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Existence and Ulam stability of initial value problem for fractional perturbed functional q-difference equations

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Abstract. In this work, we discuss the existence and uniqueness of solutions to the initial value problem for perturbed functional fractional q-difference equations involving q-derivative of the Caputo sense. By applying Banach contraction principle and Burton and Kirk's fixed point theorems. Further, we present the Ulam-Hyers and Ulam-Hyers-Rassias stabilities results by using direct analysis methods. Finally, we give two examples illustrating of the results.

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1. Introduction

Fractional calculus is a significant branch in mathematical analysis. Indeed, Leibniz and Newton developed differential calculus, it has numerous applications in various sciences, for example, mechanics, electricity, biology. Also, Fractional differential equations play a fundamental role in the modeling of a considerable number of phenomena in many areas. Currently being addressed by many researchers of various fields of science and engineering such as physics, chemistry, biology, economics, control theory, and biophysics, etc. For more details, see the books of Hilfer [18] and Tarasov *et al.* [35], Kilbas *et al.* [24] and Samko *et al.* [33], Podlubny [26] and Miller *et al.* [25].

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The q-difference calculus is an interesting and old subject. In 1910, Jackson [21, 20] introduced and developed q-difference calculus or quantum calculus in a systematic way, basic definitions and properties of q-difference calculus can be found in [16, 23]. Then, Al-Salam [11] and Agarwal [5] proposed the fractional q-difference calculus. Due to it applicability in mathematical modeling in different branches like technical sciences, engineering, physics and biomathematics, it has drawn wide attention to many researchers.

Fractional q-difference equations initiated at the beginning of the nineteenth century [4, 15] and received significant attention in recent years. While some interesting details about initial and boundary value problems of fractional q-difference equations can be found in books of Ahmad *et al.* [7] and Annaby *et al.* [12]; see the papers of Ahmed *et al.* [6], Abbas *et al.* [2, 3], Allouch *et al.* [9, 8, 10] and Samei *et al.* [32].

The stability of functional equations was originally emerged Ulam [36, 37] and Hyers [19]. Thereafter, the stability of this type is called Ulam-Hyers Stability. In 1978, Rassias [29] provided a generalization of the Hyers theorem which allows the Cauchy difference to be unbounded. Considerable attention has been given to the study of the Ulam-Hyers and Ulam-Hyers-Rassias stability of all kinds of fractional differential equations. See the papers of Rassias [29], Rus [30, 31], Abbes *et al.* [1, 3], Jung [22], Taieb *et al.* [34] and Wang *et al.* [38].

In [13], Belarbi *et al.* studied the initial value problem (IVP for short) for perturbed fractional order functional differential equations of the form:

$$D^{\alpha}y(t) = f(t, y_t) + g(t, y_t), \text{ for a.e. } t \in J = [0, b], \quad 0 < \alpha < 1,$$

$$y(t) = \phi(t), \ t \in [-r, 0],$$

where ${}^{C}D^{\alpha}$ is the Riemman-Liouville fractional derivative, $f, g : J \times C([-r, 0], \mathbb{R}) \to \mathbb{R}$ are given functions and $\phi \in C([-r, 0], \mathbb{R})$ with $\phi(0) = 0$. For any continuous function y defined on $[-r, b]$ and any $t \in J$, we denote by y_t the element of $C([-r, 0], \mathbb{R})$ defined by:

$$y_t(\theta) = y(t + \theta), \quad \theta \in [-r, 0].$$

Here $y_t(.)$ represents the history of the state from time $t - r$ up to the present time t.

Motivated by aforementioned work, in this paper, we concentrate on the existence, uniqueness and Ulam stability of solutions of the initial value problem (IVP for short) for perturbed functional fractional q-difference equations of the form:

$${}^{C}D_q^{\alpha}y(t) = f(t, y_t) + g(t, y_t), \text{ for a.e. } t \in J = [0, T], \quad 0 < \alpha < 1, \tag{1.1}$$

$$y(t) = \varphi(t), \ t \in \overline{J} = [-d, 0], \tag{1.2}$$

where $T > 0, \ d > 0, \ q \in (0, 1), \ {}^{C}D_q^{\alpha}$ is the Caputo fractional q-derivative of order α, $f, g : J \times C([-d, 0], \mathbb{R}) \to \mathbb{R}$ are given functions and $\varphi \in C([-d, 0], \mathbb{R})$ with $\varphi(0) = 0$. For any continuous function y defined on $[-d, T]$ and any $t \in J$, we denote by y_t the element of $C([-d, 0], \mathbb{R})$ defined by:

$$y_t(\theta) = y(t + \theta), \quad \theta \in [-d, 0].$$

Here $y_t(.)$ represents the history of the state from time $t-d$ up to the present time t.

The work is arranged as follows : In Section 2, we introduce some preliminary, including basic definitions and properties of fractional q-calculus. In Section 3, we prove the existence and uniqueness results for the problem (1.1)-(1.2), we give two results, the first one is based on Banach contraction principle (Theorem 3.3), the second is based on Burton and Kirk's fixed point theorem (Theorem 3.4). In Section 4, Ulam-Hyers and Ulam-Hyers-Rassias stabilities theorems are presented. In Section 5, we give two examples to illustrate the obtained results. Finally, we end with the conclusion.

2. Preliminaries

In this section, we present some basic definitions, lemmas and notations which will be used in this paper.

Let $T > 0$, $d > 0$ and define $J := [0, T]$, $\overline{J} := [-d, 0]$. By $C(J, \mathbb{R})$ we denote the Banach space of all continuous functions from J into $\mathbb{R}$ with the usual supremum norm:

$$\|y\|_\infty = \sup\{|y(t)| : 0 \leq t \leq T\}.$$

Also, $C(\overline{J}, \mathbb{R})$ is endowed with the norm $\|.\|_*$ defined by:

$$\|y\|_* = \sup\{|y(t)| : -d \leq t \leq 0\}.$$

Let $\mathcal{C} = \{y : [-d, T] \to \mathbb{R} : y|_{[-d,0]} \in C(\overline{J}, \mathbb{R})\ and\ y|_{[0,T]} \in C(J, \mathbb{R})\}$ is a Banach space with the norm:

$$\|y\|_{\mathcal{C}} = \sup\{|y(t)| : -d \leq t \leq T\}.$$

Now, we introduce some definitions and properties of fractional q-calculus [16, 23].

