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Statistical Korovkin-type theorem for monotone and sublinear operators

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Abstract. In this paper we generalize the result on statistical uniform convergence in the Korovkin theorem for positive and linear operators in $C([a,b])$, to the more general case of monotone and sublinear operators. Our result is illustrated by concrete examples.

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1. Introduction

The celebrated theorem of Korovkin [29], [30] provides a very simple test of strong operator convergence to the identity for any sequence $(T_n)_n$ of positive linear operators that map $C([0,1])$ into itself: the occurrence of this convergence for the functions $e_0(t) = 1, e_1(t) = t$ and $e_2(t) = t^2$, $t \in [0,1]$. In other words, the fact that

$$\lim_{n\to\infty} T_n(f) = f \quad \text{uniformly on } [0,1],$$

for every $f \in C([0,1])$ reduces to the status of the three aforementioned functions. Due to its simplicity and usefulness, this result has attracted a great deal of attention leading to numerous generalizations. Part of them are included in the authoritative monograph of Altomare-Campiti [7] and the excellent survey of Altomare [6].

Recently, the present authors have extended the Korovkin theorem to the framework of monotone and sublinear operators acting on function spaces endowed with the topology of uniform convergence on compact sets. See Gal-Niculescu [24], [25], [27].

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Let D be a subset of $\mathbb{N}$, the set of all natural numbers. The density of D is defined by

$$\delta(D) := \lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} \chi_D(j),$$

whenever the limit exists, where χ_D is the characteristic function of D.

The sequence $(\alpha_k)_k$ is statistically convergent to the number L if, for every $\varepsilon > 0$, we have $\delta\{k \in \mathbb{N} : |\alpha_k - L| \geq \varepsilon\} = 0$, (see Conor [10]) or equivalently, there exists a subset $K \subset \mathbb{N}$ with $\delta(K) = 1$ and $n_0(\varepsilon)$ such that $k > n_0$ and $k \in K$ imply that $|\alpha_k - L| < \varepsilon$, see Šalát [34]. In this case we write $st - \ lim \quad \alpha_k = L$. It is known that any convergent sequence is statistically convergent, but not conversely. For example, the sequence defined by $\alpha_n = \sqrt{n}$ if n is square and $\alpha_n = 0$ otherwise, has the property that $st - \ lim\alpha_n = 0$.

Some basic properties of statistical convergence are exhibited in Connor [10], Salat [34], Schoenberg [35]. Over the years this concept has been examined in number theory Erdös - Jenenbaüm [16], trigonometric series Zygmund [37], probability theory Fridy-Khan [19], optimization Pehlivan-Mamedov [33], measure theory Miller [32] and summability theory Connor [10], Fridy [18], Fridy-Orhan [20].

Korovkin type theorems for statistical convergence of positive and linear operators were obtained by many authors, to make a selection see, e.g., Gadjev [22], [21], Agratini [1]-[4], Cárdenas-Morales - Garancho [8], Dirik [13], Duman-Khan-Orhan [14], Duman [15], Akdağ [5] and the references therein.

Since evidently that a positive linear operator is monotone and sublinear, it is the purpose of this paper to generalize the result on statistical uniform convergence in the Korovkin theorem for positive and linear operators, to monotone and sublinear operators.

2. Preliminaries on weakly nonlinear operators and on Choquet integral

In what follows we denote by X a metric measure space that is, a triple (X, d, m) consisting of a space X endowed with the metric d and the measure m defined on the sigma field of Borel subsets of X. Notice that every metric space can be seen as a metric measure considering on it any finite combination (with positive coefficients) of Dirac measures.

Attached to it is the vector lattice $\mathcal{F}(X)$ of all real-valued functions defined on X, endowed with the pointwise ordering. Some important vector sublattices of $\mathcal{F}(X)$ are

$$B(X) = \{f \in \mathcal{F}(X) : \ f \text{ bounded}\}.$$

$$C(X) = \{f \in \mathcal{F}(X) : \ f \text{ continuous and bounded}\}.$$

On $B(X)$ and $C(X)$ one considers the uniform norm $\|f\| = \sup\{|f(x)|; x \in [a, b]\}$.

Suppose that X and Y are two metric spaces and E and F are respectively ordered vector subspaces (or the positive cones) of $\mathcal{F}(X)$ and $\mathcal{F}(Y)$ that contain the unity. An operator $T : E \to F$ is said to be a *weakly nonlinear operator* (respectively a *weakly nonlinear functional* when $F = \mathbb{R}$) if it satisfies the following three conditions:

(SL) (*Sublinearity*) T is subadditive and positively homogeneous, that is,

$$T(f+g) \le T(f) + T(g) \quad \text{and} \quad T(af) = aT(f)$$

for all f, g in E and $a \ge 0$;

(M) (*Monotonicity*) $f \le g$ in E implies $T(f) \le T(g)$.

