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Some aspects of a coupled system of nonlinear integral equations

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Abstract. In the present work we take a system of two integral equations and prove the existence and uniqueness of their solution. We investigate four aspects of the problem, namely, error estimation and rate of convergence of the iteration leading to the solution, Ulam-Hyers stability, well-posedness and data dependence of the solution sets. We give some new definitions pertaining to the system we analyze here. In order to establish our results we utilize the coupled contraction mapping principle due to Bhaskar and Lakshmikantham (Nonlinear Anal. TMA 65(2006), 1379-1393) and several related results which we deduce here.

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1. Introduction

In this paper, we consider a system of two coupled nonlinear Fredholm type integral equations. Coupled integral equations are of great practical value. Some examples of works are [3], [11], [12] and [20] where they have been applied to contact problems, magnetostatic problems, solidification problems and scattering of nucleons.

Problem I. The problem is to solve the coupled system of nonlinear equations

$$\left.\begin{array}{rcll} u(t) & = & g(t) + \lambda \int_a^b K(t,\ s)\, \mathfrak{h}(s,\ u(s),\ v(s))\, ds & \text{and} \\ v(t) & = & g(t) + \lambda \int_a^b K(t,\ s)\, \mathfrak{h}(s,\ v(s),\ u(s))\, ds, & \lambda \geq 0, \end{array}\right\} \tag{1.1}$$

for all $t \in [a,\ b]$ under some appropriate conditions on $g, \mathfrak{h}$ and K.

The organization of our work is following.

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• First we describe the coupled contraction mapping theorem of Bhaskar et al. [13]. This result is pivotal to our study here.

• In section 3, we solve Problem I under certain conditions. We also establish that this solution is unique if we take some extra assumptions.

• In section 4, we study the rate of convergence and error estimation for the iteration obtained in section 3.

• In section 5, we discuss the Ulam-Hyers stability of the problem. It is a stability concept of general character which is applicable to diverse domains of mathematics. The essence of the stability is to see whether a mathematical object having approximate behaviour of a given class of objects can actually be approximated by a member of that class.

• In section 6, we investigate the well-posedness aspect of the problem.

• In section 7, we obtain a data dependence result for the solution of the problem.

• In both sections 6 and 7, we offer new definitions pertaining to the problem.

In our analysis we consider the structure of partial order on a metric space.

2. Review of coupled fixed point result of Bhaskar et al.[13]

Here we review a coupled fixed point result due to Bhaskar et al. [13]. This result is instrumental to establishing our results in the following sections of the paper.

Although coupled fixed point was introduced by Guo et al. [14] some time back in 1987, it was only after Bhaskar et al. [13] produced their result in 2006, there have been wide spread interest in this subject. Some prominent references on this topic, amongst others, are [2, 7, 8, 16]. Fixed point method is well known in several areas of mathematics. Coupled fixed point theorems have also been used to solve several problems of mathematics like these discussed in [14, 15]. In the present paper we derive results by use of such methodologies.

In the paper, the notation X^2 stands for $X \times X$ and the notation $(X, d, \preceq)$ stands for a partially ordered metric space.

A coupled fixed point of a mapping $\mathfrak{F} : X^2 \to X$ is an element $(s, t) \in X^2$ satisfying $s = \mathfrak{F}(s, t)$ and $t = \mathfrak{F}(t, s)$.

Problem P. Let $(X, d, \preceq)$ be a metric space with a partial order. The problem is to find a coupled fixed point of a mapping $\mathfrak{F} : X^2 \to X$ under suitable conditions.

Definition 2.1 ([13])**.** A mapping $\mathfrak{F} : X^2 \to X$, where $(X, \preceq)$ is a partially order set, is called mixed monotonic if for any $u, v \in X$,

$$t_1, t_2 \in X,\ t_1 \preceq t_2 \text{ implies } \mathfrak{F}(t_1, v) \preceq \mathfrak{F}(t_2, v)$$

and

$$s_1, s_2 \in X,\ s_1 \preceq s_2 \text{ implies } \mathfrak{F}(u, s_2) \preceq \mathfrak{F}(u, s_1).$$

Starting with $(X, \preceq)$ we define a partial order " $\leq$ " on the product space X^2 as follows: for $(s, t), (u, v) \in X^2, (u, v) \leq (s, t) \Leftrightarrow u \preceq s$ and $t \preceq v$.

Definition 2.2 ([13])**.** A partially ordered metric space $(X, d, \preceq)$ is regular if

(i) $x_n \preceq t$, for all n, whenever $\{x_n\}$ is any nondecreasing sequence converging to t;

(ii) $t \preceq x_n$, for all n, whenever $\{x_n\}$ is any nonincreasing sequence converging to t.

Theorem 2.3 ([13])**.** *Let $(X, d, \preceq)$ be a complete metric space with a partial ordered having regular property. Let $\mathfrak{F} : X^2 \to X$ be a mixed monotonic function such that for all $(s,t),\ (u,v) \in X^2$ with $u \preceq s,\ t \preceq v$,*

$$d(\mathfrak{F}(s,t),\ \mathfrak{F}(u,v)) \leq \frac{\xi}{2}\ [d(s,\ u) + d(t,\ v)],\ \text{where } \xi \in [0,1). \tag{2.1}$$

If there exist $x_0, y_0 \in X$ satisfying $x_0 \preceq \mathfrak{F}(x_0, y_0)$ and $\mathfrak{F}(y_0, x_0) \preceq y_0$, then the sequence $\{(x_n, y_n)\}$ obtained for all $n \geq 1$ as

$$x_n = \mathfrak{F}(x_{n-1},\ y_{n-1}) = \mathfrak{F}^n(x_0,\ y_0) \quad \text{and} \quad y_n = \mathfrak{F}(y_{n-1}, x_{n-1}) = \mathfrak{F}^n(y_0,\ x_0) \tag{2.2}$$

converges to a coupled fixed point $(x,\ y)$ of $\mathfrak{F}$, that is, $x_n \to x$ and $y_n \to y$ with $x = \mathfrak{F}(x,y)$ and $y = \mathfrak{F}(y,x)$.

Theorem 2.4 ([13])**.** *The coupled fixed point is unique in Theorem 2.3 if it is further assumed that for every $(x_1, y_1),\ (x_2, y_2) \in X^2$ there exists an element $(x_3, y_3) \in X^2$ which is comparable to both (x_1, y_1) and (x_2, y_2).*

3. Existence and uniqueness of solution of Problem I

In this section we deal with system of nonlinear integral equations and we apply Theorem 2.4 ([13]) to establish the existence and uniqueness of solution of the system in a complete metric space. The system (1.1) will be considered under some suitable conditions.

In this section, we present our main finding, we take help of the coupled results discussed in previous section to prove existence of the unique solution of (1.1).