For $0 < q < 1$, we set:

$$[a]_q = \frac{1 - q^a}{1 - q}, \quad a \in \mathbb{R}.$$

The q-analogue of the power $(a - b)^{(n)}$ is expressed by:

$$(a - b)^{(0)} = 1,\ (a - b)^{(n)} = \prod_{k=0}^{n-1} (a - bq^k),\ a, b \in \mathbb{R},\ n \in \mathbb{N}.$$

More generally, if $\alpha \in \mathbb{R}$, then

$$(a - b)^{(\alpha)} = a^\alpha \prod_{k=0}^{\infty} \left(\frac{a - bq^k}{a - bq^{k+\alpha}} \right),\ a, b \in \mathbb{R}.$$

Note that if $b = 0$, then $a^{(\alpha)} = a^\alpha$.

Definition 2.1. [23] The q-gamma function is defined by:

$$\Gamma_q(\alpha) = \frac{(1 - q)^{(\alpha - 1)}}{(1 - q)^{\alpha - 1}},\ \alpha \in \mathbb{R} - \{0, -1, -2, \ldots\}.$$

Notice that the q-gamma function satisfies $\Gamma_q(\alpha + 1) = [\alpha]_q \Gamma_q(\alpha)$.

Definition 2.2. [23] The q-derivative of order $n \in \mathbb{N}$ of a function $f : J \to \mathbb{R}$, is defined by $(D_q^0 f)(t) = f(t)$,

$$(D_q f)(t) = (D_q^1 f)(t) = \frac{f(t) - f(qt)}{(1-q)t},\ t \neq 0,\ (D_q f)(0) = \lim_{t\to 0}(D_q f)(t),$$

and

$$(D_q^n f)(t) = (D_q^1 D_q^{n-1} f)(t),\ t \in J,\ n \in \{1, 2, \ldots\}.$$

Set $J_t := \{tq^n : n \in \mathbb{N}\} \cup \{0\}$.

Definition 2.3. [23] The q-integral of a function $f : J_t \to \mathbb{R}$, is given by:

$$(I_q f)(t) = \int_0^t f(s) d_q s = \sum_{n=0}^{\infty} t(1-q)q^n f(tq^n),$$

provided that the series converges.

We note that $(D_q I_q f)(t) = f(t)$, while if f is continuous at 0, then

$$(I_q D_q f)(t) = f(t) - f(0).$$

Definition 2.4. [5] Let $\alpha \in \mathbb{R}_+$ and function $f : J \to \mathbb{R}$. The fractional q-integral of the Riemann-Liouville type of order α is defined by $(I_q^0 f)(t) = f(t)$, and

$$(I_q^\alpha f)(t) = \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} f(s) d_q s,\ t \in J.$$

Note that for $\alpha = 1$, we have $(I_q^1 f)(t) = (I_q f)(t)$.

Lemma 2.5. [27] *For $\alpha \in \mathbb{R}_+$ and $\beta \in (-1, +\infty)$, we have:*

$$(I_q^\alpha (t-a)^{(\beta)})(t) = \frac{\Gamma_q(\beta+1)}{\Gamma_q(\alpha+\beta+1)}(t-a)^{(\alpha+\beta)},\ 0 < a < t < T.$$

In particular,

$$(I_q^\alpha 1)(t) = \frac{1}{\Gamma_q(\alpha+1)} t^{(\alpha)}.$$

Definition 2.6. [28] The fractional q-derivative of the Riemann-Liouville type of order $\alpha \in \mathbb{R}_+$ of a function $f : J \to \mathbb{R}$, is defined by $(D_q^0 f)(t) = f(t)$, and

$$(D_q^\alpha f)(t) = (D_q^{[\alpha]} I_q^{[\alpha]-\alpha} f)(t),\ t \in J,$$

where $[\alpha]$ is the integer part of α.

Definition 2.7. [28] The fractional q-derivative of the Caputo type of order $\alpha \in \mathbb{R}_+$ of a function $f : J \to \mathbb{R}$, is defined by $(D_q^0 f)(t) = f(t)$, and

$$({}^C D_q^\alpha f)(t) = (I_q^{[\alpha]-\alpha} D_q^{[\alpha]} f)(t),\ t \in J.$$

where $[\alpha]$ is the integer part of α.

Lemma 2.8. [28] *Let $\alpha, \beta \in \mathbb{R}_+$ and let f be a function defined on J. Then, the next identities hold:*

(i) $(I_q^\alpha I_q^\beta f)(t) = (I_q^{\alpha+\beta} f)(t)$.
(ii) $(D_q^\alpha I_q^\alpha f)(t) = f(t)$.

Lemma 2.9. [28] *Let $\alpha \in \mathbb{R}_+$ and let f be a function defined on J. Then, the following equality holds:*

$$(I_q^\alpha \, {}^C D_q^\alpha f)(t) = f(t) - \sum_{k=0}^{[\alpha]-1} \frac{t^k}{\Gamma_q(k+1)} (D_q^k f)(0).$$

In particular, if $\alpha \in (0,1)$, then

$$(I_q^\alpha \, {}^C D_q^\alpha f)(t) = f(t) - f(0).$$

Next, we introduce the main fixed point theorems.

Theorem 2.10. *(Banach Contraction Principle)* [17]
Let $\mathcal{M}$ be a non-empty closed subset of a Banach space X, then any contraction mapping F of $\mathcal{M}$ into itself has a unique fixed point.

