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Hankel and symmetric Toeplitz determinants for Sakaguchi starlike functions

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Abstract. In this paper, we consider the class of starlike functions with respect to symmetric points which are also known as Sakaguchi starlike functions. We determine best possible bounds on Zalcman conjecture $|a_n^2 - a_{2n-1}|$ and generalized Zalcman conjecture $|a_m a_n - a_{m+n-1}|$ for $n = 2$ and $n = 4$, $m = 2$, respectively for such functions. Further, we compute estimate on third order and fourth order Hankel determinants. As well, we also obtain estimates on third and fourth symmetric Toeplitz determinants.

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1. Introductory text

Let $\mathcal{A}$ be the family of all normalized analytic functions f defined on $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ with series expansion $f(z) = z + a_2 z^2 + a_3 z^3 + \cdots$. The subfamily $\mathcal{S} \subset \mathcal{A}$ contains univalent functions. Let $\mathcal{S}^*$ and $\mathcal{K}$ represent the subfamily of $\mathcal{S}$ containing starlike and convex functions, respectively. Analytically, $\mathcal{S}^* = \{f \in \mathcal{S} : \operatorname{Re}(zf'(z)/f(z)) > 0,\, z \in \mathbb{D}\}$ and $\mathcal{K} = \{f \in \mathcal{S} : 1 + \operatorname{Re}(zf''(z)/f'(z)) > 0,\, z \in \mathbb{D}\}$ [11]. The class $\mathcal{P}$ consists of all analytic functions $p : \mathbb{D} \to \mathbb{C}$ satisfying conditions $p(0) = 1$ and $\operatorname{Re} p(z) > 0$. Recent results for a more general class of $\mathcal{P}$ can be found in [3]. In 1959, Sakaguchi [33] studied the subclass $\mathcal{S}^*_{\mathcal{S}}$ of $\mathcal{S}$ consisting of starlike functions

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with respect to the symmetric points. The analytical description of these functions is

$$\mathcal{S}_{\mathcal{S}}^{*} = \left\{ f \in \mathcal{S} : \frac{2zf'(z)}{f(z) - f(-z)} \in \mathcal{P},\, z \in \mathbb{D} \right\}.$$

The functions $f \in \mathcal{S}_{\mathcal{S}}^{*}$ are also called Sakaguchi starlike functions. The coefficient estimates related literature gives the geometric properties of univalent functions. The bound on the initial coefficient a_2 contribute in growth, distortion and covering theorems. Zalcman conjecture and Hankel determinants are two of the coefficient problems that have been discussed by several authors. In recent years, many authors have studied the Toeplitz determinant $T_q(n)$ for various values of q and n for several subclasses of analytic functions. A significant problem concerning the coefficients in the series expansion of the the function $f \in \mathcal{A}$ is the Zalcman conjecture which is defined as

$$|a_n^2 - a_{2n-1}| \leq (n-1)^2,\, n \geq 2.$$

From [7], we observe that the Zalcman conjecture implies the Bieberbach conjecture. Ma [24] verified Zalcman conjecture ($n \geq 4$) for close-to-convex functions. Further, Ma [25] explored the generalized Zalcman conjecture which is defined as

$$|a_m a_n - a_{m+n-1}| \leq (m-1)(n-1); \qquad m \geq 2,\, n \geq 2$$

for the starlike functions and the univalent functions with real coefficients. In [32], Ravichandran and Verma established the generalized Zalcman conjecture for certain starlike and convex functions. In [34], the Zalcman conjecture and the generalized Zalcman conjecture for the locally univalent functions were discussed using extreme point theory. Recently, in [26] the Zalcman conjecture and the generalized Zalcman conjecture were shown for the class $\mathcal{U}$ defined as $\mathcal{U} = \{z \in \mathcal{A} : \left|(z/f(z))^2 f'(z) - 1\right| < 1,\, z \in \mathbb{D}\}$. For $q \geq 1$ and $n \geq 1$, the q^{th} Hankel determinant $H_q(n)(f)$ for a function $f \in \mathcal{S}$ is given by $H_q(n)(f) := \det\{a_{n+i+j-2}\}_{i,j}^{q}$, $1 \leq i,\, j \leq q$, where $a_1 = 1$. For $q = 2$ and $n = 1$, the Hankel determinant $H_2(1) = a_3 - a_2^2$ is the Fekete Szegö functional. The study of Hankel determinant was initiated by Pommerenke [27, 28] for the starlike functions. Since then the growth of $H_q(n)(f)$ has been studied for different subclasses of univalent functions. One of the notable results in this direction is by Hayman [12] giving the best possible upper bound as $Mn^{1/2}$ on $H_2(n)(f)$, where M is an absolute constant. For $q = 2$ and $n = 2$, Janteng *et al.*[13] obtained the sharp estimates on second order Hankel determinant $H_2(2)(f) = a_2a_4 - a_3^2$ for the classes of starlike and convex functions. However, the sharp bound for the whole class $\mathcal{S}$ is not known till now. For the class of Bazilevic functions, Krishna and RamReddy [16] determined $H_2(2)(f)$. Recently, Anand *et al.* [4] studied the second order Hankel determinant for a class of normalized analytic functions.

For $q = 3$ and $n = 1, 2, 3$, the third Hankel determinants are given as

$$H_3(1)(f) = a_3(a_2a_4 - a_3^2) + a_4(a_2a_3 - a_4) + a_5(a_3 - a_2^2) \tag{1.1}$$

$$H_3(2)(f) = a_2(a_4a_6 - a_5^2) - a_3(a_3a_6 - a_4a_5) + a_4(a_3a_5 - a_4^2) \tag{1.2}$$

$$H_3(3)(f) = a_3(a_5a_7 - a_6^2) - a_4(a_4a_7 - a_5a_6) + a_5(a_4a_6 - a_5^2). \tag{1.3}$$