We take the coupled system of nonlinear integral equations

$$\left.\begin{array}{rcl} x(t) &=& g(t) + \lambda\ \int_a^b K(t,\ s)\ \mathfrak{h}(s,\ x(s),\ y(s))\ ds \qquad \text{and} \\ y(t) &=& g(t) + \lambda\ \int_a^b K(t,\ s)\ \mathfrak{h}(s,\ y(s),\ x(s))\ ds, \quad \lambda \geq 0, \end{array}\right\}$$

where the unknown functions $x(t)$ and $y(t)$ are real valued and continuous on $[a,\ b]$. That is, we investigate the possibility of continuous solution of (1.1).

Consider the metric space $X = C[a,\ b]$, the space of all real valued continuous functions defined on $[a,b]$, endowed with the metric

$$d(x,\ y) = \sup_{t\in[a,\ b]} \mid x(t) - y(t) \mid . \tag{3.1}$$

Assume that this metric space is endowed with the following partial ordered relation $\preceq$. Let in X, the relation $x \preceq y$ holds if $x(t) \leq y(t)$, whenever $a \leq t \leq b$.

We designate the following assumptions by $I_1,\ I_2,\ I_3,\ I_4$ and I_5.

I_1: $g \in X$ and $\mathfrak{h} : [a,b] \times \mathbb{R} \times \mathbb{R} \to [0, \infty),\ K : [a,b] \times [a,b] \to [0, \infty)$ are continuous mappings.

I_2: For $x, y, u, v \in X$ and $s \in [a,b]$, $x \preceq u$ implies $\mathfrak{h}(s, x(s), y(s)) \leq \mathfrak{h}(s, u(s),\ y(s))$ and $y \preceq v$ implies $\mathfrak{h}(s, x(s), v(s)) \leq \mathfrak{h}(s, x(s), y(s))$.

I_3: $\mid \mathfrak{h}(s,x(s),y(s)) - \mathfrak{h}(s,u(s),v(s)) \mid \leq \mathcal{M}(x,y,u,v)$, for $(x,y),(u,v) \in X^2$ with $u \preceq x$ and $y \preceq v$, where

$$\mathcal{M}(x,\ y,\ u,\ v) = \sup_{s\in[a,\ b]} \frac{\mid x(s) - u(s) \mid + \mid y(s) - v(s) \mid}{2}.$$

I_4: $\mid K(t,\ s) \mid \leq m$ and $\xi = \lambda\ (b-a)\ m$ with $0 \leq \xi < 1$.

I_5: There exist $x_0, y_0 \in X$ satisfying the following two inequalities:

$$x_0(t) \leq g(t) + \lambda \int_a^b K(t,s)\ \mathfrak{h}(s,x_0(s),y_0(s))\ ds, \text{ for all } t \in [a,b]$$

and

$$g(t) + \lambda \int_a^b K(t,s)\ \mathfrak{h}(s,y_0(s),x_0(s))\ ds \leq y_0(t), \text{ for all } t \in [a,b].$$

Theorem 3.1. *Let $(X,\ d) = (C[a,\ b],\ d)$, $\mathfrak{h}$, g, $K(t,s)$ satisfy all the assumptions I_1, I_2, I_3, I_4 and I_5. Then the system of equations (1.1) has a unique solution $(x(t),\ y(t))$ in X^2 and there exist sequences $\{x_n\}$ and $\{y_n\}$ in X converging respectively to x and y uniformly in $[a,\ b]$.*

Proof. Define $\mathfrak{F} : X^2 \to X$ as

$$\mathfrak{F}(x,\ y)(t) = g(t) + \lambda \int_a^b K(t,\ s)\ \ \mathfrak{h}(s,\ x(s),\ y(s))\ ds, \quad \text{for all } \ a \leq t \leq b. \tag{3.2}$$

Take $x, y, u, v \in X$ with $x \preceq u$ and $y \preceq v$. By I_1, I_2, we obtain

$$\begin{aligned}\mathfrak{F}(x,y)(t) &= g(t) + \lambda \int_a^b K(t,s)\mathfrak{h}(s,x(s),y(s))ds \\ &\leq g(t) + \lambda \int_a^b K(t,s)\mathfrak{h}(s,u(s),y(s))ds = \mathfrak{F}(u,y)(t),\end{aligned}$$

$$\begin{aligned}\mathfrak{F}(x,y)(t) &= g(t) + \lambda \int_a^b K(t,s)\mathfrak{h}(s,x(s),y(s))ds \\ &\geq g(t) + \lambda \int_a^b K(t,s)\mathfrak{h}(s,x(s),v(s))ds = \mathfrak{F}(x,v)(t),\end{aligned}$$

that is, $\mathfrak{F}(x,y) \preceq \mathfrak{F}(u,y)$ and $\mathfrak{F}(x,v) \preceq \mathfrak{F}(x,y)$. Hence $\mathfrak{F}$ is a mixed monotonic mapping.

By assumptions I_1, I_3 and I_4, for all $(x,y),\ (u,v) \in X^2$ with $u \preceq x,\ y \preceq v$ and for all $a \leq t \leq b$, we get

$$|\mathfrak{F}(x,y)(t) - \mathfrak{F}(u,v)(t)| = \lambda \mid \int_a^b K(t,s)[\mathfrak{h}(s,x(s),y(s)) - \mathfrak{h}(s,u(s),v(s))]ds|$$

$$\leq \lambda \int_a^b m \ \mid [\mathfrak{h}(s,x(s),y(s)) - \mathfrak{h}(s,u(s),v(s))]ds \mid$$

$$\leq \lambda\ m \int_a^b \mathcal{M}(x,y,u,v)ds$$

$$= \lambda\ m \int_a^b \sup_{s\in[a,b]} \frac{|\ x(s)-u(s)\ |+|\ y(s)-v(s)\ |}{2} ds$$

$$\leq \lambda\ m\ \frac{[d(x,u)+d(y,v)]}{2} \int_a^b ds$$

$$= \lambda\ m\ (b-a)\ \frac{[d(x,u)+d(y,v)]}{2} = \frac{\xi}{2}\ [d(x,u)+d(y,v)],$$

that is,

$$d(\mathfrak{F}(x,y),\ \mathfrak{F}(u,v)) \leq \frac{\xi}{2}\ [d(x,u)+d(y,v)],$$

where $\xi = \lambda\ m\ (b-a)$ and $\xi \in [0,\ 1)$. From the definition of $\mathfrak{F}$ and the assumption I_5, we have $x_0, y_0 \in X$ satisfying $x_0 \preceq \mathfrak{F}(x_0, y_0)$ and $\mathfrak{F}(y_0, x_0) \preceq y_0$.

