinson (1981)). *Let (X, d) be a complete metric space and $f_i : X \to X$ be an l-contraction, $i = \overline{1, m}$. Then the set-to-set operator, $T_f : P_{cp}(X) \to P_{cp}(X)$ is well defined and it is an l-contraction in $(P_{cp}(X), H_d)$. Here, H_d is the Pompeiu-Hausdorff metric corresponding to d.*

From the SPC we have:

Theorem 5.2. *Let T_f be as in Theorem 5.1. Then we have:*

(i) $F_{T_f} = \{A^*\}$.

(ii) $T_f^n(A) \overset{H_d}{\to} A^*$ *as* $n \to \infty$, $\forall\ A \in P_{cp}(X)$.

(iii) $H_d(A, A^*) \leq \frac{1}{1-l} H_d(A, T_f(A))$, $\forall\ A \in P_{cp}(X)$.

(iv) *If* $A_n \in P_{cp}(X)$, $n \in \mathbb{N}$ *are such that*

$$H_d(A_n, T_f(A_n)) \to 0 \text{ as } n \to \infty,$$

then, $A_n \overset{H_d}{\to} A^*$ *as* $n \to \infty$.

(*v*) *If* $A_n \in P_{cp}(X)$, $n \in \mathbb{N}$ *are such that*

$$H_d(A_{n+1}, T_f(A_n)) \to 0 \text{ as } n \to \infty,$$

then, $A_n \overset{H_d}{\to} A^*$ *as* $n \to \infty$.

(*vi*) *Let* $U \subset P_{cp}(X)$ *be a bounded and closed subset such that* $f(U) \subset U$. *Then,* $A^* \in U$ *and*

$$\bigcap_{n \in \mathbb{N}} T_f^n(U) = \{A^*\}.$$

5.3. The SPC has applications in the variational theory of differential equations. Let us consider the following example.

In [45] (see also [8]), Radu Precup presents the following interesting result:

Theorem 5.3. *Let* X *be a Hilbert space,* $N : X \to X$ *be a contraction with the unique fixed point* u^*. *If there exists a* C^1*-functional,* $E : X \to \mathbb{R}$, *bounded from below such that*

$$E'(u) = u - N(u), \text{ for all } u \in X,$$

then u^* *minimizes the functional, i.e.,*

$$E(u^*) = \inf_X E.$$

The Precup proof for this theorem can be read as follows.

As a consequence of Bishop-Phelps' theorem, there is a sequence (u_n) with

$$E(u_n) \to \inf_X E \text{ and } E'(u_n) \to 0.$$

Since, $E'(u_n) = u_n - N(u_n) \to 0$ and N is a contraction, from conclusion (v) in SPC we have that, $u_n \to u^*$.

From this proof the following remark follows:

Remark 5.4. In Theorem 5.3 we can put instead of the operator N, an operator for which the fixed point problem is well posed.

6. Other research directions

6.1. What does it mean Saturated principle of fiber contraction ?
References: [69], [68], [67], [1].

6.2. To extend the results in sec. 2 to the case of nonself operators.
References: [6], [12], [4], [64], [17],...

6.3. To extend the results in sec. 2 to the case of multivalued operators.
References: [37], [41], [64],...

6.4. To extend the results in sec. 2 to the case of nonself multivalued operators.
References: [42], [64],...

6.5. Let (X, d) be a complete metric space, $f : X \to X$ be an operator with $F_f = \{x^*\}$ and $f_n : X \to X$ be a sequence which converges in some sense to f. Consider the iterative algorithm

$$x_{n+1} = f_n(x_n).$$

In which conditions on f and f_n this algorithm is convergent to x^* ? What estimate we have for $d(x, x^*)$?

References: [32], [4], [7], [11], [34], [59].

6.6. To extend the results in this paper to the weakly Picard operators.

References: [62], [53], [63], [64], [65], [4],...

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