(TR) (*Translatability*) $T(f + \alpha \cdot 1) = T(f) + \alpha T(1)$ for all functions $f \in E$ and all numbers $a \ge 0$.

A stronger condition than translatability is that of *comonotonic additivity*,

(CA) $T(f+g) = T(f) + T(g)$ whenever the functions $f, g \in E$ are comonotone in the sense that

$$(f(s) - f(t)) \cdot (g(s) - g(t)) \ge 0 \quad \text{for all } s, t \in X.$$

The (CA) condition occurs naturally in the context of Choquet's integral (and thus in the case of Choquet type operators). See Gal-Niculescu [25], [26] and the references therein.

Suppose that E and F are respectively closed vector sublattices of the Banach lattices $C(X)$ and $C(Y)$.

Every monotone and subadditive operator (functional when $F = \mathbb{R}$) $T : E \to F$ verifies the inequality

$$|T(f) - T(g)| \le T\left(|f-g|\right) \text{ for all } f, g. \tag{2.1}$$

Indeed, $f \le g + |f-g|$ yields $T(f) \le T(g) + T\left(|f-g|\right)$, that is,

$$T(f) - T(g) \le T\left(|f-g|\right),$$

and interchanging the role of f and g we infer that

$$-\left(T(f) - T(g)\right) \le T\left(|f-g|\right).$$

If T is linear, then the property of monotonicity is equivalent to that of *positivity*, that is, to the fact that

$$T(f) \ge 0 \quad \text{for all } f \ge 0.$$

If the operator (functional when $F = \mathbb{R}$) T is monotone and positively homogeneous, then necessarily

$$T(0) = 0.$$

The properties of weakly nonlinear operators were suggested by those of the nonlinear functional called Choquet integral. For this reason we shortly mention them below. Full details on this integral can be found in the books of D. Denneberg [12], M. Grabisch [28] and Z. Wang and G. J. Klir [36].

Let $(X, \mathcal{A})$ be an arbitrarily fixed measurable space, consisting of a nonempty abstract set X and a σ-algebra $\mathcal{A}$ of subsets of X.

Definition 2.1. (see, e.g., Denneberg [12] or Wang-Klir [36]) A set function $\mu : \mathcal{A} \to [0,1]$ is called a capacity if it verifies the following two conditions:

(a) $\mu(\emptyset) = 0$;

(b) $\mu(A) \le \mu(B)$ for all $A, B \in \mathcal{A}$, with $A \subset B$ (monotonicity).

An important class of capacities is that of probability measures (that is, the capacities playing the property of σ-additivity). Probability distortions represents a major source of nonadditive capacities. Technically, one start with a probability measure $P : \mathcal{A} \to [0,1]$ and applies to it a distortion $u : [0,1] \to [0,1]$, that is, a nondecreasing and continuous function such that $u(0) = 0$ and $u(1) = 1$;for example, one may chose $u(t) = t^a$ with $\alpha > 0$. When the distortion u is concave (for example, when $u(t) = t^a$ with $0 < \alpha < 1$), then μ is also *submodular* in the sense that

$$\mu(A \cup B) + \mu(A \cap B) \leq \mu(A) + \mu(B) \quad \text{for all } A, B \in \mathcal{A}.$$

The Choquet concept of integrability with respect to a capacity refers to the whole class of random variables, that is, to all functions $f : X \to \mathbb{R}$ such that $f^{-1}(A) \in \mathcal{A}$ for every Borel subset A of $\mathbb{R}$.

Definition 2.2. (see, e.g., Denneberg [12] or Wang-Klir [36]) The Choquet integral of a random variable f with respect to the capacity μ is defined as the sum of two Riemann improper integrals,

$$(C)\int_X f d\mu = \int_0^{+\infty} \mu\left(\{x \in X : f(x) \geq t\}\right) dt + \int_{-\infty}^0 \left[\mu\left(\{x \in X : f(x) \geq t\}\right) - 1\right] dt,$$

Accordingly, f is said to be Choquet integrable if both integrals above are finite.

If $f \geq 0$, then the last integral in the formula appearing in Definition 2.2 is 0.

The inequality sign $\geq$ in the above two integrands can be replaced by $>$; see [36], Theorem 11.1, p. 226.