The study of the third order Hankel determinant $H_3(1)(f)$ for the classes $\mathcal{S}^*$ and $\mathcal{K}$ was initiated by Babalola (2010) [6] which was later improved by Zaprawa [36]. However, the bounds obtained in [36] were not sharp. The best possible bound on third order Hankel determinant $H_3(1)(f)$ for the class of convex functions was computed by Kowalczyk *et al.* [15]. Also, Lecko *et al.* [23] computed the best possible upper bound on $H_3(1)(f)$ for the starlike functions of order 1/2. Krishna *et al.* [17] obtained the bound on $H_3(1)(f)$ for the class $\mathcal{S}^*_S$. Recently, Kumar *et al.* [20] improved the existing bound for the class $\mathcal{S}^*_S$. For more recent developments on coefficient estimates and third order Hankel determinant, see [14, 17, 29, 37, 22, 21, 35]. For $q = 4$ and $n = 1$, the fourth order Hankel determinant is given by

$$H_4(1)(f) = a_7 H_3(1)(f) - a_6\Delta_1 + a_5\Delta_2 - a_4\Delta_3 \tag{1.4}$$

where

$$\Delta_1 = (a_3a_6 - a_4a_5) - a_2(a_2a_6 - a_3a_5) + a_4(a_2a_4 - a_3^2),$$
$$\Delta_2 = (a_4a_6 - a_5^2) - a_2(a_3a_6 - a_4a_5) + a_3(a_3a_5 - a_4^2)$$

and

$$\Delta_3 = a_2(a_4a_6 - a_5^2) - a_3(a_3a_6 - a_4a_5) + a_4(a_3a_5 - a_4^2).$$

Arif *et al.* [5] obtained the bound on $H_4(1)(f)$ for the functions with bounded turning. Cho and Kumar [9] computed the bound on $H_4(1)(f)$ for starlike functions associated with a lune-shaped region. For recent results on fourth order Hankel determinant, see [19, 10]. For $q \geq 1$ and $n \geq 1$, the symmetric Toeplitz determinant $T_q(n)$ for a function $f \in \mathcal{S}$ is defined as

$$T_q(n) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_n & \cdots & a_{n+q} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n+q-1} & a_{n+q} & \cdots & a_n \end{vmatrix}$$

where $a_1 = 1$. In particular, for $q = 2$ and $n = 2, 3$ the second Toeplitz deteminants are given by $T_2(2) = a_3^2 - a_2^2$ and $T_2(3) = a_4^2 - a_3^2$.

For $q = 3$ and $n = 1, 2$ the third Toeplitz determinants are as follows

$$T_3(1) = 1 + 2a_2^2(a_3 - 1) - a_3^2 \text{ and } T_3(2) = (a_2 - a_4)(a_2^2 - 2a_3^2 + a_2a_4). \tag{1.5}$$

For $q = 4$ and $n = 2$ the fourth Toeplitz determinant is given by

$$T_4(2) = (a_2^2 - a_3^2)^2 + 2(a_3^2 - a_2a_4)(a_2a_4 - a_3a_5) - (a_2a_3 - a_3a_4)^2 + (a_4^2 - a_3a_5)^2 - (a_3a_4 - a_2a_5)^2. \tag{1.6}$$

In 2019, Zhang *et al.* [38] computed the upper bound on the Toeplitz determinant $T_3(2)$ for the starlike functions associated with the sine function. Ahuja *et al.* [1] studied the Toeplitz determinants $T_2(2)$ and $T_3(1)$ for unified class of starlike and convex functions. Recently, in [39], Zhang and Tang obtained the upper bound on fourth Toeplitz determinant $T_4(2)$ for the starlike functions associated with the sine function. For more recent details, see [2, 18]

In this manuscript, we prove Zalcman Conjecture $|a_n^2 - a_{2n-1}| \leq (n-1)^2$ for $n = 2$ and generalised Zalcman Conjecture $|a_m a_n - a_{m+n-1}| \leq (m-1)(n-1)$ for

$m = 2$, $n = 4$. Further, we obtain the estimates on the third order Hankel determinant $H_3(1)(f)$ for such functions which is an improvement to the existing estimate computed in [20]. In addition, we compute the bounds on third order Hankel determinants $H_3(2)(f)$, $H_3(3)(f)$ and the fourth order Hankel determinant $H_4(1)(f)$. Moreover, bounds on the symmetric Toeplitz determinants $T_2(2), T_2(3), T_3(1), T_3(2)$ and $T_4(2)$ are also determined.

2. Inductive lemmas

In order to establish the main results, we need following lemmas related to coefficient estimates.

Lemma 2.1. [30] *Let $w(z) = c_1z + c_2z^2 + \cdots$ be a Schwarz function. Then*

$$|c_3 + \mu c_1c_2 + \nu c_1^3| \leq 1,$$

where $1/2 \leq |\mu| \leq 2,\ 4(|\mu| + 1)^3/27 - (|\mu| + 1) \leq \nu \leq 1$.

Let $\mathcal{B}$ be the class of functions $f \in \mathcal{A}$ satisfying $|f(z)| < 1$ for all $z \in \mathbb{D}$.

Lemma 2.2. [8] *Let $f(z) = a_0 + \sum_{n=1}^{\infty} a_nz^n$ be in $\mathcal{B}$. Then*

$$|a_{2n+1}| \leq 1 - |a_0|^2 - |a_1|^2 - \cdots - |a_n|^2,\ n = 0, 1, \cdots \tag{2.1}$$

and

$$|a_{2n}| \leq 1 - |a_0|^2 - |a_1|^2 - \cdots - |a_{n-1}|^2 - \frac{|a_n|^2}{1 + |a_0|},\ n = 1, 2, \cdots. \tag{2.2}$$

Equality in (2.1) holds for

$$f(z) = \frac{a_0 + a_1z + \cdots + a_nz^n + \varepsilon z^{2n+1}}{1 + (\overline{a_n}z^{n+1} + \overline{a_{n-1}}z^{n+2} + \cdots + \overline{a_0}z^{2n+1})\varepsilon},\ |\varepsilon| = 1$$

and in (2.2) for

$$f(z) = \frac{a_0 + a_1z + \cdots + a_{n-1}z^{n-1} + \frac{a_n}{1+|a_0|} + \varepsilon z^{2n}}{1 + (\frac{\overline{a_n}}{1+|a_0|}z^n + \overline{a_{n-1}}z^{n+1} + \cdots + \overline{a_0}z^{2n})\varepsilon},\ |\varepsilon| = 1$$

where $a_0\overline{a_n}^2\varepsilon$ is non-positive real.