Let $\{x_n\}$ be a sequence in X such that $x_n \to x \in X$ as $n \to \infty$. If $\{x_n\}$ is nondecreasing then $x_n \preceq x_{n+1}$, for $n > 0$, that is, $x_n(s) \leq x_{n+1}(s)$, for all n and $s \in [a,\ b]$. Then $x_n(s) \leq x(s)$, for $n > 0$ and $s \in [a,b]$, that is, $x_n \preceq x$, for $n > 0$. If $\{x_n\}$ is nonincreasing then $x_{n+1} \preceq x_n$, for $n > 0$, that is, $x_{n+1}(s) \leq x_n(s)$, for $n > 0$ and $s \in [a,b]$. Then $x(s) \leq x_n(s)$, for $n > 0$ and $s \in [a,b]$, that is, $x \preceq x_n$, for $n > 0$. Therefore, X has regular property.

By application of Theorem 2.3, we get $x, y \in X$ satisfying

$$x(t) = \mathfrak{F}(x,\ y)(t) = g(t) + \lambda \int_a^b K(t,\ s)\ \mathfrak{h}(s,\ x(s),\ y(s))\ ds$$

and

$$y(t) = \mathfrak{F}(y,\ x)(t) = g(t) + \lambda \int_a^b K(t,\ s)\ \mathfrak{h}(s,\ y(s),\ x(s)) ds,$$

for all $t \in [a,b]$, and corresponding to (2.2) there exist two sequences $\{x_n\}$ and $\{y_n\}$ such that

$$\left.\begin{aligned} x_{n+1}(t) = \mathfrak{F}(x_n,\ y_n)(t) = g(t) + \lambda\ \textstyle\int_a^b K(t,\ s)\ \mathfrak{h}(s,\ x_n(s),\ y_n(s))\ ds, \\ y_{n+1}(t) = \mathfrak{F}(y_n,\ x_n)(t) = g(t) + \lambda\ \textstyle\int_a^b K(t,\ s)\ \mathfrak{h}(s,\ y_n(s),\ x_n(s))\ ds, \end{aligned}\right\} \quad (3.3)$$

and $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} y_n = y$ in X. Then

$$\sup_{t\in[a,\ b]} |\ x_n(t) - x(t)\ | \to 0 \text{ and } \sup_{t\in[a,\ b]} |\ y_n(t) - y(t)\ | \to 0, \text{ as } n \to \infty,$$

that is, $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} y_n = y$ uniformly on $[a,\ b]$, as $n \to \infty$.

Let $x, y \in X$. Define $z(t) = \max\ \{x(t),\ y(t)\}$ and $w(t) = \min\ \{x(t),\ y(t)\}$, for $t \in [a,b]$. Then $x \preceq z$, $y \preceq z$ and $w \preceq x$, $w \preceq y$. Therefore, for any $x, y \in X$, there exist z and $w \in X$ such that z is upper bound of x, y and w is lower bound of x, y.

By application of Theorem 2.4, we have that $(x(t),\ y(t))$ is the unique coupled fixed point of $\mathfrak{F}$, that is, $(x(t),\ y(t))$ is the unique solution of the system (1.1).

Example 3.2. Consider the metric space $X = C[0,\ 1]$ with the metric

$$d(x,\ y) = \sup_{t\in[0,\ 1]} |\ x(t) - y(t)\ |$$

and with a partial ordered relation $\preceq$ defined as $x \preceq y$ if and only if $x(t) \leq y(t)$, whenever $x, y \in X$ and $0 \leq t \leq 1$. Let $h : [0,1] \times \mathbb{R} \times \mathbb{R} \to [0, \infty)$, $K : [0,1] \times [1,0] \to [0, \infty)$, and $g \in X$ be defined respectively as follows:

$$h(s,\ u,\ v) = \begin{cases} \frac{u-v}{3}, & \text{if } \ u \geq v \\ 0, & \text{otherwise,} \end{cases}$$

$$K(x,\ y) = y, \text{ for } x, y \in [0,\ 1] \text{ and } \ g(t) = 0, \text{ for } t \in [0, 1].$$

Take $m = 1$ and $\lambda = \frac{1}{2}$. Let $x_0 = 0$ and $y_0 = c(> 0)$ be two points in X. Then all the conditions of Theorem 3.1 are satisfied and here $(x(t),\ y(t)) = (0,\ 0)$ is the unique solution of the system of equations (1.1). Consider the sequence $\{x_n\}$ and $\{y_n\}$, where $x_n = 0$ for all $n \geq 0$ and $y_0 = c$, $y_n = \frac{c}{3\ 4^n}$ for all $n \geq 1$. Here the sequences $\{x_n\}$ and $\{y_n\}$ in X converge respectively to $x = 0$ and $y = 0$ uniformly in $[0,\ 1]$.

4. Error estimation and rate of convergence

We investigate some aspects of the coupled fixed point problem considered by Bhaskar et al. [13] in this section. We make an error estimation of the coupled fixed point iteration which we construct in this paper. We also investigate the rate of convergence of the iteration process. Such considerations have appeared in the fixed point theory through works like [4].

We now study the rate at which the iteration method of finding the coupled fixed point of Problem P converges if the initial approximation of the coupled fixed point is sufficiently close to the desired coupled fixed point. For this purpose we first define the order of convergence of the Problem P.

Definition 4.1. Problem P is said to be of order r or has the rate of convergence r with respect to $\{(x_n, y_n)\}$ given by equation (2.2) if (i) $\mathfrak{F}$ admits a unique coupled fixed point (x, y), (ii) r is a positive real number for which there exists a finite fixed $C > 0$ for which $R_{n+1} \leq C\ (R_n)^r$, where $R_n = d(x, x_n) + d(y, y_n)$ is the error in n-th iterate and (x_n, y_n) is the n-th approximation of the coupled fixed point (x, y). The constant C is called the asymptotic error.

We study here the rate at which the iteration method of finding the solution of system of integral equations converges if the initial approximation of the solution of the system is sufficiently close to the desired solution of the system. For this purpose we define the order of convergence of the solution of system of integral equations.

Definition 4.2. Problem I is said to be of order r or has the rate of convergence r with respect to $\{(x_n, y_n)\}$ given by equation (3.3) if (i) the system of integral equations (1.1) has a unique solution (x, y), (ii) r is a positive real number for which there exists a finite fixed $C > 0$ for which $R_{n+1} \leq C\ (R_n)^r$, where $R_n = \sup_{s \in [a,\ b]}[\mid x(s) - x_n(s) \mid + \mid y(s) - y_n(s) \mid]$ is the error in n-th iterate and (x_n, y_n) is the n-th approximation of the solution (x, y) of the system of integral equations (1.1). The constant C is called the asymptotic error.

Theorem 4.3. *Let* $(x_0, y_0) \in X^2$ *be the initial approximation of the unique coupled fixed point* (x, y) *of* $\mathfrak{F}$ *in Theorem 2.4. Then* $R_{n+1} \leq \dfrac{\xi^{n+1}}{(1-\xi)}\ [d(x_1, x_0) + d(y_1, y_0)]$, *where* $R_n = d(x, x_n) + d(y, y_n)$ *is the error in n-th iterate and* (x_n, y_n) *is the n-th approximation of the coupled fixed point* (x, y).