The Choquet integral coincides with the Lebesgue integral when the underlying set function μ is a σ-additive measure.

As usually, a function f is said to be Choquet integrable on a set $A \in \mathcal{A}$ if $f\chi_A$ is integrable in the sense of Definition 2.2. We denote

$$(C)\int_A f d\mu = (C)\int_X f\chi_A d\mu.$$

The basic properties of the Choquet integral, seen as a functional are as follows: it is monotone, positive homogenous, comonotonic additive and subadditive (if μ is submodular).

Remark 2.3. Several extensions of Korovkin's theorem in the case of weakly nonlinear operators acting on a sublattice of a space $C(X)$ and the uniform convergence on compact sets can be found in the papers Gal-Niculescu [24], [25], [27].

In the next section we discuss an analogue in the space $C([a,b])$ and for statistical uniform convergence.

3. Main result, uniform convergence case

In this section we obtain an analogue result with the classical Korovkin theorem in $C([a,b])$ for the statistical uniform convergence of a sequence of monotone and sublinear operators.

Theorem 3.1. *If the sequence of monotone and sublinear operators* $A_n : C([a,b]) \to B([a,b])$ *satisfies the conditions*

$$st - \lim \|A_n(e_0) - e_0\| = 0; st - \lim \|A_n(e_1) - e_1\| = 0,$$

$$st - \lim \|A_n(e_2) - e_2\| = 0; st - \lim \|A_n(-e_1) + e_1\| = 0, \tag{3.1}$$

then for any nonnegative function $f \in C([a;b])$*, we have*

$$st - \lim \|A_n(f) - f\| = 0. \tag{3.2}$$

If, in addition, all A_n *are translatable, then the above convergence holds for all* $f \in C([a,b])$*. Here* $\|\cdot\|$ *denotes the uniform norm.*

Proof. Suppose firstly that $f \in C([a,b])$ is nonnegative on $[a,b]$. Since f is bounded, we can write

$$|f(t) - f(x)| \leq 2M, \text{ for all } t, x \in [a,b].$$

Also, since f is continuous on $[a,b]$, it follows that there exists a $\delta > 0$ (depending on ε) such that $|f(t) - f(x)| < \varepsilon$ for all $t, x \in [a,b]$ satisfying $|x - t| < \delta$, which implies that for all $t, x \in [a,b]$ we obtain

$$|f(t) - f(x)| \leq \varepsilon + \frac{2M}{\delta^2}(t-x)^2 = \varepsilon + \frac{2M}{\delta^2}(t^2 - 2xt + x^2). \tag{3.3}$$

We have two cases:
Case 1. $x \in [a,b],\ x \leq 0.$
Case 2. $x \in [a,b],\ x > 0.$

Applying A_n to (3.3), by the sublinearity of A_n and by the property (2.1), since $-2x \geq 0$, in Case 1 it follows

$$A_n(|f - f(x)|)(x) \leq \varepsilon A_n(e_0)(x) + \frac{2M}{\delta^2} A_n((e_1 - x)^2)(x) \leq \varepsilon A_n(e_0)(x)$$

$$+\frac{2M}{\delta^2}(A_n(e_2)(x) - x^2 + x^2 - 2xA_n(e_1)(x) + 2x^2 - 2x^2 + x^2 A_n(e_0)(x) - x^2 + x^2)$$

$$\leq \varepsilon\|A_n(e_0) - e_0\| + \varepsilon$$

$$+\frac{2M}{\delta^2}(\|A_n(e_2) - e_2\| + 2|x| \cdot \|A_n(e_1) - e_1\| + x^2\|A_n(e_0) - e_0\|),$$

which by

$$A_n(f)(x) - f(x) = A_n(f)(x) - A_n(f(x))(x) + f(x)(A_n(e_0)(x) - e_0(x)),$$

immediately implies

$$|A_n(f)(x) - f(x)| \leq |A_n(f)(x) - A_n(f(x))(x)| + |f(x)| \cdot |A_n(e_0)(x) - e_0(x)|,$$

and therefore

$$\|A_n(f) - f\| \leq \|A_n(|f - f(x)|)\| + M \cdot \|A_n(e_0) - e_0\|$$

$$\leq (\varepsilon + M + \frac{2M\alpha^2}{\delta^2})\|A_n(e_0) - e_0\| + \frac{4M\alpha}{\delta^2}\|A_n(e_1) - e_1\| + \frac{2M}{\delta^2}\|A_n(e_2) - e_2\|$$

$$\leq C(\|A_n(e_0) - e_0\| + \|A_n(e_1) - e_1\| + \|A_n(e_2) - e_2\|), \tag{3.4}$$

where $C = \max\{\varepsilon + M + \frac{2M\alpha^2}{\delta^2}, \frac{4M\alpha}{\delta^2}\}$ and $\alpha = \max\{|a|, |b|\}$.