In view of Lemma 2.2, for a Schwarz function $w(z) = c_1z + c_2z^2 + \cdots$, we have

$$|c_2| \leq 1 - |c_1|^2,\ |c_3| \leq 1 - |c_1|^2 - \frac{|c_2|^2}{1 + |c_1|} \text{ and } |c_4| \leq 1 - |c_1|^2 - |c_2|^2. \tag{2.3}$$

Lemma 2.3. [33] *Let $f(z) = z + \sum_{n=2}^{\infty} a_nz^n$ be univalent and starlike with respect to symmetric points in $\mathbb{D}$. Then*

$$|a_n| \leq 1,\ n \geq 2$$

equality being attained by the function $z/(1 + \varepsilon z)$, $|\varepsilon| < 1$.

Lemma 2.4. [31] *If* $p(z) = 1 + p_1 z + p_2 z^2 + p_3 z^3 + \cdots \in \mathcal{P}$ *then for all* $n, m \in \mathbb{N}$

$$|\mu p_n p_m - p_{m+n}| \leq \begin{cases} 2, & 0 \leq \mu \leq 1 \\ 2|2\mu - 1|, & \textit{elsewhere.} \end{cases}$$

If $0 < \mu < 1$, *the inequality is sharp for the function* $p(z) = (1 + z^{n+m})/(1 - z^{n+m})$. *In other cases, the inequality is sharp for the function* $p(z) = (1 + z)/(1 - z)$.

3. Zalcman conjecture

In this section, we first prove Zalcman conjecture ($n = 2$) for starlike functions with respect to the symmetric space.

Theorem 3.1. *If the function* $f \in \mathcal{S}^*_{\mathcal{S}}$ *is of the form* $f(z) = z + a_2 z^2 + a_3 z^3 + \cdots$. *Then*

$$|a_2^2 - a_3| \leq 1.$$

The inequality is sharp.

Proof. Let $f \in \mathcal{S}^*_{\mathcal{S}}$. Then we have

$$\frac{2zf'(z)}{f(z) - f(-z)} \prec \frac{1+z}{1-z}$$

for all $z \in \mathbb{D}$ so that

$$\frac{2zf'(z)}{f(z) - f(-z)} = p(z)$$

where $\prec$ denotes subordination and $p(z) = 1 + p_1 z + p_2 z^2 + \cdots \in \mathcal{P}$. On comparing the coefficients of like power terms on both sides, we get

$$a_2 = \frac{p_1}{2}; \tag{3.1}$$

$$a_3 = \frac{p_2}{2}; \tag{3.2}$$

$$a_4 = \frac{1}{8}(p_1 p_2 + 2p_3); \tag{3.3}$$

$$a_5 = \frac{1}{8}(p_2^2 + 2p_4); \tag{3.4}$$

$$a_6 = \frac{1}{48}(4p_2 p_3 + p_1(p_2^2 + 2p_4) + 8p_5); \tag{3.5}$$

$$a_7 = \frac{1}{48}(p_2^3 + 6p_2 p_4 + 8p_6); \tag{3.6}$$

It follows from (3.1) and (3.2) that

$$a_2^2 - a_3 = \frac{p_1^2}{4} - \frac{p_2}{2}.$$

By using Lemma 2.4, we get

$$|a_2^2 - a_3| = \frac{1}{2}\left|\frac{1}{2}p_1^2 - p_2\right| \leq 1.$$

The inequality is sharp for the function

$$\frac{2zf'(z)}{f(z)-f(-z)} = \frac{1+z^2}{1-z^2} = 1+2z^2+2z^4+\cdots \tag{3.7}$$

by noting the fact $a_2 = 0, a_3 = 1$ implies $|a_2^2 - a_3| = 1$. □

Next we prove the generalized Zalcman conjecture for $m = 2$ and $n = 4$.

Theorem 3.2. *Let the function $f(z) = z + a_2z^2 + a_3z^3 + \cdots \in \mathcal{S}^*_{\mathcal{S}}$. Then*

$$|a_2a_4 - a_5| \le 1.$$

The inequality is sharp.

Proof. If the function $f \in \mathcal{S}^*_{\mathcal{S}}$, then using (3.1),(3.3) and (3.4), we get

$$\begin{aligned} a_2a_4 - a_5 &= \frac{1}{16}p_1(p_1p_2 + 2p_3) - \frac{1}{8}(p_2^2 + 2p_4) \\ &= \frac{1}{8}p_2\left(\frac{1}{2}p_1^2 - p_2\right) + \frac{1}{4}\left(\frac{1}{2}p_1p_3 - p_4\right). \end{aligned}$$

Using triangle inequality

$$|a_2a_4 - a_5| \le \frac{1}{8}|p_2|\left|\frac{1}{2}p_1^2 - p_2\right| + \frac{1}{4}\left|\frac{1}{2}p_1p_3 - p_4\right|.$$

Applying Lemma 2.4 and the fact $|p_n| \le 2$, we get

$$|a_2a_4 - a_5| \le 1.$$

The inequality is sharp for the function f defined by (3.7). □

4. Hankel determinants

Using the technique discussed in [37], the following theorem gives an improved estimate on $H_3(1)$ for the functions f in the class $\mathcal{S}^*_{\mathcal{S}}$.

Theorem 4.1. *Let the function $f \in \mathcal{S}^*_{\mathcal{S}}$ be of the form $f(z) = z + a_2z^2 + a_3z^3 + \cdots$. Then*

$$|H_3(1)(f)| \le \frac{329}{400} \simeq 0.8225.$$

Proof. Let $f \in \mathcal{S}^*_{\mathcal{S}}$. Then we have

$$\frac{2zf'(z)}{f(z)-f(-z)} \prec \frac{1+z}{1-z}$$

so that

$$\frac{2zf'(z)}{f(z)-f(-z)} = \frac{1+w(z)}{1-w(z)}, \qquad z \in \mathbb{D}$$

where $\prec$ denotes subordination and $w(z) = c_1 z + c_2 z^2 + \cdots$ is a Schwarz function. On comparing the coefficients of like powers of z, we get

$$a_2 = c_1,\ a_3 = c_1^2 + c_2,\ a_4 = \frac{1}{2}(c_3 + 3c_1c_2 + 2c_1^3) \tag{4.1}$$

$$a_5 = \frac{1}{2}(c_4 + 2c_1c_2 + 5c_1^2c_2 + 2c_1^4 + 2c_2^2). \tag{4.2}$$

Therefore, in view of (1.1), (4.1) and (4.2) the third order Hankel determinant $H_3(1)$ becomes

$$\begin{aligned} H_3(1)(f) &= \frac{1}{4}(c_1^2c_2^2 + 2c_1c_2c_3 - c_3^2 + 2c_2c_4) \\ &= \frac{1}{4}(-2c_3(c_3 - c_1c_2) + c_3^2 + c_1^2c_2^2 + 2c_2c_4). \end{aligned}$$