Theorem 2.11. *(Burton and Kirk)* [14]
Let X be a Banach space, and $A, B : X \to X$ two operators satisfying:

(i) *A is a contraction, and*
(ii) *B is completely continuous.*

Then either

(a) *the operator equation $y = A(y) + B(y)$ has a solution, or*
(b) *the set $\Omega = \left\{ y \in X : \lambda A\left(\frac{y}{\lambda}\right) + \lambda B(y) = y \right\}$ is unbounded for $\lambda \in (0,1)$.*

Finally, we state the following generalization of **Gronwall**'s lemma.

Lemma 2.12. *(Gronwall lemma)* [39]
Let $u : J \to [0, +\infty)$ be a real function and $v(.)$ is a nonnegative, locally integrable function on J. Assume that there is a constant $c > 0$ and $0 < \alpha < 1$ such that

$$u(t) \le v(t) + c \int_0^t (t-s)^{-\alpha} u(s) ds.$$

Then, there exists a constant $\delta = \delta(\alpha)$ such that

$$u(t) \le v(t) + \delta c \int_0^t (t-s)^{-\alpha} v(s) ds, \text{ for every } t \in J.$$

3. Existence and uniqueness results

In this section, we present the existence and uniqueness of solutions for the problem (1.1)-(1.2).

Let us start by defining what we mean by a solution of the problem (1.1)-(1.2).

Definition 3.1. A function $y \in \mathcal{C}$ is said to be a solution of the problem (1.1)-(1.2) if y satisfies the equation $({}^C D_q^\alpha y)(t) = f(t, y_t) + g(t, y_t)$ on J, and satisfies the condition $y(t) = \varphi(t)$ on $\overline{J}$.

For the existence of solutions for the problem (1.1)-(1.2), we need the following auxiliary lemma.

Lemma 3.2. *Let $h : J \to \mathbb{R}$ be continuous, the solution of the initial value problem:*

$$({}^C D_q^\alpha y)(t) = h(t), \, t \in J = [0, T], \quad 0 < \alpha < 1, \tag{3.1}$$

$$y(t) = \varphi(t), \; t \in \overline{J} = [-d, 0], \tag{3.2}$$

is given by:

$$y(t) = \begin{cases} \phi(t), & t \in \overline{J} = [-d, 0], \\ \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} h(s) d_q s, & t \in J = [0, T]. \end{cases} \tag{3.3}$$

Proof. Applying the Riemann-Liouville fractional q-integral of order α to both sides of equation (3.1), and by using Lemma 2.9, we have:

$$y(t) = \int_0^t \frac{(t - qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} h(s) d_q s + c_0.$$

Using the initial condition of the problem (3.1)-(3.2) and $y(0) = \phi(0) = 0$, we obtain:

$$c_0 = 0.$$

So,

$$y(t) = \int_0^t \frac{(t - qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} h(s) d_q s, \; t \in J = [0, T],$$

and

$$y(t) = \varphi(t), \; t \in \overline{J} = [-d, 0].$$

The proof is completed. □

In the following subsection, we prove uniqueness and existence results for the problem (1.1)-(1.2) by means fixed point theorems.

The first result is based on Banach contraction principle (Theorem 2.10).

Theorem 3.3. *Assume that the following hypotheses hold:*

($H1$) *The functions $f, g : J \times C(\overline{J}, \mathbb{R}) \to \mathbb{R}$ are continuous.*

($H2$) *There exist $L_f > 0$, such that for each $t \in J$ and each $y, x \in \mathbb{R}$, we have:*

$$|f(t, y) - f(t, x)| \leq L_f |y - x|.$$

($H3$) *There exist $L_g > 0$, such that for each $t \in J$ and each $y, x \in \mathbb{R}$, we have:*

$$|g(t, y) - g(t, x)| \leq L_g |y - x|.$$

If

$$\frac{(L_f + L_g) T^{(\alpha)}}{\Gamma_q(\alpha + 1)} < 1. \tag{3.4}$$

Then, the problem (1.1)-(1.2) has a unique solution.

Proof. Transform the problem (1.1)-(1.2) into a fixed point problem, we consider the operator

$$F : \mathcal{C} \longrightarrow \mathcal{C}$$

Defined by:

$$(Fy)(t) = \begin{cases} \phi(t), & t \in \overline{J} = [-d, 0], \\ \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} f(s, y_s) d_q s + \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} g(s, y_s) d_q s, & t \in J = [0, T]. \end{cases}$$

Clearly, the fixed points of operator F are solution of the problem (1.1)-(1.2).

Now, we shall prove that F is a contraction mapping on $\mathcal{C}$.

Let $y, x \in \mathcal{C}$, if $t \in \overline{J}$, then we have:

$$|(Fy)(t) - (Fx)(t)| = |\phi(t) - \phi(t)| = 0.$$

Hence,

$$\|(Fy)(t) - (Fx)(t)\|_* = 0. \tag{3.5}$$

For $t \in J$, we have:

$$\begin{aligned} |(Fy)(t) - (Fx)(t)| &= \left| \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} \left(f(s, y_s) - f(s, x_s)\right) d_q s \right. \\ &\quad \left. + \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} \left(g(s, y_s) - g(s, x_s)\right) d_q s \right|. \end{aligned}$$

Therefore,

$$\begin{aligned} |(Fy)(t) - (Fx)(t)| &\leq \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} \Big(|f(s, y_s) - f(s, x_s)| \\ &\quad + |g(s, y_s) - g(s, x_s)| \Big) d_q s. \end{aligned}$$

By hypothesis $(H2)$-$(H3)$, we get:

$$\begin{aligned} |(Fy)(t) - (Fx)(t)| &\leq \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} \Big(L_f |y_s - x_s| + L_g |y_s - x_s| \Big) d_q s, \\ &\leq (L_f + L_g) \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} \|y_s - x_s\|_* \, d_q s. \end{aligned}$$

Thus,

$$\|(Fy)(t) - (Fx)(t)\|_\infty \leq \frac{(L_f + L_g) T^{(\alpha)}}{\Gamma_q(\alpha + 1)} \|y - x\|_{\mathcal{C}}. \tag{3.6}$$

From equations (3.5) and (3.6), we conclude that:

$$\|(Fy)(t) - (Fx)(t)\|_{\mathcal{C}} \leq \frac{(L_f + L_g) T^{(\alpha)}}{\Gamma_q(\alpha + 1)} \|y - x\|_{\mathcal{C}}.$$

By condition (3.4), F is a contraction operator, and by Banach contraction mapping principle, we deduce that the operator F has a unique fixed point, which is the unique solution of the problem (1.1)-(1.2). ∎

The second result is based on Burton and Kirk's fixed point theorem (Theorem 2.11).