Proof. Following the same techniques used in establishing Theorem 2.4 (see [13]), we have the sequence $\{(x_n, y_n)\}$ in X^2 given by equation (2.2). Also,

- $x_n \preceq x_{n+1}$ and $y_{n+1} \preceq y_n$, for all $n \geq 0$,
- both $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences in X and $\{(x_n,\ y_n)\}$ converges to a coupled fixed point of $\mathfrak{F}$ in X^2.

As, we consider that $(x,\ y)$ is the unique coupled fixed point of $\mathfrak{F}$, we have $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} y_n = y$. By equation (2.2) and the regularity assumption, $x_n \preceq x$ and $y \preceq y_n$, for $n \geq 0$. Using (2.1), we have

$$d(x,\ x_{n+1}) = d(\mathfrak{F}(x,\ y),\ \mathfrak{F}(x_n,\ y_n)) \leq \frac{\xi}{2}\ [d(x, x_n) + d(y, y_n)].$$

Similarly,

$$d(y, y_{n+1}) = d(\mathfrak{F}(y, x), \mathfrak{F}(y_n, x_n)) \leq \frac{\xi}{2}\ [d(x, x_n) + d(y, y_n)].$$

Therefore,

$$R_{n+1} = d(x,\ x_{n+1}) + d(y,\ y_{n+1}) \leq \xi\ [d(x_n, x) + d(y_n, y)] = \xi\ R_n. \tag{4.1}$$

Let,

$$r_n = d(x_n, x_{n+1}) + d(y_n, y_{n+1}).$$

It follows from (4.1) that

$$\begin{aligned} R_{n+1} &= d(x,\ x_{n+1}) + d(y,\ y_{n+1}) \leq \xi\ [d(x_n, x) + d(y_n, y)] \\ &\leq \xi\ [d(x_n, x_{n+1}) + d(y_n, y_{n+1}) + d(x_{n+1}, x) + d(y_{n+1}, y)] = \xi\ [R_{n+1} + r_n], \end{aligned} \tag{4.2}$$

which implies that

$$R_{n+1} \leq \frac{\xi}{(1-\xi)}\ r_n. \tag{4.3}$$

Using (2.2), we obtain

$$\begin{aligned} r_{n+1} &= d(x_{n+1}, x_{n+2}) + d(y_{n+1}, y_{n+2}) \\ &= d(\mathfrak{F}(x_n, y_n),\ \mathfrak{F}(x_{n+1}, y_{n+1})) + d(\mathfrak{F}(y_n, x_n),\ \mathfrak{F}(y_{n+1}, x_{n+1})) \\ &= d(\mathfrak{F}(x_{n+1}, y_{n+1}),\ \mathfrak{F}(x_n, y_n)) + d(\mathfrak{F}(y_n, x_n),\ \mathfrak{F}(y_{n+1}, x_{n+1})) \\ &\leq \frac{\xi}{2}\ [d(x_{n+1}, x_n) + d(y_{n+1}, y_n)] + \frac{\xi}{2}\ [d(y_n, y_{n+1}) + d(x_n, x_{n+1})] \\ &= \xi\ [d(x_n, x_{n+1}) + d(y_n, y_{n+1})] = \xi\ r_n. \end{aligned}$$

Applying (4.3) repeatedly and using the above inequality, we get

$$R_{n+1} \leq \frac{\xi^{n+1}}{(1-\xi)}\ r_0 = \frac{\xi^{n+1}}{(1-\xi)}\ [d(x_1,\ x_0) + d(y_1,\ y_0)]. \tag{4.4}$$

Remark 4.4. In general, the speed of the iteration depends on the value of ξ; the smaller is the value of ξ, the faster would be the convergence.

Remark 4.5. Above theorem shows that if $0 < \xi < 1$, the error in n-th iterate does not exceed $\frac{\xi^n}{1-\xi}\ [d(x_1,\ x_0) + d(y_1,\ y_0)]$. This error can be made less than a preassigned real number $\varepsilon > 0$ by taking $n \geq \max\left\{\left[\frac{\log(\frac{d(x_1,x_0)+d(y_1,y_0)}{\varepsilon\ (1-\xi)})}{\log\ (\frac{1}{\xi})}\right],\ \ 0\right\} + 1$, where $[y]$ denotes the greatest integer function. This gives the number of iterations n needed to bring the point $(x_n,\ y_n)$ within ε distance of the actual coupled fixed point.

Theorem 4.6. *Assume that the conditions of Theorem 3.1 are satisfied and R_n is the error at n^{th} stage of approximation of solution of the system (1.1). Then*

$$R_{n+1} \leq \frac{[\lambda\ (b-a)m]^{n+1}}{1-\lambda\ (b-a)m} \sup_{s\in[a,b]} [|\ x_1(s) - x_0(s)\ | + |\ y_1(s) - y_0(s)\ |],$$

where (x_0, y_0) is the initial approximation of the solution of (1.1),

$$x_1(t) = g(t) + \lambda \int_a^b K(t,s)\ \mathfrak{h}(s, x_0(s), y_0(s))ds,$$

$$y_1(t) = g(t) + \lambda \int_a^b K(t,s)\ \mathfrak{h}(s, y_0(s), x_0(s))ds$$

and m is given in I_4.

Proof. Define $\mathfrak{F} : C[a,\ b] \times C[a,\ b] \to C[a,\ b]$ as in (3.2). Applying Theorems 3.1 and 4.3, we have

$$\begin{aligned} R_{n+1} &\leq \frac{\xi^{n+1}}{(1-\xi)}\ \ [d(x_1,\ x_0) + d(y_1,\ y_0)] \\ &= \frac{[\lambda\ (b-a)\ m]^{n+1}}{[1-\lambda\ (b-a)\ m]} \sup_{s\in[a,\ b]} [\ |\ x_1(s) - x_0(s)\ | + |\ y_1(s) - y_0(s)\ |\ ]\ \ [\text{ by } I_4]. \end{aligned}$$

5. Ulam-Hyers stability

In present section, we investigate Ulam-Hyers stability of the above fixed point problem of coupled mapping. It is a type of stability which was initiated by a mathematical question by Ulam [27] and subsequent partial answer by Hyers [17] and Rassias [23]. The investigation of such stability has been of profound interest in various contexts of mathematics like functional equations, isometries [24], etc.

We consider the issue of stability of the afore-mentioned coupled fixed points. The kind of stability we consider is known as Hyers - Ulam stability which is also known as Hyers - Ulam - Rassias stability or H-U-S stability in contemporary literatures. It has its origin in the work of Ulam [27] and was extended by Hyers [17], Rassias [23] and many others. Its generality makes it applicable to a wide variety of domains like functional equations [9], isometries [24], group homomorphisms [18] and the like. In fixed point theory this kind of stability was considered in recent works like [5, 6, 19]. In [25] one can find the following definition as well as some related notions concerning the Ulam-Hyers stability which is relevant to the present consideration.