Hence, applying Lemma 2.1 ($\mu = -1, \nu = 0$) and inequalities given in (2.3), we get

$$\begin{aligned} |H_3(1)(f)| &\le \frac{1}{4}(2\left(1 - |c_1|^2 - \frac{|c_2|^2}{1+|c_1|}\right) + \left(1 - |c_1|^2 - \frac{|c_2|^2}{1+|c_1|}\right)^2 \\ &\qquad + |c_1|^2|c_2|^2 + 2|c_2|\left(1 - |c_1|^2 - |c_2|^2\right)) \\ &= \frac{1}{4}G(|c_1|, |c_2|). \end{aligned}$$

The function $G(x, y)$ is given by

$$G(x, y) = g_1(x, y) + g_2(x, y) + g_3(x, y),$$

where

$$\begin{aligned} g_1(x, y) &= \left(\frac{y}{(1+x)^2} - 1\right) y^3 \\ g_2(x, y) &= 2(1 - x^2)y - \frac{4 - 3x^2 - x^3}{1+x} y^2 \\ g_3(x, y) &= -y^3 + x^4 - 4x^2 + 3 \end{aligned}$$

where $x = |c_1|$ and $y = |c_2|$. In view of $|c_2| \le 1 - |c_1|^2$, we maximize the function $G(x, y)$ in the region

$$\Omega = \{(x, y) : x \ge 0, y \ge 0, y \le 1 - x^2\}.$$

It is noted that

$$g_1(x, y) \le 0. \tag{4.3}$$

Since $g_2(x, y)$ is a quadratic expression in y, so it attains its maximum value at

$$y_0 = \frac{(1 - x^2)(1 + x)}{4 - 3x^2 - x^3}.$$

Also $y_0 < 1 - x^2$ for all $x \in [0, 1]$ and thus we have

$$g_2(x, y) \le g_2(x, y_0) = \frac{(1 - x)(1 + x)^3}{(2 + x)^2} =: f(x).$$

A simple calcultaion shows that $x_2 = 0.3$ is a critical point of the function f in $(0, 1)$. Hence,

$$g_2(x, y) \le f(x_2) = \frac{29}{100}. \tag{4.4}$$

For the function $g_3(x, y)$, it is evident that

$$g_3(x, y) \le g_3(x, 0) = x^4 - 4x^2 + 3 =: h(x).$$

Now $h'(x) = 4x(x^2 - 2)$, so $h(x) \le h(0)$. This gives

$$g_3(x, y) \le h(0) = 3. \tag{4.5}$$

Using (4.3), (4.4) and (4.5), we get $G(x, y) \le 329/100$.
Therefore, we have $|H_3(1)(f)| \le 329/400 \simeq 0.8225$. □

Remark 4.2. The obtained upper bound $\frac{329}{400} \simeq 0.8225$ on $H_3(1)(f)$ (4.1) improves the existing bound $\frac{5}{4} \simeq 1.25$ [20, Theorem 2.3, p.227] for the functions $f \in \mathcal{S}^*_{\mathcal{S}}$.

Next theorem gives bound on $H_3(2)$ for the functions $f \in \mathcal{S}^*_{\mathcal{S}}$.

Theorem 4.3. *If $f \in \mathcal{S}^*_{\mathcal{S}}$ is of the form $f(z) = z + a_2z^2 + a_3z^3 + \cdots$. Then*

$$|H_3(2)(f)| < \frac{83}{24} \simeq 3.45.$$

Proof. On substituting the values of a_4, a_5 and a_6 from (3.3), (3.4) and (3.5), respectively in the expression $a_4a_6 - a_5^2$, we have

$$\begin{aligned}
a_4a_6 - a_5^2 &= \frac{1}{384}(p_1p_2 + 2p_3)(4p_2p_3 + p_1p_2^2 + 2p_1p_4 + 8p_5) - \frac{1}{64}(p_2^4 + 4p_4^2 + 4p_2^2p_4) \\
&= \frac{1}{384}(4p_1p_2^2p_3 + p_1^2p_2^3 + 2p_1^2p_2p_4 + 8p_1p_2p_5 + 8p_2p_3^2 + 2p_1p_2^2p_3 + 4p_1p_3p_4 \\
&\qquad + 16p_3p_5) - \frac{1}{64}p_2^4 - \frac{1}{16}p_4^2 - \frac{1}{16}p_2^2p_4 \\
&= \frac{1}{384}p_1^2p_2^3 + \frac{1}{64}p_1p_2^2p_3 + \frac{1}{48}p_2p_3^2 + \frac{1}{192}p_1^2p_2p_4 + \frac{1}{96}p_1p_4p_3 + \frac{1}{48}p_1p_2p_5 \\
&\qquad + \frac{1}{24}p_3p_5 - \frac{1}{64}p_2^4 - \frac{1}{16}p_4^2 - \frac{1}{16}p_2^2p_4 \\
&= \frac{1}{64}p_2^3\left(\frac{1}{6}p_1^2 - p_2\right) + \frac{1}{16}p_2^2\left(\frac{1}{4}p_1p_3 - p_4\right) + \frac{1}{24}p_3\left(\frac{1}{2}p_2p_3 + p_5\right) \\
&\quad + \frac{1}{16}p_4\left(\frac{1}{6}p_1p_3 - p_4\right) + \frac{1}{48}p_1p_2\left(\frac{1}{4}p_1p_4 + p_5\right).
\end{aligned}$$

By triangle inequality, we get

$$\begin{aligned}
|a_4a_6 - a_5^2| \le \frac{1}{64}|p_2^3|\left|\frac{1}{6}p_1^2 - p_2\right| + \frac{1}{16}|p_2^2|\left|\frac{1}{4}p_1p_3 - p_4\right| + \frac{1}{24}|p_3|\left|\frac{1}{2}p_2p_3 + p_5\right| \\
+ \frac{1}{16}|p_4|\left|\frac{1}{6}p_1p_3 - p_4\right| + \frac{1}{48}|p_1||p_2|\left|\frac{1}{4}p_1p_4 + p_5\right|.
\end{aligned}$$