Theorem 3.4. *Assume that the hypotheses* $(H1)$-$(H2)$-$(H3)$ *are satisfied and*

$(H4)$ *There exists constant* $M_g > 0$*, such that for each* $t \in J$ *and each* $y \in \mathbb{R}$*, we have:*

$$|g(t,y)| \leq M_g.$$

If

$$\frac{L_f T^{(\alpha)}}{\Gamma_q(\alpha+1)} < 1. \tag{3.7}$$

Then, the problem (1.1)-(1.2) has at least one solution.

Proof. Consider the operators

$$F_1, F_2 : \mathcal{C} \longrightarrow \mathcal{C}$$

Defined by:

$$(F_1 y)(t) = \begin{cases} 0, & t \in \overline{J} = [-d,0], \\ \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} f(s,y_s) d_q s, & t \in J = [0,T]. \end{cases}$$

And

$$(F_2 y)(t) = \begin{cases} \phi(t), & t \in \overline{J} = [-d,0], \\ \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} g(s,y_s) d_q s, & t \in J = [0,T]. \end{cases}$$

Then, the problem of finding the solution of the initial problem (1.1)-(1.2) is reduced to finding the solution of the operator equation $(F_1 y)(t) + (F_2 y)(t) = y(t)$, $t \in [-d,T]$.

Next, we shall show that the operators F_1 and F_2 satisfy all the conditions of Theorem 2.11. For better readability, we break the proof into a sequence of steps.
Step 1: F_1 *is contraction operator.*
Let $y, x \in \mathcal{C}$, if $t \in \overline{J}$, then we have:

$$|(F_1 y)(t) - (F_1 x)(t)| = 0.$$

Hence,

$$\|(F_1 y)(t) - (F_1 x)(t)\|_* = 0. \tag{3.8}$$

For $t \in J$, we have:

$$|(F_1 y)(t) - (F_1 x)(t)| = \left| \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} \left(f(s,y_s) - f(s,x_s)\right) d_q s \right|.$$

Therefore,

$$|(F_1 y)(t) - (F_1 x)(t)| \leq \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} |f(s,y_s) - f(s,x_s)| \, d_q s.$$

By hypothesis $(H2)$, we get:

$$\begin{aligned} |(F_1y)(t) - (F_1x)(t)| &\le \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} L_f\, |y_s - x_s|\, d_qs, \\ &\le \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} L_f\, \|y_s - x_s\|_*\, d_qs. \end{aligned}$$

Thus,

$$\|(F_1y)(t) - (F_1x)(t)\|_\infty \le \frac{L_f T^{(\alpha)}}{\Gamma_q(\alpha+1)} \|y - x\|_{\mathcal{C}}. \tag{3.9}$$

From equations (3.8) and (3.9), we conclude that:

$$\|(F_1y)(t) - (F_1x)(t)\|_{\mathcal{C}} \le \frac{L_f T^{(\alpha)}}{\Gamma_q(\alpha+1)} \|y - x\|_{\mathcal{C}}.$$

Consequently, the operator F_1 is contraction.
Step 2: *F_2 is continuous.*
Let $\{y_n\}_{n\in\mathbb{N}}$ be a sequence such that $y_n \to y$ in $\mathcal{C}$. If $t \in \overline{J}$, then we have:

$$|(F_2y_n)(t) - (F_2y)(t)| = |\varphi(t) - \varphi(t)| = 0.$$

Hence,

$$\|(F_2y_n)(t) - (F_2y)(t)\|_* = 0. \tag{3.10}$$

For each $t \in J$, we have:

$$|(F_2y_n)(t) - (F_2y)(t)| \le \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} |g(s, y_{ns}) - g(s, y_s)|\, d_qs.$$

Thus,

$$\|(F_2y_n)(t) - (F_2y)(t)\|_\infty \le \frac{T^{(\alpha)}}{\Gamma_q(\alpha+1)} \|g(.,y_{n.}) - g(.,y_.)\|_\infty .$$

Since g is a continuous function, we get:

$$\|F_2(y_n) - F_2(y)\|_\infty \to 0 \ \ as \ \ n \to \infty. \tag{3.11}$$

From equations (3.10) and (3.11), we conclude that:

$$\|F_2(y_n) - F_2(y)\|_{\mathcal{C}} \to 0 \ \ as \ \ n \to \infty.$$

Consequently, F_2 is continuous in $\mathcal{C}$.
Step 3: *F_2 maps bounded sets into bounded sets in $\mathcal{C}$.*
Indeed, it is enough to show that for any $r > 0$, there exists a positive constant R such that for each $y \in B_r = \{y \in \mathcal{C} : \|y\|_{\mathcal{C}} \le r\}$ we have $\|F_2(y)\|_{\mathcal{C}} \le R$.
Let $y \in B_r$. If $t \in \overline{J}$, then we have:

$$|(F_2y)(t)| = |\varphi(t)|.$$

Hence,

$$\|(F_2y)\|_* \le \|\varphi\|_*. \tag{3.12}$$

For each $t \in J$, we have:

$$\begin{aligned} |(F_2y)(t)| &= \left| \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} g(s,y_s) d_qs \right|, \\ &\leq \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} |g(s,y_s)| d_qs. \end{aligned}$$

By hypothesis $(H4)$, we get:

$$|(F_2y)(t)| \leq M_g \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} d_qs.$$

Thus,

$$\|(F_2y)\|_\infty \leq \frac{M_g T^{(\alpha)}}{\Gamma_q(\alpha+1)} := l. \tag{3.13}$$

From equations (3.12) and (3.13), we conclude that:

$$\|(F_2y)\|_{\mathcal{C}} \leq \max\{\|\varphi\|_*, l\} := R. \tag{3.14}$$

Consequently, the operator F_2 is uniformly bounded in B_r.