Let $S : M \to M$, where (M, d) be a metric space. We say that the fixed point problem $x = Sx$ is Ulam-Hyers stable if for each $\epsilon > 0$ and $y \in X$ satisfying $d(y, Sy) \leq \epsilon$ there exists $x_0 \in X$ for which $x_0 = Sx_0$ and $d(y, x_0) \leq \epsilon$. The essence of the problem of stability is to investigate the fact whether approximate fixed points are approximations of actual fixed points at the same level of accuracy as is evident from the above statement.

Definition 5.1 ([6]). Problem P is Ulam-Hyers stable if for each $\epsilon > 0$ and for each solution $(u^*,\ v^*) \in X^2$ of the inequalities $d(x,\ \mathfrak{F}(x,\ y)) \leq \epsilon$ and $d(y,\ \mathfrak{F}(y,\ x)) \leq \epsilon$ there exists a solution $(x^*,\ y^*) \in X^2$ of Problem P satisfying $\max\{d(u^*,\ x^*),\ d(v^*,\ y^*)\} \leq \phi(\epsilon)$, where $\phi : [0,\ \infty) \to [0,\ \infty)$ is monotone increasing and continuous at 0 with $\phi(0) = 0$.

Being inspired by above definition we give the following definition in case of the system of equations (1.1).

Definition 5.2. Coupled system of nonlinear equations (1.1) is called Ulam-Hyers stable if for each $\epsilon > 0$ and for each solution $(u^*,\ v^*)$ of the inequations

$$\left.\begin{array}{l} \sup_{t\in[a,b]} \mid x(t) - g(t) - \lambda\ \int_a^b K(t,\ s)\ \mathfrak{h}(s,\ x(s),\ y(s))\ ds \mid< \epsilon \ \text{ and} \\ \sup_{t\in[a,b]} \mid y(t) - g(t) - \lambda\ \int_a^b K(t,\ s)\ \mathfrak{h}(s,\ y(s),\ x(s))\ ds \mid< \epsilon,\ \lambda \geq 0, \end{array}\right\}$$

there exists a solution $(x^*,\ y^*)$ of (1.1) satisfying

$$\sup_{s\in[a,\ b]} \max\{\mid u^*(s) - x^*(s) \mid, \mid v^*(s) - y^*(s) \mid\} \leq \phi(\epsilon),$$

where $\phi : [0,\ \infty) \to [0,\ \infty)$ is monotone increasing and continuous at 0 with $\phi(0) = 0$.

We use the following assumption to assure the Ulam-Hyers stability of fixed point problem of mixed monotone mapping:

(A1): If $(x^*,\ y^*)$ be any solution of Problem P, then $x \preceq x^*,\ \ y^* \preceq y$, for any $(x,\ y) \in X^2$.

Theorem 5.3. *Problem P is Ulam-Hyers stable if the assumption (A1) is included in Theorem 2.4.*

Proof. By Theorem 2.4, $\mathfrak{F}$ has a unique coupled fixed point (x^*, y^*) (say). Therefore, (x^*, y^*) is a solution of Problem P. Let $\epsilon > 0$ and $(u^*, v^*) \in X^2$ be a solution of the inequalities $d(x, \mathfrak{F}(x,\ y)) \leq \epsilon$ and $d(y, \mathfrak{F}(y,\ x)) \leq \epsilon$. Then $d(u^*,\ \mathfrak{F}(u^*,\ v^*)) \leq \epsilon$ and $d(v^*,\ \mathfrak{F}(v^*,\ u^*)) \leq \epsilon$. By the assumption (A1), we have $u^* \preceq x^*,\ \ y^* \preceq v^*$. Using (2.1), we have

$$\begin{aligned} d(x^*, u^*) &= d(\mathfrak{F}(x^*, y^*), u^*) \leq d(\mathfrak{F}(x^*, y^*),\ \mathfrak{F}(u^*, v^*)) + d(\mathfrak{F}(u^*, v^*), u^*) \\ &\leq \frac{\xi}{2}\ [d(x^*,\ u^*) + d(y^*,\ v^*)] + \epsilon. \end{aligned}$$

Similarly, we have

$$d(y^*,\ v^*) \leq \frac{\xi}{2}\ [d(x^*,\ u^*) + d(y^*,\ v^*)] + \epsilon.$$

Therefore,

$$\begin{aligned} \max\ \{d(x^*,\ u^*),\ d(y^*,\ v^*)\} &\leq \frac{\xi}{2}\ [d(x^*,\ u^*) + d(y^*,\ v^*)] + \epsilon \\ &\leq \xi \max\ \{d(x^*,\ u^*),\ d(y^*,\ v^*)\} + \epsilon, \end{aligned}$$

which implies that

$$\max\ \{d(x^*, u^*),\ d(y^*, v^*)\} \leq \frac{\epsilon}{(1-\xi)}. \tag{5.1}$$

Define $\phi : [0,\ \infty) \to [0,\ \infty)$ as $\phi(t) = \dfrac{t}{(1-\xi)}$. Then

$$\max\ \{d(x^*, u^*),\ d(y^*, v^*)\} \leq \frac{\epsilon}{(1-\xi)} = \phi(\epsilon). \tag{5.2}$$

Since ϕ is monotone increasing, continuous at 0 with $\phi(0) = 0$. Therefore, Problem P is Ulam-Hyers stable.

Now we establish Ulam-Hyers stability of the system (1.1).

Take the following system of integral inequations

$$\left.\begin{array}{c} \sup_{t\in[a,\ b]} \mid x(t) - g(t) - \lambda \int_a^b K(t,s)\ \mathfrak{h}(s, x(s), y(s))\ ds \mid \leq \epsilon, \\ \sup_{t\in[a,\ b]} \mid y(t) - g(t) - \lambda \int_a^b K(t,s)\ \mathfrak{h}(s, y(s), x(s))\ ds \mid \leq \epsilon, \\ \text{where } \lambda \geq 0, \quad t \in [a,b] \text{ and } \epsilon > 0. \end{array}\right\} \tag{5.3}$$

In the next theorem, we take an extra assumption for assuring the Ulam-Hyers stability of (1.1).

(I_6) If $(x^*,\ y^*)$ is any solution of (1.1), then $u \preceq x^*$ and $y^* \preceq v$ for any $(u, v) \in X \times X$.