Using Lemma 2.4 and the inequality $|p_n| \leq 2$, we get

$$|a_4a_6 - a_5^2| \leq \frac{19}{12}. \tag{4.6}$$

Again, on substituting the values of a_3, a_4, a_5 and a_6 from (3.2), (3.3), (3.4) and (3.5), respectively in the expression $a_3a_6 - a_4a_5$, we have

$$\begin{aligned} a_3a_6 - a_4a_5 &= \frac{1}{96}(4p_2^2p_3 + p_1p_2^3 + 2p_1p_2p_4 + 8p_2p_5) \\ &\quad - \frac{1}{64}(p_1p_2^3 + 2p_1p_2p_4 + 2p_2^2p_4 + 2p_2^2p_3 + 4p_3p_4) \\ &= \frac{1}{96}p_2^2p_3 - \frac{1}{192}p_1p_2^3 - \frac{1}{96}p_1p_2p_4 + \frac{1}{12}p_2p_5 - \frac{1}{16}p_3p_4. \end{aligned}$$

So that

$$|a_3a_6 - a_4a_5| \leq \frac{1}{16}|p_3|\left|\frac{1}{12}p_2^2 - p_4\right| + \frac{1}{192}|p_2^2||p_1p_2 - p_3| + \frac{1}{12}|p_2|\left|\frac{1}{8}p_1p_4 - p_5\right|.$$

By Lemma 2.4 and the fact $|p_n| \leq 2$, we have

$$|a_3a_6 - a_4a_5| \leq \frac{5}{8}. \tag{4.7}$$

On substituting the values of a_3, a_4 and a_5 from (3.2), (3.3) and (3.4), respectively in the expression $a_3a_5 - a_4^2$, we have

$$a_3a_5 - a_4^2 = -\frac{1}{16}p_2^2\left(\frac{1}{4}p_1^2 - p_2\right) - \frac{1}{8}p_2\left(\frac{1}{2}p_1p_3 - p_4\right) - \frac{1}{16}p_3^2$$

so that

$$|a_3a_5 - a_4^2| \leq \frac{1}{16}|p_2|^2\left|\frac{1}{4}p_1^2 - p_2\right| + \frac{1}{8}|p_2|\left|\frac{1}{2}p_1p_3 - p_4\right| + \frac{1}{16}|p_3|^2.$$

Using Lemma 2.4 and the inequality $|p_n| \leq 2$, we have

$$|a_3a_5 - a_4^2| \leq \frac{5}{4}. \tag{4.8}$$

It follows from (1.2) that

$$|H_3(2)(f)| \leq |a_2||a_4a_6 - a_5^2| + |a_3||a_3a_6 - a_4a_5| + |a_4||a_3a_5 - a_4^2|.$$

Using inequality (4.6), (4.7) and (4.8) and Lemma 2.3, we have $|H_3(2)(f)| \leq 83/24 \simeq 3.45$. □

In the next theorem we estimate third order Hankel determinant $H_3(3)$ for $f \in \mathcal{S}^*_{\mathcal{S}}$.

Theorem 4.4. *If* $f(z) = z + \sum_{n=2}^{\infty} a_nz^n \in \mathcal{S}^*_{\mathcal{S}}$, *then*

$$|H_3(3)(f)| \leq \frac{89}{24} \simeq 3.7.$$

Proof. On substituting (3.4),(3.5) and (3.6), we have

$$\begin{aligned} a_5a_7 - a_6^2 &= \frac{1}{384}(p_2^5 + 6p_2^3p_4 + 8p_2^2p_6 + 2p_2^3p_4 + 12p_2p_4^2 + 16p_4p_6) - \frac{1}{2304}(16p_2^2p_3^2 \\ &\quad + p_1^2p_2^4 + 4p_1^2p_4^2 + 64p_5^2 + 8p_1p_2^3p_3 + 16p_1p_2p_3p_4 + 64p_2p_3p_5 + 4p_1^2p_2^2p_4 \\ &\quad + 16p_1p_2^2p_5 + 32p_1p_4p_5) \\ &= \frac{1}{96}p_2^2\left(p_6 - \frac{2}{3}p_3^2\right) + \frac{1}{384}p_2^4\left(p_2 - \frac{1}{6}p_1^2\right) + \frac{1}{192}p_4^2\left(p_2 - \frac{1}{3}p_1^2\right) \\ &\quad + \frac{1}{64}p_2^3\left(p_4 - \frac{2}{9}p_1p_3\right) + \frac{5}{192}p_2p_4\left(p_4 - \frac{4}{15}p_1p_3\right) + \frac{1}{192}p_2^2p_4 \\ &\quad \left(p_2 - \frac{1}{3}p_1^2\right) + \frac{1}{24}p_4\left(p_6 - \frac{1}{3}p_1p_5\right) + \frac{1}{96}p_2^2\left(p_6 - \frac{2}{3}p_1p_5\right) - \frac{1}{36}p_5^2 \\ &\quad - \frac{1}{36}p_2p_3p_5. \end{aligned}$$

Using triangle inequality, Lemma 2.4 and the fact $|p_n| \leq 2$, we get

$$|a_5a_7 - a_6^2| \leq \frac{4}{3}. \tag{4.9}$$

Again in view of (3.3),(3.4),(3.5) and (3.6), we have

$$\begin{aligned} a_4a_7 - a_6a_5 &= \frac{1}{384}(p_1p_2^4 + 6p_1p_2^2p_4 + 8p_1p_2p_6 + 2p_2^3p_3 + 12p_2p_3p_4 + 16p_3p_6) \\ &\quad - \frac{1}{384}(4p_2^3p_3 + 8p_2p_3p_4 + p_1p_2^4 + 4p_1p_2^2p_4 + 4p_1p_4^2 + 8p_2^2p_5 + 16p_4p_5) \\ &= \frac{1}{48}p_2^2\left(\frac{1}{4}p_1p_4 - p_5\right) + \frac{1}{24}p_4\left(\frac{1}{8}p_2p_3 - p_5\right) + \frac{1}{192}p_2p_3(p_4 - p_2^2) \\ &\quad + \frac{1}{24}p_6\left(\frac{1}{2}p_1p_2 + p_3\right) + \frac{1}{96}p_1p_4^2. \end{aligned}$$

Using triangle inequality, by Lemma 2.4 and $|p_n| \leq 2$, we have

$$|a_4a_7 - a_6a_5| \leq \frac{19}{24}. \tag{4.10}$$

It follows from (1.3) that

$$|H_3(3)| \leq |a_3||a_5a_7 - a_6^2| + |a_4||a_4a_7 - a_6a_5| + |a_5||a_4a_6 - a_5^2|.$$

Using (4.6),(4.9), and (4.10) and Lemma 2.3, we have $|H_3(3)(f)| \leq 89/24 \simeq 3.7$. □

Next we compute an estimate on the fourth Hankel determinant $H_4(1)$.