Step 4: *F_2 maps bounded sets into equicontinuous sets in $\mathcal{C}$.*

Let $t_1, t_2 \in J,\ t_1 < t_2$ and let B_r be a bounded set of $\mathcal{C}$ as in Step 2. Let $y \in B_r$, then we have:

$$\begin{aligned} |(F_2y)(t_2) - (F_2y)(t_1)| &= \left| \int_0^{t_2} \frac{(t_2-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} g(s,y_s) d_qs \right. \\ &\quad \left. - \int_0^{t_1} \frac{(t_1-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} g(s,y_s) d_qs \right|, \\ &= \left| \int_0^{t_1} \frac{\left((t_2-qs)^{(\alpha-1)} - (t_1-qs)^{(\alpha-1)}\right)}{\Gamma_q(\alpha)} g(s,y_s) d_qs \right. \\ &\quad \left. + \int_{t_1}^{t_2} \frac{(t_2-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} g(s,y_s) d_qs \right|, \\ &\leq \int_0^{t_1} \frac{\left((t_2-qs)^{(\alpha-1)} - (t_1-qs)^{(\alpha-1)}\right)}{\Gamma_q(\alpha)} |g(s,y_s)| d_qs \\ &\quad + \int_{t_1}^{t_2} \frac{(t_2-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} |g(s,y_s)| d_qs. \end{aligned}$$

By hypothesis $(H4)$, we get:

$$\begin{aligned} |(F_2y)(t_2) - (F_2y)(t_1)| &\leq \frac{M_g}{\Gamma_q(\alpha)} \int_0^{t_1} \left((t_2-qs)^{(\alpha-1)} - (t_1-qs)^{(\alpha-1)}\right) d_qs \\ &\quad + \frac{M_g}{\Gamma_q(\alpha)} \int_{t_1}^{t_2} (t_2-qs)^{(\alpha-1)} d_qs, \\ &\leq \frac{M_g}{\Gamma_q(\alpha+1)} \left(t_2^{(\alpha)} - t_1^{(\alpha)}\right). \end{aligned} \tag{3.15}$$

And if $t_1, t_2 \in \overline{J}$, then we have:

$$|(F_2y)(t_2) - (F_2y)(t_1)| = |\varphi(t_2) - \varphi(t_1)|. \tag{3.16}$$

The right hand sides of equations (3.15) and (3.16) tend to zero independently of $y \in B_r$ as $t_1 \to t_2$.
As a consequence of Steps 2 to 4, together with the Arzela-Ascoli theorem, we can conclude that the operator F_2 is completely continuous.
Step 5: *A priori bound.*
the set $\Omega = \{y \in \mathcal{C} : y = \lambda F_1\left(\frac{y}{\lambda}\right) + \lambda F_2(y)\}$ is bounded.
Let $y \in \Omega$, then $y = \lambda F_1\left(\frac{y}{\lambda}\right) + \lambda F_2(y)$ for some $0 < \lambda < 1$. If $t \in \overline{J}$, then

$$y(t) = \lambda\varphi(t).$$

For each $t \in J$, we have:

$$y(t) = \lambda \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} f(s, \frac{y_s}{\lambda}) d_qs + \lambda \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} g(s, y_s) d_qs.$$

Thus,

$$\begin{aligned}
|y(t)| &\leq \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} |f(s, \frac{y_s}{\lambda})| d_qs + \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} |g(s, y_s)| d_qs, \\
&\leq \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} |f(s, \frac{y_s}{\lambda}) - f(s,0)| d_qs + \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} |f(s,0)| d_qs \\
&\quad + \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} |g(s, y_s)| d_qs.
\end{aligned}$$

This implies by hypothesis $(H2)$ and $(H4)$ that for each $t \in J$, we get:

$$\begin{aligned}
|y(t)| &\leq \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} L_f |y_s| d_qs + \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} |f(s,0)| d_qs \\
&\quad + M_g \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} d_qs, \\
&\leq L_f \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} \|y_s\|_* d_qs + f^* \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} d_qs \\
&\quad + M_g \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} d_qs, \\
&\leq \frac{(M_g + f^*)T^{(\alpha)}}{\Gamma_q(\alpha+1)} + L_f \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} \|y_s\|_* d_qs,
\end{aligned}$$

where $f^* = \sup_{s \in J} |f(s,0)|$.
Now, we consider the function ρ defined by:

$$\rho(t) = \sup\{|y(s)| : 0 \leq s \leq t\}, \ t \in J.$$

Then, there exists $t^* \in [-d, t]$ be such that $\rho(t) = |y(t^*)|$.
If $t^* \in \overline{J}$, then

$$\rho(t) = \|\varphi\|_*. \tag{3.17}$$

If $t^* \in J$, then by the previous inequality we have for $t \in J$:

$$\rho(t) \leq \frac{(M_g + f^*)T^{(\alpha)}}{\Gamma_q(\alpha+1)} + L_f \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} \rho(s) d_q s.$$

Then, from Lemma 2.12, there exists $\delta = \delta(\alpha)$ such that we get:

$$\rho(t) \leq \frac{(M_g + f^*)T^{(\alpha)}}{\Gamma_q(\alpha+1)} + L_f \delta \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} \frac{(M_g + f^*)T^{(\alpha)}}{\Gamma_q(\alpha+1)} d_q s.$$

Thus,

$$\rho(t) \leq \frac{(M_g + f^*)T^{(\alpha)}}{\Gamma_q(\alpha+1)} \left(1 + \frac{L_f \delta T^{(\alpha)}}{\Gamma_q(\alpha+1)}\right) =: k. \tag{3.18}$$

Thus for any $t \in J$, $\|y\|_{\mathcal{C}} \leq \rho(t)$, from (3.17) and (3.18), we conclude that:

$$\|y\|_{\mathcal{C}} \leq \max(\|\varphi\|_{\mathcal{C}}, k).$$

This shows that the set Ω is bounded.
As a consequence of Theorem 3.4, we deduce that $F_1(y) + F_2(y)$ has a fixed point which is an integral solution of the problem (1.1)-(1.2). □

4. Ulam stability results

In this section, we will define and study some types of Ulam stability for problem (1.1)-(1.2). The following definitions were adopted from [31].

