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Advanced versions of the inverse function theorem

David Shoikhet

Dedicated to the memory of Professor Gabriela Kohr

Abstract. This short opus is dedicated to the bright memory of the distinguished mathematician *Gabriela Kohr* and her mathematical heritage. *Gabriela Kohr*'s contribution to analysis of one and several complex variables brought new knowledge into the modern theory as well as new colors to the subject. During our meetings with Gabriela at various conferences she always proposed some interesting and often nonstandard questions related to classical issues as well as new directions. It is worth to be mentioned her excellent book [50] together with Ian Graham on classical and modern problems in Geometric Function Theory in complex spaces (see also, [46], [56], [27], [22], [24], [49] and [48]).

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1. Introduction

A number of the results presented in this manuscript is based mostly on the joint and works with Filippo Bracci, Mark Elin, Victor Khatskevich, Marina Levenstein, Simeon Reich and Toshiyuki Sugawa [58], [14], [19], [35], [33], [68], [95] as well as addendum from Vladimir Mazi'ya and Gregory Kresin [64], [63], [66] who had many joint mathematical interests with *Gabriela Kohr*. Also we would like to mentioned a great contribution to the theory of semigroups of holomorphic mappings and complex dynamical systems developed by Leonardo Arosio, Filippo Bracci, Manuel D. Contreras and Santiago Diaz-Madrigal and Hidetaka Hamada (see [7], [12], [14], [13] and references therein).

A deep understanding and knowledge of *Gabriela Kohr* in various topics related to generalizations of the Loewner chains to higher dimensions is presented in her joint book in with Ian Graham [50]. In a parallel way Filippo Bracci, Manuel D.

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Contreras, Santiago Diaz-Madrigal and Hidetaka Hamada [15], [13], [20], [17] and [7] have developed geometrical aspects of this theory; some of them were probably waft by Gabrela's work. They have produced very interesting questions and problems (as well as their solutions) which in our opinion will give a push for further investment the Loewner Theory to general complex analysis. In particular, Leandro Arosio, Filippo Bracci, Hidetaka Hamada and *Gabriela Kohr* [7] have presented a new geometric construction of Loewner chains in one and several complex variables which holds on a complete hyperbolic complex manifold M and proved that there is essentially a one-to-one correspondence between evolution families of order d and Loewner chains of the same order. As a consequence they obtained a solution for any Loewner-Kufarev PDE, given by univalent mappings.

Finally, we would like to highlight some questions and problems inspired by *Gabriela Kohr* which for the one-dimensional case have discussed and developed independently by Mark Elin and Fiana Jacobson [31] and [30].

As far as we will see below that actually the inverse function is an element of the so-called *resolvent family* of a discrete (or continuous) semigroup of holomorphic mappings. By using this fact and previous investigations in [39] and [40] in order to answer some *Gabriela Kohr's* questions one can employ the results in [31] to establish new features of nonlinear resolvents of holomorphic generators of one-parameter semigroups acting in the open unit disk. Since the class of nonlinear resolvents consists of univalent functions, it can be studied in the frameworks of classical and modern geometric function theories. In this way in works [31] and [30] the authors establish some distortion and covering results as well as pointed out the order of starlikeness and strong starlikeness of resolvents. It is shown that any resolvent admits quasiconformal extension to the complex plane $\mathbb{C}$. Also, they obtain some characteristics of semigroups generated by these resolvents.

Also, we have to mention a recent work of Xiu-Shuang Ma, Saminathan Ponnusamy and Toshiyuki Sugawa on spirallikeness and strongly starlikeness of harmonic functions.

2. Preliminary notions and results

It often happens in mathematics, in examinations of classical issues, that one can discover (sometimes surprisingly) a number of renewed problems and questions.

In this short survey we trace some traits and relationships between invertibility and the numerical range of holomorphic mappings in the one dimensional and (partially) higher dimensional cases.

It is well known that for holomorphic mappings in Banach spaces the *Inverse Function Theorem, the Implicit Function Theorem and the Fixed Point Theorem* are closely related each other.