Theorem 4.5. *Let $f \in \mathcal{S}_{\mathcal{S}}^*$ be of the form $f(z) = z + a_2z^2 + a_3z^3 + \cdots$. Then*

$$|H_4(1)(f)| \leq 1.84.$$

Proof. Since $f \in \mathcal{S}^*_{\mathcal{S}}$, then in view of (1.4), (3.1), (3.2), (3.3), (3.4),(3.5) and (3.6), we get

$$\begin{aligned}
36864H_4(1)(f) &= p_1^4(p_2^2-4p_4)^2 + 8p_1^3(p_2^2-4p_4)(p_2p_3-4p_5)\\
&- 32p_1(2p_2^4p_3 - 2p_2^2p_3p_4 - 2p_2^3p_5 + 12p_3(-2p_4^2+p_3p_5)\\
&+ p_2(-3p_3^3 + 20p_4p_5 - 12p_3p_6)) + 8(3p_2^6 - 6p_2^4p_4 + 4p_2^2(9p_4^2+20p_3p_5)\\
&- 4p_2^3(p_3^2+12p_6) + 6(3p_3^4 - 12p_4^3 + 16p_3p_4p_5 - 8p_3^2p_6)\\
&- 32p_2(3p_3^2p_4 + 2p_5^2 - 3p_4p_6)) - 8p_1^2(p_2^5 - 8p_2^3p_4\\
&+ 16p_2(p_4^2+p_3p_5) - p_2^2(5p_3^2+12p_6) + 4(3p_3^2p_4 - 8p_5^2 + 12p_4p_6)).
\end{aligned}$$

$$\begin{aligned}
36864H_4(1)(f) &= p_1^4p_2^4 - 8p_1^2p_2^5 + 24p_2^6 + 8p_1^3p_2^3p_3 - 64p_1p_2^4p_3 + 40p_1^2p_2^2p_3^2\\
&- 32p_2^3p_3^2 + 96p_1p_2p_3^3 + 144p_3^4 - 8p_1^4p_2^2p_4 + 64p_1^2p_2^3p_4 - 48p_2^4p_4\\
&- 32p_1^3p_2p_3p_4 + 64p_1p_2^2p_3p_4 - 96p_1^2p_3^2p_4 - 768p_2p_3^2p_4 + 16p_1^4p_4^2\\
&- 128p_1^2p_2p_4^2 + 288p_2^2p_4^2 + 768p_1p_3p_4^2 - 576p_4^3 - 32p_1^3p_2^2p_5 + 64p_1p_2^3p_5\\
&- 128p_1^2p_2p_3p_5 + 640p_2^2p_3p_5 - 384p_1p_3^2p_5 + 128p_1^3p_4p_5\\
&- 640p_1p_2p_4p_5 + 768p_3p_4p_5 + 256p_1^2p_5^2 - 512p_2p_5^2 + 96p_1^2p_2^2p_6\\
&- 384p_2^3p_6 + 384p_1p_2p_3p_6 - 384p_3^2p_6 - 384p_1^2p_4p_6 + 768p_2p_4p_6.
\end{aligned}$$

A simple calculation gives

$$\begin{aligned}
36864H_4(1)(f) &= 8p_1^4p_2^2\left(\frac{1}{8}p_2^2-p_4\right) + \frac{1}{2}p_1^3\left(\frac{1}{4}p_2^2-p_4\right)\left(\frac{1}{4}p_2p_3-p_5\right)\\
&- 64p_1^2p_2^3\left(\frac{1}{8}p_2^2-p_4\right) - 64p_1p_2^2p_3(p_2^2-p_4) + 32p_2^2p_3^2\left(\frac{5}{4}p_1^2-p_2\right)\\
&+ 48p_2^4\left(\frac{1}{2}p_2^2-p_4\right) - 96p_1p_3p_4(p_1p_3-p_4) + 576p_4^2(p_1p_3-p_4)\\
&- 768p_3p_4(p_2p_3-p_5) - 96p_1p_4^2(p_1p_2-p_3) - 640p_2p_3p_5\left(\frac{1}{5}p_1^2-p_2\right)\\
&- 640p_2p_4(p_1p_5-p_6) + 384p_3p_6(p_1p_2-p_3) - 128p_4p_6(3p_1^2-p_2)\\
&+ 512p_5^2\left(\frac{1}{2}p_1^2-p_2\right) + 192p_2^2p_6\left(\frac{1}{2}p_1^2-p_2\right) + 192p_2^3\left(\frac{1}{3}p_1p_5-p_6\right)\\
&- 384p_1p_3^2\left(\frac{1}{8}p_2p_3-p_5\right) - 288p_2p_4^2\left(\frac{1}{9}p_1^2-p_2\right)\\
&- 144p_3^3\left(-\frac{1}{3}p_1p_2-p_3\right)
\end{aligned}$$

which implies

$$\begin{aligned}36864|H_4(1)(f)| \leq\ & 8p_1^4p_2^2\left|\frac{1}{8}p_2^2-p_4\right|+\frac{1}{2}p_1^3\left|\frac{1}{4}p_2^2-p_4\right|\left|\frac{1}{4}p_2p_3-p_5\right| \\ & +64p_1^2p_2^3\left|\frac{1}{8}p_2^2-p_4\right|+64p_1p_2^2p_3|p_2^2-p_4|+32p_2^2p_3^2\left|\frac{5}{4}p_1^2-p_2\right| \\ & +48p_2^4\left|\frac{1}{2}p_2^2-p_4\right|+96p_1p_3p_4|p_1p_3-p_4|+576p_4^2|p_1p_3-p_4| \\ & +768p_3p_4|p_2p_3-p_5|+96p_1p_4^2|p_1p_2-p_3|+640p_2p_3p_5\left|\frac{1}{5}p_1^2-p_2\right| \\ & +640p_2p_4|p_1p_5-p_6|+384p_3p_6|p_1p_2-p_3|+128p_4p_6|3p_1^2-p_2| \\ & +512p_5^2\left|\frac{1}{2}p_1^2-p_2\right|+192p_2^2p_6\left|\frac{1}{2}p_1^2-p_2\right|+192p_2^3\left|\frac{1}{3}p_1p_5-p_6\right| \\ & +384p_1p_3^2\left|\frac{1}{8}p_2p_3-p_5\right|+288p_2p_4^2\left|\frac{1}{9}p_1^2-p_2\right|+144p_3^3\left|-\frac{1}{3}p_1p_2-p_3\right|.\end{aligned}$$