Theorem 5.4. *The solution of (1.1) is Ulam-Hyers stable if the assumption (I_6) is included in Theorem 3.1.*

Proof. With the help of Theorem 3.1 we get a unique point $(x^*,\ y^*) \in X^2$ which satisfies (1.1). Hence it is the unique coupled fixed point of $\mathfrak{F}$ defined in (3.2). Let $(u^*,\ v^*)$ be a solution of the system of integral inequation (5.3). Hence $(u^*,\ v^*)$ is a solution of $d(x,\ \mathfrak{F}(x, y)) \leq \epsilon$ and $d(y,\ \mathfrak{F}(y, x)) \leq \epsilon$. Also by ($I_6$), we have $u^* \preceq x^*$ and $y^* \preceq v^*$. By (5.2) of Theorem 5.3, we obtain

$$\sup_{s\in[a,b]} \max\{\mid u^*(s) - x^*(s) \mid, \mid v^*(s) - y^*(s) \mid\} = \max\ \{d(x^*, u^*), d(y^*, v^*)\}$$

$$\leq \frac{\epsilon}{(1-\xi)} = \frac{\epsilon}{1 - \lambda\ (b-a)\ m} = \phi(\epsilon).$$

Therefore, the solution of (1.1) is Ulam-Hyers stable.

6. Well-Posedness

In the current section, we also investigate the well-posedness of the fixed point problem considered here. The study of well-posedness has appeared in several recent works related to fixed point theory as, for instances, in [21, 26].

The notion of well-posedness of a fixed point problem has evoked interests of several mathematicians (see for example [1, 19, 22]). Let $S : M \to M$, where (M, d) is a metric space. The fixed point problem $x = Sx$ is well-posed if S admits a unique fixed point $x \in X$ and $d(x_n, x) \to 0$ as $n \to \infty$ for any sequence $\{x_n\}$ in X with $d(x_n,\ Sx_n) \to 0$ as $n \to \infty$.

Incorporating the ideas and technicalities described above, the followings are the corresponding concepts for coupled mapping and also for system of equations (1.1).

Definition 6.1 ([6])**.** Problem P is well-posed if (i) $\mathfrak{F}$ has a unique coupled fixed point $(x^*,\ y^*)$, (ii) $x_n \to x^*$ and $y_n \to y^*$ as $n \to \infty$, whenever $\{(x_n,\ y_n)\}$ is any sequence in X^2 satisfying $d(x_n,\ \mathfrak{F}(x_n,\ y_n)) \to 0$ and $d(y_n,\ \mathfrak{F}(y_n,\ x_n)) \to 0$, as $n \to \infty$.

Definition 6.2. The coupled system of nonlinear integral equations (1.1) is well-posed if

(i) the system has a unique unique solution $(x^*,\ y^*)$,

(ii) $x_n \to x^*$ and $y_n \to y^*$ as $n \to \infty$, whenever $\{(x_n,\ y_n)\}$ is any sequence of functions satisfying

$$\sup_{t\in[a,\ b]} \mid x_n(t) - g(t) - \lambda \int_a^b K(t,\ s)\ \mathfrak{h}(s,\ x_n(s),\ y_n(s))\ ds \mid \to 0$$

and

$$\sup_{t\in[a,\ b]} \mid y_n(t) - g(t) - \lambda \int_a^b K(t,\ s)\ \mathfrak{h}(s,\ y_n(s),\ x_n(s))\ ds \mid \to 0,$$

as $n \to \infty$, where $\lambda \geq 0$.

We consider the following condition for the well-posedness of mixed monotone mapping.

(A2): If $(x^*,\ y^*)$ is any solution of Problem P and $\{(x_n,\ y_n)\}$ is any sequence in X^2 with $\lim_{n\to\infty} d(x_n, \mathfrak{F}(x_n,\ y_n)) = 0$ and $\lim_{n\to\infty} d(y_n, \mathfrak{F}(y_n,\ x_n)) = 0$, then $x^* \preceq x_n,\ \ y_n \preceq y^*$, for $n > 0$.

Theorem 6.3. *Problem P is well-posed, if (A2) is taken as the additional assumption in Theorem 2.4.*

Proof. By Theorem 2.4, $\mathfrak{F}$ has a unique coupled fixed point $(x^*,\ y^*)$ (say). Then $(x^*,\ y^*)$ is a solution of Problem P. Let $\{(x_n,\ y_n)\} \in X^2$ be a sequence for which $d(x_n,\ \mathfrak{F}(x_n,\ y_n)) \to 0$ and $d(y_n,\ \mathfrak{F}(y_n,\ x_n)) \to 0$ as $n \to \infty$. By the assumption (A2), we have $x^* \preceq x_n,\ \ y_n \preceq y^*$, for all n. Using (2.1), we have

$$\begin{aligned} d(x_n, x^*) &= d(x_n,\ \mathfrak{F}(x^*, y^*)) \leq d(x_n,\ \mathfrak{F}(x_n, y_n)) + d(\mathfrak{F}(x_n, y_n),\ \mathfrak{F}(x^*, y^*)) \\ &\leq d(x_n,\ \mathfrak{F}(x_n,\ y_n)) + \frac{\xi}{2}\ [d(x_n, x^*) + d(y_n, y^*)]. \end{aligned}$$

Similarly,

$$d(y_n, y^*) \leq d(y_n,\ \mathfrak{F}(y_n, x_n)) + \frac{\xi}{2}\ [d(x_n, x^*) + d(y_n, y^*)].$$

Therefore,

$$\begin{aligned} d(x_n, x^*) + d(y_n, y^*) &\leq d(x_n,\ \mathfrak{F}(x_n,\ y_n)) + d(y_n,\ \mathfrak{F}(y_n, x_n)) \\ &\quad + \xi\ [d(x_n, x^*) + d(y_n, y^*)], \end{aligned}$$

which implies that

$$d(x_n, x^*) + d(y_n, y^*) \leq \frac{d(x_n,\ \mathfrak{F}(x_n,\ y_n)) + d(y_n,\ \mathfrak{F}(y_n, x_n))}{(1-\xi)}.$$

Taking limit as $n \to \infty$, we have

$$\begin{aligned} \lim_{n\to\infty} [d(x_n, x^*) + d(y_n, y^*)] &\leq \lim_{n\to\infty} \frac{d(x_n,\ \mathfrak{F}(x_n, y_n)) + d(y_n, \mathfrak{F}(y_n, x_n))}{(1-\xi)} \\ &= 0, \end{aligned}$$

which implies that $\lim_{n\to\infty}[d(x_n, x^*) + d(y_n, y^*)] = 0$, that is, $\lim_{n\to\infty} d(x_n, x^*) = 0$ and $\lim_{n\to\infty} d(y_n, y^*) = 0$, that is, $x_n \to x^*$ and $y_n \to y^*$, as $n \to \infty$. Hence Problem P is well-posed.

In the next theorem, we take an assumption for assurance the well-posedness for the system (1.1).