The classical *Inverse Function Theorem* says.

Theorem 2.1. *Let* $\mathbb{X}$ *be a complex Banach space and let* F *be a holomorphic mapping in a neighborhood of the origin such that*

$$F(0) = 0 \text{ and } F'(0) \text{ is the invertible linear operator on } \mathbb{X}.$$

Then there are positive numbers r and ρ such that $F(\mathcal{B}_r) \supset \mathcal{B}_\rho$ and $F^{-1} : \mathcal{B}_\rho \to \mathcal{B}_r$ is a well defined holomorphic mapping on D_ρ .These numbers r and ρ are often called the Bloch radii for F (cf. for example, [52], [53], [55], [82], [50], [35] *and* [37]*).*

Note in passing that the pair (r, ρ) is not uniquely defined. See details in [59], [55], [50], [82], [35] and [37]. This manuscript in a sense can be considered an additional chapter to the book [37].

Let $\mathbb{X}^*$ denote the dual of the Banach space $\mathbb{X}$ and let $\langle z, z^* \rangle$ denote the duality pairing of $z^* \in \mathbb{X}^*$ and $z \in \mathbb{X}$. For each $z \in \mathbb{X}$ the set $J(z)$ defined by

$$J(z) = \left\{ z^* \in \mathbb{X}^* : \ \langle z, z^* \rangle = \|z\|^2 = \|z^*\|^2 \right\}$$

is not empty by virtue of the Hanh-Banach theorem and is a closed and convex bounded subset of $\mathbb{X}^*$. The mapping $J : z \mapsto z^*$ is in general multi-valued, however it is single-valued if $\mathbb{X}^*$ is strictly convex. For a Hilbert space $\mathbb{X} = \mathbb{H}$ the semi-scalar product $\langle \cdot, \cdot \rangle$ in $\mathbb{X} * \mathbb{X}$, can be just identify with the standard inner product in $\mathbb{H}$.

Let D and Ω be domains in $\mathbb{X}$ and let $Hol(D, \Omega)$ be the set of all holomorphic mappings on D with values in Ω. If $D = \Omega$, then we list write $Hol(D)$ for the set $Hol(D, D)$ of holomorphic self-mappings of D.

2.1. Holomorphically accretive and dissipative mappings

Let $\mathbb{X}$ be a complex Banach space with its dual $\mathbb{X}^*$. By $\langle \cdot, \cdot \rangle$ we denote the semi-scalar product in $\mathbb{X} * \mathbb{X}^*$, so that $\langle z, z^* \rangle = \|z\|^2$ and $|\langle z, w^* \rangle| \leq \|z\| \, \|w\|$.

Let $\mathcal{B}$ be the open unit ball in $\mathbb{X}$ and let $f : \mathcal{B} \to \mathbb{X}$ be a *holomorphic* mapping on $\mathcal{B}$.

Definition 2.2. (cf. [34]) Let $f \in \mathrm{Hol}(\mathcal{B}, \mathbb{X})$. We say that f is (holomorphically) accretive on $\mathcal{B}$ if

$$\overline{\lim_{s \to 1^-}} \inf \mathrm{Re} \, \langle f_s(z), z^* \rangle \geq \varepsilon \geq 0,$$

where $f_s(z) = f(sz)$, $0 \leq s < 1$, $\|z\| = 1$. It is called to be *strongly (holomorphically) accretive* on $\mathcal{B}$ if $\varepsilon > 0$. Respectively, a holomorphic mapping $g : \mathcal{B} \to \mathbb{X}$ is called *(holomorphically) dissipative if* $f = -g$ is *(holomorphically) accretive* on $\mathcal{B}$.

We call these conditions *one side estimates* (see, for example, [3]).