Using Lemma 2.4 and the fact $|p_n| \leq 2$, we get

$$|H_4(1)(f)| \leq \frac{4241}{2304} \simeq 1.84.$$

Thus, we have the required bound for $|H_4(1)(f)|$. □

5. Toeplitz determinants

In this section, we first compute the bound on second Toeplitz determinant $T_2(2)$.

Theorem 5.1. *If $f \in \mathcal{S}^*_{\mathcal{S}}$ be of the form $f(z) = z + a_2z^2 + a_3z^3 + \cdots$. Then*

$$|T_2(2)(f)| \leq 2.$$

The inequality is sharp.

Proof. Since $f \in \mathcal{S}^*_{\mathcal{S}}$, then on putting the values of a_2 and a_3 from (3.1) and (3.2) in expression $T_2(2) = a_3^2 - a_2^2$, we get

$$|T_2(2)| = |a_3^2 - a_2^2| = \left|\frac{p_2^2}{4} - \frac{p_1^2}{4}\right|.$$

Applying triangle inequality and using the fact $|p_n| \leq 2$, we get

$$|T_2(2)| \leq 2.$$

The inequality is sharp for the function $f : \mathbb{D} \to \mathbb{C}$ defined as

$$f(z) = \frac{z}{1 - iz}.$$

It is noted that $a_2 = i, a_3 = -1$ and thus $|a_3^2 - a_2^2| = 2$. □

Next, we obtain an estimate for second Toeplitz determinant $T_2(3)$.

Theorem 5.2. *Let $f \in \mathcal{S}^*_{\mathcal{S}}$ be of the form $f(z) = z + a_2z^2 + a_3z^3 + \cdots$. Then*

$$|T_2(3)(f)| \leq 2.$$

The inequality is sharp.

Proof. For $f \in \mathcal{S}^*_{\mathcal{S}}$, then on putting the values of a_3 and a_4 from (3.2) and (3.3) in expression $T_2(3) = a_4^2 - a_3^2$, we get

$$\begin{aligned}|T_2(3)| = |a_4^2 - a_3^2| &= \left|\frac{1}{64}(p_1p_2 + 2p_3)^2 - \frac{p_2^2}{4}\right| \\ &= \left|\frac{1}{16}\left(-\frac{1}{2}p_1p_2 - p_3\right)^2 - \frac{p_2^2}{4}\right|.\end{aligned}$$

Applying triangle inequality, Lemma 2.4 and the fact $|p_n| \leq 2$, we get

$$|T_2(3)| \leq 2.$$

To prove the sharpness, consider the function $f : \mathbb{D} \to \mathbb{C}$ defined as

$$f(z) = \frac{z}{1 - iz}.$$

Here $a_3 = -1$ and $a_4 = -i$ and thus $|a_4^2 - a_3^2| = 2$. □

In the next theorem we obtain an estimate for the bound on third Toeplitz determinant $T_3(1)$.

Theorem 5.3. *If $f \in \mathcal{S}^*_{\mathcal{S}}$ be of the form $f(z) = z + a_2z^2 + a_3z^3 + \cdots$. Then*

$$|T_3(1)(f)| \leq 4.$$

The inequality is sharp.

Proof. Let $f \in \mathcal{S}^*_{\mathcal{S}}$. Then in view of (1.5), (3.1) and (3.2), we get

$$\begin{aligned}|T_3(1)| = |1 + 2a_2^2(a_3 - 1) - a_3^2| &= \left|1 + 2\frac{p_1^2}{4}\left(\frac{p_2}{2} - 1\right) - \left(\frac{p_2^2}{4}\right)\right| \\ &= \frac{1}{4}\left|4 + p_1^2p_2 - 2p_1^2 - p_2^2\right| \\ &= \frac{1}{4}\left|4 + p_2(p_1^2 - p_2) - 2p_1^2\right|.\end{aligned}$$

Using triangle inequality, we obtain

$$|T_3(1)| \leq \frac{1}{4}(4 + |p_2||p_1^2 - p_2| + 2|p_1^2|).$$

Applying Lemma 2.4 and using the fact that $|p_n| \leq 2$, we get

$$|T_3(1)| \leq 4.$$

For the function $f(z) = \dfrac{z}{1 - iz}$, we have $a_2 = i$ and $a_3 = -1$. Thus, we get

$$|1 + 2a_2^2(a_3 - 1) - a_3^2| = 4.$$

This proves the sharpness of the result. □

Next we compute the bound on third Toeplitz determinant $T_3(2)$.

Theorem 5.4. *Let $f \in \mathcal{S}^*_{\mathcal{S}}$ be of the form $f(z) = z + a_2z^2 + a_3z^3 + \cdots$. Then*

$$|T_3(2)(f)| = |(a_2 - a_4)(a_2^2 - 2a_2^3 + a_2a_4)| \leq 6.$$

Proof. In view of (3.1) and (3.3), we get

$$\begin{aligned}
&|T_3(2)(f)| \\
&= |(a_2 - a_4)(a_2^2 - 2a_2^3 + a_2a_4)| \\
&= \left|\left(\frac{p_1}{2} - \frac{1}{8}(p_1p_2 + 2p_3)\right)\left(\frac{p_1^2}{4} - \frac{p_1^3}{4} + \frac{p_1}{16}(p_1p_2 + 2p_3)\right)\right| \\
&= \left|\frac{p_1^3}{8} - \frac{p_1^4}{8} + \frac{1}{32}p_1^4p_2 - \frac{1}{128}p_1^3p_2^2 + \frac{1}{16}p_1^3p_3 - \frac{1}{32}p_1^2p_2p_3 - \frac{1}{32}p_1p_3^2\right| \\
&= \left|\frac{p_1^3}{8} - \frac{p_1^4}{8} - \frac{1}{16}p_1^3\left(-\frac{1}{2}p_1p_2 - p_3\right) + \frac{1}{32}p_1^2p_2\left(-\frac{1}{4}p_1p_2 - p_3\right) - \frac{1}{32}p_1p_3^2\right|.
\end{aligned}$$