(I_7) For any sequence $\{(x_n,\ y_n)\}$,

$$\sup_{t\in[a,\ b]}\ |\ x_n(t) - g(t) - \lambda \int_a^b K(t,s)\ \mathfrak{h}(s,\ x_n(s),\ y_n(s))\ ds\ | \to 0 \quad \text{and}$$

$$\sup_{t\in[a,\ b]}\ |\ y_n(t) - g(t) - \lambda \int_a^b K(t,s)\ \mathfrak{h}(s,\ y_n(s),\ x_n(s))\ ds\ | \to 0,$$

as $n \to \infty$ imply $x^* \preceq x_n$ and $y_n \preceq y^*$, for all n, where (x^*, y^*) is a solution of (1.1).

Theorem 6.4. *The system (1.1) is well-posed if* (I_7) *holds in Theorem 3.1.*

Proof. Applying Theorem 3.1, we get a unique point (x^*, y^*) in X^2 which satisfies (1.1). Hence it is a unique coupled fixed point of $\mathfrak{F}$ defined in (3.2). Let $\{(x_n,\ y_n)\}$ be a sequence such that

$$\sup_{t\in[a,\ b]}\ |\ x_n(t) - g(t) - \lambda \int_a^b K(t,s)\ \mathfrak{h}(s,\ x_n(s),\ y_n(s))\ ds\ | \to 0 \qquad \text{and}$$

$$\sup_{t\in[a,\ b]}\ |\ y_n(t) - g(t) - \lambda \int_a^b K(t,s)\ \mathfrak{h}(s,\ y_n(s),\ x_n(s))\ ds\ | \to 0,$$

as $n \to \infty$. By the assumption (I_7), we have $x^* \preceq x_n$ and $y_n \preceq y^*$, for all n. Hence we have $d(x_n,\ \mathfrak{F}(x_n,\ y_n)) \to 0$ and $d(y_n,\ \mathfrak{F}(y_n,\ x_n)) \to 0$, as $n \to \infty$ with $x^* \preceq x_n$ and $y_n \preceq y^*$, for all n, where $\mathfrak{F}$ defined in (3.2). As (by application of Theorem 6.3) Problem P is well-posed, the coupled system of nonlinear equations (1.1) is also so.

7. Data dependence result

Let $S_1,\ S_2 : M \to M$ be two mappings, where (M, d) is a metric space such that $d(S_1x,\ S_2x) \leq \eta$ for all $x \in X$, where η is some positive number. Then the problem of data dependence is to estimate the distance between the fixed points of these two mappings. Several research papers on data dependence have been published in recent literatures of which we mention a few in references [6, 10, 25].

Our problem of data dependence is with coupled mappings and their coupled fixed point sets. Such problems for coupled fixed point sets have already appeared in work of Chifu et al. [6]. We formulate a version of the problem suitable to our needs.

Being inspired of the aforesaid ideas we give definitions of data dependence for the case of aforementioned system of integral equations.

Definition 7.1. Let $(x^*,\ y^*)$ be a solution of (1.1) and $(u^*,\ v^*)$ be a solution of the following system

$$\left.\begin{array}{rcll} x(t) & = & f(t) + \lambda \int_a^b K_1(t,\ s)\ \mathfrak{h}_1(s,\ x(s),\ y(s))\, ds & \text{and} \\ y(t) & = & f(t) + \lambda \int_a^b K_1(t,\ s)\ \mathfrak{h}_1(s,\ y(s),\ x(s))\, ds, & \lambda \geq 0, \end{array}\right\}$$

for all $t \in [a, b]$. The problem of data dependence is to find

$$\sup_{t \in [a,b]} [\mid x^*(t) - u^*(t) \mid + \mid y^*(t) - v^*(t) \mid].$$

Theorem 7.2. *Let $(X, d, \preceq)$ be a complete and partially ordered metric space having regular property and $\mathfrak{F} : X^2 \to X$. Suppose that all the assumptions of Theorem 2.4 are satisfied. Then $\mathfrak{F}$ has a unique coupled fixed point (x^*, y^*). Moreover, let $T : X^2 \to X$ has nonempty coupled fixed point set. Assume that there exists $M > 0$ for which $d(\mathfrak{F}(x, y), T(x, y)) \leq M$, whenever $(x, y) \in X^2$ and for any coupled fixed point (x, y) of the mapping T, $x \preceq \mathfrak{F}(x, y)$ and $\mathfrak{F}(y, x) \preceq y$ hold. Then*

$$d(x, x^*) + d(y, y^*) \leq \frac{4M}{(1 - \xi)},$$

whenever (x, y) is any coupled fixed point of T.

Proof. From Theorem 2.4, $\mathfrak{F}$ has a unique coupled fixed point (x^*, y^*). Suppose that (x, y) is a coupled fixed point of T. Take $x_0 = x$ and $y_0 = y$. Then

$$x_0 = T(x_0,\ y_0) \ \text{ and } \ \ y_0 = T(y_0,\ x_0). \tag{7.1}$$

Let $x_1 = \mathfrak{F}(x_0, y_0)$ and $y_1 = \mathfrak{F}(y_0, x_0)$. Then

$$\left.\begin{array}{c} d(x_0, x_1) = d(T(x_0, y_0),\ \mathfrak{F}(x_0, y_0)) \leq M \\ \text{and} \\ d(y_0, y_1) = d(T(y_0, x_0),\ \mathfrak{F}(y_0, x_0)) \leq M. \end{array}\right\} \tag{7.2}$$

Applying the assumption of the theorem, we get $x_0 \preceq x_1$ and $y_1 \preceq y_0$. Let $x_2 = \mathfrak{F}(x_1, y_1)$ and $y_2 = \mathfrak{F}(y_1, x_1)$. Then by a property of $\mathfrak{F}$, it follows that $x_1 \preceq x_2$ and $y_2 \preceq y_1$. Then taking the technicalities as in establishing of Theorem 2.4 (see [13]), we have a sequence $\{(x_n, y_n)\}$ in X^2 given by equation (2.2) and

- $x_n \preceq x_{n+1}$ and $y_{n+1} \preceq y_n$, for all $n \geq 0$;
- both $\{x_n\}$ and $\{y_n\}$ are two Cauchy sequences in (X, d) and there exist $u, v \in X$

such that $\lim_{n\to\infty} x_n = u$ and $\lim_{n\to\infty} y_n = v$;
• (u, v) is a coupled fixed point of $\mathfrak{F}$. As coupled fixed point of $\mathfrak{F}$ unique, we have $u = x^*$, $v = y^*$.

Using (2.1), we obtain

$$\begin{aligned} r_{n+1} &= d(x_{n+1}, x_{n+2}) + d(y_{n+1}, y_{n+2}) \\ &= d(\mathfrak{F}(x_n, y_n),\ \mathfrak{F}(x_{n+1}, y_{n+1})) + d(\mathfrak{F}(y_n, x_n),\ \mathfrak{F}(y_{n+1}, x_{n+1})) \\ &= d(\mathfrak{F}(x_{n+1}, y_{n+1}),\ \mathfrak{F}(x_n, y_n)) + d(\mathfrak{F}(y_n, x_n),\ \mathfrak{F}(y_{n+1}, x_{n+1})) \\ &\le \frac{\xi}{2}\ [d(x_{n+1}, x_n) + d(y_{n+1}, y_n)] + \frac{\xi}{2}\ [d(y_n, y_{n+1}) + d(x_n, x_{n+1})] \\ &= \xi\ [d(x_n, x_{n+1}) + d(y_n, y_{n+1})] = \xi\ r_n, \end{aligned}$$

where $r_n = d(x_n, x_{n+1}) + d(y_n, y_{n+1})$.