Let $f \in Hol(\mathcal{B}, \mathbb{X})$ admit a continuous extension onto $\overline{\mathcal{B}}$-the closure of $\mathcal{B}$ and be such that

$$\mathrm{Re} \, \langle f(z), z^* \rangle \geq 0$$

for all $z \in \partial\mathcal{B}$ -the boundary of $\mathcal{B}$. Then $f : \mathcal{B} \to \mathbb{X}$ is obviously *(holomoprphically) accretive* on $\mathcal{B}$. *It is strongly holomorphically accretive if*

$$\mathrm{Re} \, \langle f(z), z^* \rangle \geq \varepsilon > 0, \ z \in \partial\mathcal{B}.$$

In this connection we recall the Bohl - Poincare´- Krasnoselskii fixed point theorem.

Theorem 2.3. [62] *Let $\mathcal{B}$ be the open unit ball of a real Hilbert space $\mathbb{H}$ with the inner product $\langle \cdot, \cdot \rangle$ and let $\Phi : \overline{\mathcal{B}} \to \mathcal{B}$ be a* completely continuous (compact) *mapping on $\overline{\mathcal{B}}$ (not necessarily holomorphic). If condition*

$$\langle \Phi(z), z \rangle \leq 1, \ z \in \partial\mathcal{B},$$

holds, then Φ *has at least one fixed point in* $\overline{\mathcal{B}}$*. If* $\langle \Phi(z), z \rangle < 1$*,* $z \in \partial\mathcal{B}$*, then* Φ *has a unique fixed point in* $\mathcal{B}$*.*

Analogously, if $\mathcal{B}$ is the open unit ball in a complex Hilbert space $\mathbb{H}$ and $f : \mathcal{B} \to \mathbb{H}$ is holomorphically accretive (respectively, dissipative) *completely continuous vector field which does not vanish on* $\partial\mathcal{B}$ then it has at least one null point in $\mathcal{B}$. With some additional restrictions a similar result holds also for Banach spaces.

Theorem 2.3 has many applications to the solvability of nonlinear equations. One-sided estimates of such type have been systematically used in many fields. For example, in [62] it is mentioned Galerkin's approximation methods, the theory of equations with potential operators, monotone operator theory and nonlinear integral and partial differential equations. One of the main points in Theorem 2.3 is, of course, the compactness of the mapping Φ (or more generally the complete continuity of the vector field defined by $I - \Phi$) which allows us to use the methods of the rotation theory of vector fields or degree theory [62]. Since we are interested in the class of holomorphic vector fields, we note that in *infinite dimensional spaces* this class is not contained in the class of completely continuous vector fields. Moreover, in this case the intersection of these classes is quite narrow.

Despite this lack of compactness, there exists a well-developed fixed point theory for holomorphic mappings in Hilbert spaces and Banach spaces (see, for example, [3], [13], [37], [35], [44], [45], [66] and [50]). In particular, for a complex Hilbert space one can reach more information.

Theorem 2.4. [4] *Let* $\mathbb{H}$ *be a complex Hilbert space and let* $\mathcal{B}$ *be the open unit ball in* $\mathbb{H}$*. Suppose that* f *is a holomorphic mapping in* $\mathcal{B}$ *which has a uniformly continuous extension onto* $\overline{\mathcal{B}}$ *and satisfies the boundary condition*

$$\operatorname{Re}\langle f(z), z\rangle \geq 0, \quad \textit{respectively,} \quad \operatorname{Re}\langle f(z), z\rangle \leq 0,$$

for all $z \in \partial\mathcal{B}$*. The following assertions hold:*

1. Null $f/\overline{\mathcal{B}} \neq \varnothing$;
2. If Null $f/\mathcal{B} \neq \varnothing$, *then it is an affine sub-manifold of* $\mathcal{B}$*.*

Corollary 2.5. *If* f *satisfies one of the above boundary conditions and has no null point on* $\partial\mathcal{B}$*, then it has a unique null point in* $\mathcal{B}$*. In particular, if* $\operatorname{Re}\langle f(z), z\rangle > 0$*, (respectively,* $\operatorname{Re}\langle f(z), z\rangle < 0$ *),* $z \in \partial\mathcal{B}$*, then* f *has a unique null point in* $\mathcal{B}$*.*

2.2. One sided estimates in Banach spaces

Let $\mathcal{B}$ be the open unit ball in a complex Banach space $\mathbb{X}$. The following result can be easily obtained from [4] (Theorem 3).