In the Case 2, since $-2x<0$ and applying the positive homogeneity of A_n too, it follows

$$A_n(|f-f(x)|)(x)\le \varepsilon A_n(e_0)(x)+\frac{2M}{\delta^2}A_n((e_1-x)^2)(x)\le \varepsilon A_n(e_0)(x)$$
$$+\frac{2M}{\delta^2}(A_n(e_2)(x)-x^2+x^2+2xA_n(-e_1)(x)+2x^2-2x^2+x^2A_n(e_0)(x)-x^2+x^2)$$
$$\le \varepsilon\|A_n(e_0)-e_0\|+\varepsilon$$
$$+\frac{2M}{\delta^2}(\|A_n(e_2)-e_2\|+2|x|\cdot\|A_n(-e_1)+e_1\|+x^2|A_n(e_0)-e_0\|),$$

which immediately implies

$$\|A_n(f)-f\|\le\|A_n(|f-f(x)|)\|+M\cdot\|A_n(e_0)-e_0\|$$
$$\le(\varepsilon+M+\frac{2M\alpha^2}{\delta^2})\|A_n(e_0)-e_0\|+\frac{4M\alpha}{\delta^2}\|A_n(-e_1)+e_1\|+\frac{2M}{\delta^2}\|A_n(e_2)-e_2\|$$
$$\le C(\|A_n(1,x)-1\|+\|A_n(-t,x)+x\|+\|A_n(t^2,x)-x^2\|), \qquad (3.5)$$

where again $C=\max\{\varepsilon+M+\frac{2M\alpha^2}{\delta^2},\frac{4M\alpha}{\delta^2}\}$ and $\alpha=\max\{|a|,|b|\}$.

Denoting

$$E=\left\{n:\|A_n(e_0)-e_0\|+\|A_n(e_1)-e_1\|+\|A_n(-e_1)+e_1\|+\|A_n(e_2)-e_2\|\ge\frac{\eta}{C}\right\},$$
$$E_1:=\left\{n:\|A_n(e_0)-e_0\|\ge\frac{\eta}{4C}\right\},$$
$$E_2:=\left\{n:\|A_n(e_1)-e_1\|\ge\frac{\eta}{4C}\right\},$$
$$E_3:=\left\{n:\|A_n(-e_1)+e_1\|\ge\frac{\eta}{4C}\right\},$$
$$E_4:=\left\{n:\|A_n(e_2)-e_2\|\ge\frac{\eta}{4C}\right\},$$

the inequalities (3.4) and (3.5) show that $E\subset E_1\cup E_2\cup E_3\cup E_4$, which implies

$$\chi_{E(j)}\le\chi_{E_1(j)}+\chi_{E_2(j)}+\chi_{E_3(j)}+\chi_{E_4(j)},\ \text{for all } j\in\mathbb{N}.$$

Therefore, denoting $D=\{n\in\mathbb{N};\|A_n(f)-f\|\ge\eta\}$, it is immediate that

$$\delta(D)\le\delta(E)\le\delta(E_1)+\delta(E_2)+\delta(E_3)+\delta(E_4)$$

and by using (3.1), we get (3.2) and therefore it follows that the proof is complete for nonnegative f.

Suppose now that f is of arbitrary sign. It follows that $f+\|f\|$ is nonnegative and therefore form the above conclusion we get

$$st-\lim\|A_n(f+\|f\|)-f-\|f\|\quad\|=0.$$

Since all A_n are translatable, it follows

$$A_n(f+\|f\|)(x)=A_n(f)(x)+\|f\|\cdot A_n(e_0)(x)$$

for all $n\in\mathbb{N}$ and therefore

$$\|A_n(f+\|f\|)-f-\|f\|\quad\|=\|A_n(f)-f+\|f\|\cdot(A_n(e_0)-e_0)\|,$$

which immediately leads to the desired conclusion. □

4. Concrete examples in Theorem 1

In this section we present three concrete examples illustrating Theorem 1.