Definition 4.1. The problem (1.1)-(1.2) is Ulam-Hyers stable if there exists a real number $C > 0$ such that for each $\varepsilon > 0$ and for each solution $x \in \mathcal{C}$ of the following inequality:

$$|({}^C D_q^\alpha x)(t) - f(t, x_t) - g(t, x_t)| \leq \varepsilon, \; t \in J = [0, T], \tag{4.1}$$

there exists a solution $y \in \mathcal{C}$ of the problem (1.1)-(1.2) with the norm:

$$\|x - y\|_{\mathcal{C}} \leq C\varepsilon.$$

Definition 4.2. The problem (1.1)-(1.2) is generalized Ulam-Hyers stable if there exists $\vartheta \in C(\mathbb{R}_+, \mathbb{R}_+)$ with $\vartheta(0) = 0$, such that for each $\varepsilon > 0$, and for each solution $x \in \mathcal{C}$ of the inequality (4.1), there exists a solution $y \in \mathcal{C}$ of the problem (1.1)-(1.2) with the norm:

$$\|x - y\|_{\mathcal{C}} \leq \vartheta(\varepsilon).$$

Definition 4.3. The problem (1.1)-(1.2) is Ulam-Hyers-Rassias stable with respect to ϕ if there exists $C_\phi > 0$ such that for each $\varepsilon > 0$ and for each solution $x \in \mathcal{C}$ of the following inequality:

$$|({}^C D_q^\alpha x)(t) - f(t, x_t) - g(t, x_t)| \leq \varepsilon\phi(t), \; t \in J = [0, T], \tag{4.2}$$

there exists a solution $y \in \mathcal{C}$ of the problem (1.1)-(1.2) with the norm:

$$\|x - y\|_{\mathcal{C}} \leq C_\phi \varepsilon \phi(t), \; t \in J = [0, T].$$

Remark 4.4. A function $x \in \mathcal{C}$ is a solution of the inequality

$$|({}^C D_q^\alpha x)(t) - f(t, x_t) - g(t, x_t)| \leq \varepsilon, \ t \in J = [0, T],$$

if and only if there exists a function $k \in C([0,T], \mathbb{R})$ (which depend on y) such that:

(i) $|k(t)| \leq \varepsilon, \ t \in J = [0, T]$.
(ii) $({}^C D_q^\alpha x)(t) = f(t, x_t) + g(t, x_t) + k(t), \ t \in J = [0, T]$.

Theorem 4.5. *Assume that the hypotheses* $(H1)$*-*$(H2)$*-*$(H3)$ *and condition (3.4) are satisfied. Then, the problem (1.1)-(1.2) is Ulam-Hyers stable.*

Proof. Let $\varepsilon > 0$ and let $x \in \mathcal{C}$ be a solution of the inequality (4.1) and let $y \in \mathcal{C}$ be the solution of the problem (1.1)- (1.2). Then, we have:

$$y(t) = \begin{cases} \phi(t), & t \in \overline{J} = [-d, 0], \\ \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} f(s, y_s) d_q s + \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} g(s, y_s) d_q s, & t \in J = [0, T]. \end{cases}$$

From the inequality (4.1) for each $t \in J$, we obtain:

$$\begin{aligned} \left| x(t) - \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} f(s, x_s) d_q s - \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} g(s, x_s) d_q s \right| &\leq I_q^\alpha \varepsilon, \\ &\leq \frac{t^{(\alpha)}}{\Gamma_q(\alpha+1)} \varepsilon. \end{aligned}$$

Using the hypotheses $(H1)$-$(H2)$ and $(H3)$, for each $t \in J$, we can write:

$$\begin{aligned} |x(t) - y(t)| &\leq \left| x(t) - \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} f(s, y_s) d_q s - \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} g(s, y_s) d_q s \right|, \\ &\leq \left| x(t) - \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} f(s, x_s) d_q s - \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} g(s, x_s) d_q s \right. \\ &\quad + \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} \left(f(s, x_s) - f(s, y_s) \right) d_q s \\ &\quad \left. + \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} \left(g(s, x_s) - g(s, y_s) \right) d_q s \right|, \\ &\leq \left| x(t) - \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} f(s, x_s) d_q s - \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} g(s, x_s) d_q s \right| \\ &\quad + \left| \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} \left(f(s, x_s) - f(s, y_s) \right) d_q s \right. \\ &\quad \left. + \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} \left(g(s, x_s) - g(s, y_s) \right) d_q s \right|, \end{aligned}$$

Thus,

$$\begin{aligned}|x(t)-y(t)| &\leq \frac{t^{(\alpha)}}{\Gamma_q(\alpha+1)}\varepsilon+\int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)}\left|f(s,x_s)-f(s,y_s)\right| d_qs \\ &\quad +\int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)}\left|g(s,x_s)-g(s,y_s)\right| d_qs, \\ &\leq \frac{t^{(\alpha)}}{\Gamma_q(\alpha+1)}\varepsilon+L_f\int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)}\left\|x_s-y_s\right\|_* d_qs \\ &\quad +L_g\int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)}\left\|x_s-y_s\right\|_* d_qs.\end{aligned}$$