Theorem 2.6. *Let* $f : \mathcal{B} \to \mathbb{X}$ *be a holomorphic mapping on* $\mathcal{B}$ *which admits a uniformly continuous extension to the boundary* $\partial\mathcal{B}$*. Assume also that* f *is* strongly holomorphically accretive on $\mathcal{B}$*. Then* f *has a unique null point in* $\mathcal{B}$*.*

Clearly this result can be rephrased in the terms of fixed points. To do this we first recall the following version of the famous Earle-Hamilton Theorem [29]: for the unit ball in a complex Banach space.

If $F \in Hol(\mathcal{B})$ is such that $F(\mathcal{B})$ lies strictly inside $\mathcal{B}$, that is

$$\inf\left(\|F(x) - y\| \geq \sigma > 0,\ x \in \mathcal{B},\ y \in \partial\mathcal{B}\right),$$

then F has a unique fixed point in $\mathcal{B}$.

The standard proof of this theorem is based on the construction of pseudo-metric ρ on $\mathcal{B}$ such that F is a strict contraction with respect to ρ, i.e.,

$$\rho(F(x), F(y)) \leq k\rho(x, y)$$

for some $k \in (0, 1)$. For some generalized versions of the Earl-Hamilton Theorem and additional information see [54] and references therein.

Theorem 2.7. [91] *(cf.* [85]*) Let G be a holomorphic self-mapping of $\mathcal{B}$ which admits a uniformly continuous extension onto the boundary $\partial\mathcal{B}$ and satisfies the following boundary condition*

$$\mathrm{Re}\langle G(z), z^*\rangle \leq 1 - \delta,$$

for some $\delta > 0$ and all $z \in \partial\mathcal{B}$. Then G has a unique fixed point in $\mathcal{B}$.

It can be easily seen that for the unit ball in a complex Banach space theorem 2.7 is a generalization of the Earle-Hamilton Theorem. For more details see also [55]. The latter theorem can be also extended to a wider class of pseudo-contractive mappings [68], [19] and [37].

The results in Theorem 2.6 and Corollary 2.5 can be completed as follows.

Theorem 2.8. [91] *Let $\mathcal{B}$ be the open unit ball in a complex Banach space $\mathbb{X}$ and let $F : \mathcal{B} \to \mathbb{X}$ be a holomorphic mapping on $\mathcal{B}$. Assume that F admits a continuous extension onto $\overline{\mathcal{B}}$ and for some $\varepsilon > 0$ the condition of* **strong accretivity** *holds:*

$$\mathrm{Re}\langle F(z), z^*\rangle \geq \varepsilon, \quad z \in \partial\mathcal{B}.$$

Then the inverse mapping $z(w) = F^{-1}(w)$ is well-defined and holomorphic on the ball $\|w\| < \varepsilon$. In other words, the numbers $R = 1$ and $r = \varepsilon$ are the Bloch radii for F. Moreover, for each $w : \|w\| < \varepsilon$ and $z_0 \in \mathcal{B}$ the sequence $z_{n+1} = z_n - F(z_n) + w$, $n = 0, 1, ...,$converges locally uniformly to this solution $z(w)$ on $\mathcal{B}$.

Clearly this result implies the classical inverse function theorem mentioned above. Earlier results in this theme see also in [16].

In the next part we get down to the one-dimensional case which itself has non-standard particular qualities and has been developed in various directions. We start this part quoting the estimates suggested in [65]. Actually those evaluations can be also obtained by employing the results below.

Assume that $F : \Delta \to \mathbb{C}$ is a holomorphic mapping on the open unit disk Δ normalized by the conditions $F(0) = 0$ and $F'(0) = 1$ and admits a continuous extension onto $\partial\Delta$ the boundary of Δ.

Define a holomorphic mapping f on Δ by

$$f(z) = z - F(z),$$

so that $f(0) = 0$ and $f'(0) = 0$.