Example 4.1. Firstly, let us consider the Bernstein-Kantorovich-Choquet polynomial operators for functions of one real variable defined by the formula

$$K_{n,\mu}(f)(x) = \sum_{k=0}^{n} p_{n,k}(x) \cdot \frac{(C)\int_{k/(n+1)}^{(k+1)/(n+1)} f(t)\mathrm{d}\mu(t)}{\mu([k/(n+1),(k+1)/(n+1)])},$$

with $\mu = \sqrt{m}$, m the Lebesgue measure and

$$p_{n,k}(t) = \binom{n}{k} t^k (1-t)^{n-k}, \quad \text{for } t \in [0,1] \text{ and } n \in \mathbb{N}.$$

According to the results in Section 3 in Gal-Niculescu [24], $K_{n,\mu}(e_k) \to e_k$, $k \in \{0,1,2\}$ and $K_{n,\mu}(-e_1) \to -e_1$, uniformly on $[0,1]$. Also, according to Section 5 in Gal-Niculescu [27], these operators are monotone, sublinear and translatable.

Define now the sequence

$$P_n(f)(x) = (1+\alpha_n)K_{n,\mu}(f)(x), x \in [0,1], n \in \mathbb{N},$$

where α_n is a sequence statistically convergent to zero but not convergent to zero in classical sense.

Therefore $P_n(f)(x)$ satisfies the conditions in Theorem 1 and consequently $P_n(f)$ converges statistically to f, for any $f \in C([a,b])$.

Example 4.2. We consider below an example which does not involve the Choquet integral, namely they are the so-called possibilistic Kantorovich operators introduced in Gal [23], defined by

$$T_n(f)(x) = \sum_{k=0}^{n} p_{n,k}(x) \cdot \sup\{f(x); x \in [k/(n+1),(k+1)/(n+1)]\}.$$

It is easy to see that each T_n is a monotone, sublinear and translatable operator and

$$T_n(e_0)(x) = e_0(x).$$

Also we have

$$\begin{aligned} T_n(e_1)(x) &= \sum_{k=0}^{n} p_{n,k}(x)\frac{k+1}{n+1} \\ &= \frac{n}{n+1} \cdot \sum_{k=0}^{n} p_{n,k}(x)\left[\frac{k}{n} + \frac{1}{n}\right] \\ &= \frac{n}{n+1}x + \frac{1}{n+1} \to e_1(x), \end{aligned}$$

$$\begin{aligned} T_n(-e_1)(x) &= \sum_{k=0}^{n} p_{n,k}(x)\left[-\frac{k}{n+1}\right] \\ &= \frac{n}{n+1}\sum_{k=0}^{n} p_{n,k}(x) - \frac{k}{n} \\ &= -\frac{n}{n+1}x \to -e_1(x), \end{aligned}$$

for $n \to \infty$, uniformly on $[0,1]$.

Moreover,

$$\begin{aligned} T_n(e_2)(x) &= \sum_{k=0}^{n} p_{n,k}(x)\cdot\left(\frac{k+1}{n+1}\right)^2 \\ &= \left(\frac{n}{n+1}\right)^2 \cdot \sum_{k=0}^{n} p_{n,k}(x)\frac{k^2+2k+1}{n^2} \\ &= \left(\frac{n}{n+1}\right)^2 x^2 + \frac{2n}{(n+1)^2}x + \frac{1}{(n+1)^2} \to e_2(x), \end{aligned}$$

for $n \to \infty$, uniformly on $[0,1]$.

Now, the sequence

$$Q_n(f)(x) = (1+\alpha_n)\cdot T_n(f)(x), x \in [a,b], n \in \mathbb{N},$$

where α_n is the sequence mentioned in Example 4.1 too, satisfies Theorem 1.

Example 4.3. Define now the sequence

$$Q_n(f)(x) = \sum_{k=0}^{n} p_{n,k}(x)\cdot\frac{(C)\int_{k/(n+1)}^{(k+1)/(n+1)} f(\beta_n t)\mathrm{d}\mu(t)}{\mu([k/(n+1),(k+1)/(n+1)])},$$

where $\mu = \sqrt{m}$ and $0 \le \beta_n \le 1$, $n \in \mathbb{N}$, is statistically convergent to 1 but not convergent in the classical sense.

These operators are monotone, sublinear and translatable and we easily seen that we have

$$\begin{aligned} Q_n(e_0)(x) &= 1, \\ Q_n(e_1)(x) &= \beta_n K_{n,\mu}(e_1)(x), \\ Q_n(-e_1)(x) &= \beta_n K_{n,\mu}(-e_1)(x), \\ Q_n(e_2)(x) &= \beta_n^2 K_{n,\mu}(e_2)(x). \end{aligned}$$

Therefore, the hypothesis of Theorem 1 are satisfied.

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