Applying the above inequality repeatedly, we get

$$r_{n+1} \le \xi\ r_n \le \xi^2\ r_{n-1} \le ...\xi^n\ r_1 \le \xi^{n+1}\ r_0.$$

Using (7.2) and the above inequality, we have

$$\begin{aligned} d(x_0, x^*) = d(x_0, u) &\le \sum_{i=0}^{n} d(x_i, x_{i+1}) + d(x_{n+1},\ u) \\ &\le \sum_{i=0}^{n} r_i + d(x_{n+1}, u) \le \sum_{i=0}^{n} \xi^i\ r_0 + d(x_{n+1}, u) \end{aligned}$$

and

$$\begin{aligned} d(y_0, y^*) = d(y_0, v) &\le \sum_{i=0}^{n} d(y_i, y_{i+1}) + d(y_{n+1},\ v) \\ &\le \sum_{i=0}^{n} r_i + d(y_{n+1}, v) \le \sum_{i=0}^{n} \xi^i\ r_0 + d(y_{n+1}, u). \end{aligned}$$

Using (7.2), we obtain

$$d(x_0,\ u) \le \sum_{i=0}^{\infty} \xi^i r_0 = \frac{r_0}{(1-\xi)} = \frac{d(x_0, x_1) + d(y_0, y_1)}{(1-\xi)} \le \frac{2\ M}{(1-\xi)} \quad \text{and}$$

$$d(y_0,\ v) \le \sum_{i=0}^{\infty} \xi^i r_0 = \frac{r_0}{(1-\xi)} = \frac{d(x_0, x_1) + d(y_0, y_1)}{(1-\xi)} \le \frac{2\ M}{(1-\xi)}.$$

Hence, $d(x_0, u) + d(y_0, v) \le \dfrac{4\ M}{(1-\xi)}$, that is, $d(x, x^*) + d(y, y^*) \le \dfrac{4\ M}{(1-\xi)}$.

Theorem 7.3. *In Theorem 3.1, we also assume that if* $(x,\ y)$ *is any solution of the following system*

$$\left.\begin{aligned} x(t) &= f(t) + \lambda\ \int_a^b K_1(t,\ s)\ \mathfrak{h}_1(s,\ x(s),\ y(s))\ ds \qquad \textit{and} \\ y(t) &= f(t) + \lambda\ \int_a^b K_1(t, s)\ \mathfrak{h}_1(s, y(s), x(s))\ ds,\ \ \lambda \ge 0 \textit{ for all } t \in [a, b], \end{aligned}\right\} \quad (7.3)$$

then for $a \leq t \in b$,

$$x(t) \leq g(t) + \lambda \int_a^b K(t,s)\ \mathfrak{h}(s, x(s), y(s))\ ds$$

and

$$g(t) + \lambda \int_a^b K(t,s)\ \mathfrak{h}(s, y(s), x(s))\ ds \leq y(t).$$

Further suppose that there exist $\nu, \eta > 0$ *such that*

$$\sup_{t\in[a,b]} |K_1(t,s)\ \mathfrak{h}_1(s, x(s), y(s)) - K(t,s)\ \mathfrak{h}(s, x(s), y(s))| \leq \eta$$

and

$$\sup_{t\in[a,b]} |f(t) - g(t)| \leq \nu.$$

If (x, y) *is any solution of the system (7.3) and* $(x^*,\ y^*)$ *is any solution of the system (1.1), then*

$$\sup_{t\in[a,\ b]} [\ |\ x(t) - x^*(t)\ |\ + |\ y(t) - y^*(t)\ |\ ]\ \leq \frac{4\ [\ \nu + \lambda\ \eta\ (b-a)\ ]}{(1-\xi)},$$

where ξ *is given in* I_4.

Proof. Applying Theorem 3.1, we get that the system (1.1) has a unique solution (x^*, y^*) (say). Define $T : X^2 \to X$, where $X = C[a,\ b]$, by

$$T(x,\ y)(t) = g(t) + \lambda \int_a^b K_1(t,\ s)\ \mathfrak{h}_1(s,\ x(s),\ y(s))\ ds, \qquad \text{for all } a \leq t \leq b. \quad (7.4)$$

Since $(x,\ y)$ is a solution of (7.3), it is a coupled fixed point of T. By the assumptions of the theorem, we have $x(t) \leq \mathfrak{F}(x,\ y)(t)$ and $\mathfrak{F}(y,\ x)(t) \leq y(t)$, for all $t \in [a,b]$, which imply that $x \preceq \mathfrak{F}(x,\ y)$ and $\mathfrak{F}(y,\ x) \preceq y$. Also,

$$\begin{aligned}
&|\ \mathfrak{F}(x,y)(t) - T(x,y)(t)\ | \\
&= |\ f(t) - g(t) + \lambda \int_a^b [K(t,s)\ \mathfrak{h}(s,x(s),y(s)) - K_1(t,s)\ \mathfrak{h}_1(s,x(s),y(s))]ds\ | \\
&\leq |\ f(t) - g(t)\ | + |\ \lambda \int_a^b [K(t,s)\ \mathfrak{h}(s,x(s),y(s)) - K_1(t,s)\ \mathfrak{h}_1(s,x(s),y(s))]ds\ | \\
&\leq \nu + \lambda \int_a^b |\ [K(t,s)\ \mathfrak{h}(s,x(s),y(s)) - K_1(t,\ s)\ \mathfrak{h}_1(s,\ x(s),y(s))]\ |\ ds \\
&\leq \nu + \lambda \int_a^b \eta\ ds = \nu + \lambda\ \eta\ (b-a) = M \quad (\text{say }),\ \text{for all } t \in [a,\ b],
\end{aligned}$$

which means that $\sup_{t\in[a,\ b]}\ |\ \mathfrak{F}(x,\ y)(t) - T(x,\ y)(t)\ | \leq M$, whenever $(x,y) \in X^2$, that is, $d(\mathfrak{F}(x,y), T(x,y)) \leq M$, whenever $(x,y) \in X^2$. By application of Theorem 7.2, we have

$$\sup_{t\in[a,b]} [\ |\ x(t) - x^*(t)\ | + |\ y(t) - y^*(t)\ |\ ] = d(x, x^*) + d(y, y^*)$$
$$\leq \frac{4M}{(1-\xi)} = \frac{4\ [\ \nu + \lambda\ \eta\ (b-a)\ ]}{(1-\xi)}.$$

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