Using triangle inequality, we obtain

$$\begin{aligned}
|T_3(2)(f)| \leq{}& \frac{1}{8}|p_1|^3 + \frac{1}{8}|p_1|^4 + \frac{1}{16}|p_1|^3\left|-\frac{1}{2}p_1p_2 - p_3\right| + \frac{1}{32}|p_1||p_3|^2 \\
&+ \frac{1}{32}|p_1|^2|p_2|\left|-\frac{1}{4}p_1p_2 - p_3\right|.
\end{aligned}$$

By using Lemma 2.4 and the inequality $|p_n| \leq 2$, we get $|T_3(2)(f)| \leq 6$. □

The following theorem gives an estimate on fourth Toeplitz deteminant $T_4(2)$.

Theorem 5.5. *Let $f \in \mathcal{S}^*_{\mathcal{S}}$ be of the form $f(z) = z + a_2z^2 + a_3z^3 + \cdots$. Then*

$$\begin{aligned}
|T_4(2)(f)| ={}&|(a_2^2 - a_3^2)^2 + 2(a_3^2 - a_2a_4)(a_2a_4 - a_3a_5) - (a_2a_3 - a_3a_4)^2 \\
&+ (a_4^2 - a_3a_5)^2 - (a_3a_4 - a_2a_5)^2| \leq 15.12.
\end{aligned}$$

Proof. In view of (1.6), (3.1), (3.2), (3.3) and (3.4) and on rearranging the terms, we have

$$\begin{aligned}
&4096T_4(2)(f)\\
&= 256(p_1^2-p_2^2)^2 - 16p_2^2(p_1(-4+p_2)+2p_3)^2 - 64(p_2p_3-p_1p_4)^2\\
&\quad + ((p_1p_2+2p_3)^2 - 4p_2(p_2^2+2p_4))^2 - 32(p_1^2p_2-4p_2^2+2p_1p_3)\\
&\quad (p_1^2p_2+2p_1p_3-p_2(p_2^2+2p_4))\\
&= 256p_1^4 - 768p_1^2p_2^2 - 32p_1^4p_2^2 + 256p_1^2p_2^3 + 256p_2^4 + 16p_1^2p_2^4 + p_1^4p_2^4\\
&\quad - 128p_2^5 - 8p_1^2p_2^5 + 16p_2^6 - 128p_1^3p_2p_3 + 512p_1p_2^2p_3 + 8p_1^3p_2^3p_3\\
&\quad - 32p_1p_2^4p_3 - 128p_1^2p_3^2 - 128p_2^2p_3^2 + 24p_1^2p_2^2p_3^2 - 32p_2^3p_3^2 + 32p_1p_2p_3^3\\
&\quad + 16p_3^4 + 64p_1^2p_2^2p_4 - 256p_2^3p_4 - 16p_1^2p_2^3p_4 + 64p_2^4p_4 + 256p_1p_2p_3p_4\\
&\quad - 64p_1p_2^2p_3p_4 - 64p_2p_3^2p_4 - 64p_1^2p_4^2 + 64p_2^2p_4^2\\
&= -256p_1^2p_2^2\left(\frac{1}{8}p_1^2-p_2\right) + 128p_2^4\left(\frac{1}{8}p_1^2-p_2\right) - 16p_2^5\left(\frac{1}{2}p_1^2-p_2\right)\\
&\quad - 512p_1p_2p_3\left(\frac{1}{4}p_1^2-p_2\right) - 64p_2^4\left(\frac{1}{2}p_1p_3-p_4\right) + 32p_2^2p_3^2\left(\frac{3}{4}p_1^2-p_2\right)\\
&\quad + 16p_1^2p_2^3\left(\frac{1}{2}p_1p_3-p_4\right) + 64p_2p_3^2\left(\frac{1}{2}p_1p_3-p_4\right) + 64p_1^2p_4(p_2^2-p_4)\\
&\quad - 64p_2^2p_4(p_1p_3-p_4) + 256p_1^4 - 768p_1^2p_2^2 + 256p_2^4 + p_1^4p_2^4 - 128p_1^2p_3^2\\
&\quad - 128p_2^2p_3^2 + 16p_3^4 - 256p_2^3p_4 + 256p_1p_2p_3p_4.
\end{aligned}$$

Using triangle inequality, we get

$$\begin{aligned}
4096|T_4(2)(f)| &\le 256|p_1|^2|p_2|^2\left|\frac{1}{8}p_1^2-p_2\right| + 128|p_2|^4\left|\frac{1}{8}p_1^2-p_2\right|\\
&\quad + 16|p_2|^5\left|\frac{1}{2}p_1^2-p_2\right| + 512|p_1||p_2||p_3|\left|\frac{1}{4}p_1^2-p_2\right| + 64|p_2|^4\left|\frac{1}{2}p_1p_3-p_4\right|\\
&\quad + 32|p_2|^2|p_3|^2\left|\frac{3}{4}p_1^2-p_2\right| + 16|p_1|^2|p_2|^3\left|\frac{1}{2}p_1p_3-p_4\right|\\
&\quad + 64|p_2||p_3|^2\left|\frac{1}{2}p_1p_3-p_4\right| + 64|p_1|^2|p_4||p_2^2-p_4|\\
&\quad + 64|p_2|^2|p_4||p_1p_3-p_4| + 256|p_1|^4 + 768|p_1|^2|p_2|^2 + 256|p_2|^4\\
&\quad + |p_1|^4|p_2|^4 + 128|p_1|^2|p_3|^2 + 128|p_2|^2|p_3|^2 + 16|p_3|^4 + 256|p_2|^3|p_4|\\
&\quad + 256|p_1||p_2||p_3||p_4|.
\end{aligned}$$

Applying Lemma 2.4 and the fact that $|p_n| \le 2$, we get $|T_4(2)(f)| \le 15.12$. □

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