Hence,

$$\|x-y\|_{\mathcal{C}} \leq \frac{T^{(\alpha)}}{\Gamma_q(\alpha+1)}\varepsilon+\frac{(L_f+L_g)\,T^{(\alpha)}}{\Gamma_q(\alpha+1)}\|x-y\|_{\mathcal{C}}.$$

By condition (3.4), we get:

$$\begin{aligned}\|x-y\|_{\mathcal{C}} &\leq \frac{\frac{T^{(\alpha)}}{\Gamma_q(\alpha+1)}}{1-\frac{(L_f+L_g)T^{(\alpha)}}{\Gamma_q(\alpha+1)}}\varepsilon, \\ &:= C\varepsilon.\end{aligned}$$

Consequently, the problem (1.1)-(1.2) is Ulam-Hyers stable.
Taking $\vartheta(\varepsilon)=C\varepsilon$; $\vartheta(0)=0$, we can state that the problem (1.1)-(1.2) is generalized Ulam-Hyers stable. □

Theorem 4.6. *Assume that the hypotheses* $(H1)$*-*$(H2)$*-*$(H3)$ *and condition (3.4) are satisfied and*

$(H5)$ *Let* $\phi\in C(J,R_+)$ *be an increasing function. There exists* $\lambda_\phi>0$ *such that for each* $t\in J$*, we have:*

$$I_q^\alpha\phi(t)\leq\lambda_\phi\phi(t).$$

Then, problem (1.1)-(1.2) is Ulam-Hyers-Rassias stable.

Proof. Let $\varepsilon>0$ and let $x\in\mathcal{C}$ be a solution of the inequality (4.2) and let $y\in\mathcal{C}$ be the solution of the problem (1.1)-(1.2). Then, we have:

$$y(t)=\begin{cases}\phi(t), & t\in\overline{J}=[-d,0], \\ \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)}f(s,y_s)d_qs+\int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)}g(s,y_s)d_qs, & t\in J=[0,T].\end{cases}$$

From the inequality (4.2) and $(H5)$, for each $t\in J$, we obtain:

$$\begin{aligned}\left|x(t)-\int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)}f(s,x_s)d_qs-\int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)}g(s,x_s)d_qs\right| &\leq \varepsilon I_q^\alpha\phi(t), \\ &\leq \varepsilon\lambda_\phi\phi(t).\end{aligned}$$

Using the hypotheses $(H1)$-$(H2)$ and $(H3)$, for each $t \in J$, we can write:

$$\begin{aligned}
|x(t)-y(t)| &\leq \left| x(t) - \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} f(s,y_s)d_qs - \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} g(s,y_s)d_qs \right|, \\
&\leq \left| x(t) - \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} f(s,x_s)d_qs - \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} g(s,x_s)d_qs \right. \\
&\quad + \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} \left(f(s,x_s)-f(s,y_s)\right) d_qs \\
&\quad \left. + \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} \left(g(s,x_s)-g(s,y_s)\right) d_qs \right|, \\
&\leq \left| x(t) - \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} f(s,x_s)d_qs - \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} g(s,x_s)d_qs \right| \\
&\quad + \left| \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} \left(f(s,x_s)-f(s,y_s)\right) d_qs \right. \\
&\quad \left. + \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} \left(g(s,x_s)-g(s,y_s)\right) d_qs \right|, \\
&\leq \varepsilon\lambda_\phi\phi(t) + \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} \left|f(s,x_s)-f(s,y_s)\right| d_qs \\
&\quad + \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} \left|g(s,x_s)-g(s,y_s)\right| d_qs, \\
&\leq \varepsilon\lambda_\phi\phi(t) + L_f \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} \|x_s-y_s\|_* \, d_qs \\
&\quad + L_g \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} \|x_s-y_s\|_* \, d_qs.
\end{aligned}$$

Hence,

$$\|x-y\|_{\mathcal{C}} \leq \varepsilon\lambda_\phi\phi(t) + \frac{(L_f+L_g)\,T^{(\alpha)}}{\Gamma_q(\alpha+1)}\|x-y\|_{\mathcal{C}}.$$

By condition (3.4), we get:

$$\begin{aligned}
\|x-y\|_{\mathcal{C}} &\leq \frac{\varepsilon\lambda_\phi\phi(t)}{1-\frac{(L_f+L_g)T^{(\alpha)}}{\Gamma_q(\alpha+1)}} \\
&:= C_\phi\varepsilon\phi(t).
\end{aligned}$$

Consequently, the problem (1.1)-(1.2) is Ulam-Hyers-Rassias stable. □

5. Examples

In this section, we give two examples illustrating our main results.

Example 5.1. Consider the following initial value problem of perturbed functional fractional q-difference equations:

$$({}^{C}D^{\frac{2}{3}}_{\frac{1}{3}}y)(t) = 1 + 2t + \frac{\sin(y_t)}{e^{-t}+1} + \frac{e^{-t}\sin(y_t)}{t+6}, \quad t \in J = [0,1],\ 0 < \alpha \leq 1, \tag{5.1}$$

$$y(t) = t^2,\ t \in \overline{J} = [-1,0], \tag{5.2}$$

where $\varphi(t) = t^2$ and $\alpha = \frac{2}{3}$, $q = \frac{1}{3}$, $d = 1$, $T = 1$, and

$$f(t,y) = 1 + 2t + \frac{\sin(y)}{e^{-t}+1},\ (t,y) \in J \times \mathbb{R},$$

$$g(t,y) = \frac{e^{-t}\sin(y)}{t+6},\ (t,y) \in J \times \mathbb{R}.$$

Clearly, the function f, g are continuous.
Let $y, x \in \mathbb{R}$ and $t \in J$. Then, we have:

$$\begin{aligned} |f(t,y) - f(t,x)| &= \left|\frac{\sin(y) - \sin(x)}{e^{-t}+1}\right|, \\ &\leq \frac{1}{2}|y - x|, \\ |g(t,y) - g(t,x)| &\leq \frac{1}{6}|y - x|. \end{aligned}$$

Hence, the hypothesis $(H2)$-$(H3)$ are satisfied with $L_f = \frac{1}{2}$ and $L_g = \frac{1}{6}$. Now, we shall check that the condition (3.4) is satisfied with $T = 1$. Indeed,

$$\begin{aligned} \frac{(L_f + L_g)\,T^{(\alpha)}}{\Gamma_q(\alpha+1)} &= \frac{\left(\frac{1}{2} + \frac{1}{6}\right)}{\Gamma_{\frac{1}{3}}(\frac{5}{3})}, \\ &= 0.7028 < 1. \end{aligned}$$

Then, by Theorem 3.3, the problem (5.1)-(5.2) has a unique solution on $[-1,1]$, and from Theorem 4.5, the problem (5.1)-(5.2) is Ulam-Hyers stable on $[0,1]$.
On the other hand, we have:

$$|g(t,y)| \leq \frac{1}{6},\ (t,y) \in J \times \mathbb{R}.$$

Thus, the condition $(H4)$ holds. Next, we shall check that the condition (3.7) is satisfied with $T = 1$. Indeed,

$$\begin{aligned} \frac{L_f T^{(\alpha)}}{\Gamma_q(\alpha+1)} &= \frac{1}{2\Gamma_{\frac{1}{3}}(\frac{5}{3})}, \\ &= 0.5271 < 1. \end{aligned}$$

Then, by Theorem 3.4, the problem (5.1)-(5.2) has at least one solution on $[-1,1]$.

Example 5.2. Consider the following initial value problem of perturbed functional fractional q-difference equations:

$$({}^C D^{\frac{1}{2}}_{\frac{1}{4}} y)(t) = \frac{t^2}{e^t+5}\left(1+\frac{|y_t|}{1+|y_t|}\right) + \frac{t^2 sin(y_t)}{3}, \quad t \in J = [0,1],\ 0 < \alpha \leq 1, \quad (5.3)$$

$$y(t) = \frac{t}{6+t},\ t \in \overline{J} = [-2,0], \quad (5.4)$$

where $\varphi(t) = \frac{t}{6+t}$ and $\alpha = \frac{1}{2}$, $q = \frac{1}{2}$, $d = 2$, $T = 1$, and

$$f(t,y) = \frac{t^2}{e^t+5}\left(1+\frac{|y|}{1+|y|}\right),\ (t,y) \in J \times \mathbb{R},$$

$$g(t,y) = \frac{t^2 sin(y)}{3} \in J \times \mathbb{R}.$$

Clearly, the function f, g are continuous.
Let $y, x \in \mathbb{R}$ and $t \in J$. Then, we have:

$$\begin{aligned} |f(t,y) - f(t,x)| &= \left|\frac{1}{e^t+5}\left(\frac{|y|}{1+|y|} - \frac{|x|}{1+|x|}\right)\right|, \\ &\leq \frac{1}{6}|y-x|, \\ |g(t,y) - g(t,x)| &\leq \frac{1}{3}|y-x|. \end{aligned}$$

Hence, the hypothesis $(H2)$-$(H3)$ are satisfied with $L_f = \frac{1}{6}$ and $L_g = \frac{1}{3}$. Now, we shall check that the condition (3.4) is satisfied with $T = 1$. Indeed,

$$\begin{aligned} \frac{(L_f + L_g)\, T^{(\alpha)}}{\Gamma_q(\alpha+1)} &= \frac{\left(\frac{1}{6}+\frac{1}{3}\right)}{\Gamma_{\frac{1}{4}}(\frac{3}{2})}, \\ &= 0.5275 < 1. \end{aligned}$$

Then, by Theorem 3.3, the problem (5.3)- (5.4) has a unique solution on $[-2,1]$, and from Theorem 4.5, the problem (5.3) is Ulam-Hyers stable on $[0,1]$.
On the other hand, we have:

$$|g(t,y)| \leq \frac{1}{3},\ (t,y) \in J \times \mathbb{R}.$$

Thus, the condition $(H4)$ holds. Next, we shall check that the condition (3.7) is satisfied with $T = 1$. Indeed,

$$\begin{aligned} \frac{L_f T^{(\alpha)}}{\Gamma_q(\alpha+1)} &= \frac{1}{6\Gamma_{\frac{1}{4}}(\frac{3}{2})}, \\ &= 0.1758 < 1. \end{aligned}$$

Then, by Theorem 3.4, the problem (5.3)-(5.4) has at least one solution on $[-2,1]$.
Now, let $\phi(t) = t^2$ for each $t \in J$, we have:

$$I^{\frac{1}{2}}_{\frac{1}{4}}\phi(t) = \frac{\Gamma_{\frac{1}{4}}(3)}{\Gamma_{\frac{1}{4}}(\frac{7}{2})} t^{2+\frac{1}{2}} \leq \frac{5}{4\Gamma_{\frac{1}{4}}(\frac{7}{2})} t^2 = \lambda_\phi \phi(t). \quad (5.5)$$

Thus, the condition $(H5)$ is satisfied with $\phi(t) = t^2$ and $\lambda_\phi = \frac{5}{4\Gamma_{\frac{1}{4}}(\frac{7}{2})}$. Then, it follows from Theorem 4.6 that the problem (5.3)- (5.4) is Ulam-Hyers-Rassias stable on $[0, 1]$.

6. Conclusions

In this work, we have provided sufficient conditions for the existence of solutions for the initial value problem (IVP for short) for perturbed functional fractional q-difference equations involving the Caputo's fractional q-derivative. The uniqueness result is obtained by applying the Banach contraction mapping principle, while the existence result is obtained by using Burton and Kirk's fixed point theorem. In addition, we presented some results for Ulam-Hyers stability and Ulam-Hyers-Rassias stability. For the justification, examples are we given to illustrate the main results.

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