We also suppose that for some real numbers θ and N the following condition is satisfied

$$\max_{z\in\partial\Delta} \operatorname{Re}\left(e^{i\theta} f(z)\,\overline{z}\right) \le N.$$

Theorem 2.9. *Under the above conditions the inverse mapping mapping F^{-1} is a well-defined holomorphic mapping in the disk*

$$\Omega = \left\{ w \in \mathbb{C}\colon\ |w| < \left((1+2N)^{\frac{1}{2}} - (2N)^{\frac{1}{2}}\right)^2 \right\}$$

and satisfies the following modulus estimate

$$\left|F^{-1}(w)\right| \le 1 - \left(\frac{2N}{1+2N}\right)^{\frac{1}{2}}.$$

Thus, $\Phi(N) = \left((1+2N)^{\frac{1}{2}} - (2N)^{\frac{1}{2}}\right)^2$ is a lower estimate for the Bloch radius R_B.

Remark 2.10. Recently G. Kresin by using some more delicate calculations has shown that the latter estimate can be improved as follows.

$$R_B \ge \pi^{-1}\left((4N+\pi)^{\frac{1}{2}} - 2N^{\frac{1}{2}}\right)^2 > \left((1+2N)^{\frac{1}{2}} - (2N)^{\frac{1}{2}}\right)^2.$$

Further we discuss some geometrical aspects of the inverse functions.

Definition 2.11. Let $\mathcal{D}$ be a circular domain in $\mathbb{X}$. A locally biholomorphic mapping $G : \mathcal{D} \to \mathbb{X}$, $G(0) = 0$, is called star-like if for each $z \in \mathcal{D}$ and $t \in [0,1]$ the line $tG(z) \in G(\mathcal{D})$.

In the one dimensional case where $\mathcal{D} = \cdot$ -the open unit disk in $\mathbb{C}$, a locally univalent mapping $G : \cdot \to \mathbb{C}$, $G(0) = 0$, is star-like if and only if it satisfies the inequality

$$\frac{zG'(z)}{G(z)} \ge \alpha \ge 0.$$

If $\alpha > 0$ the mapping G is characterized as *star-like of order α* (see additional details and certain sources in [57] and [50], [51], [94], [97], [98] and [35].

Theorem 2.12. *(cf.* [40]*) Let $f \in \operatorname{Hol}(\Delta,\mathbb{C})$ with $f'(0) \ne 0$. Then there exist numbers r $(0 < r < 1)$ such that f^{-1} is a star-like self-mapping of Δ_r.*

In a parallel way for a domain D in $\mathbb{C}$ we consider the *resolvent equation*

$$z - \lambda f(z) = w \tag{2.1}$$

where, in general, λ and w are elements in $\mathbb{C}$.

Definition 2.13. One says that $f \in \operatorname{Hol}(\Delta,\mathbb{C})$ is a *locally semi-complete vector field* on Δ if there is r $(0 < r \le 1)$ such that for each $w : |w| < r$ and each $\lambda > 0$ the equation $z - \lambda f(z) = w$ has a unique solution $z = \Phi(\lambda, w) \in \Delta_r$. If $r = 1$, then f is just said to be *semi-complete* on Δ.

Remark 2.14. Another way to define a semi-complete vector field is through ordinary differential equations and the Cauchy problem (see, the next Section). As a matter of fact, for bounded convex domains both definitions are equivalent.

Remark 2.15. Sometimes it is more convenient in place of equation (2.1) to define the resolvent family $\Psi_\lambda\,(=\Psi\,(\lambda,w))$ by solving equation $z+\lambda f\,(z)=w$, $\lambda>0$, $|w|<r\le 1$ (see, for example, [82], [80] and [35]).

Clearly, if $h\in \mathrm{Hol}(\Delta,\mathbb{C})$ with $h\,(0)=0$ and $h'\,(0)\ne -1$ is a locally semi-complete vector field on Δ, then setting $f\,(z)=z+h\,(z)$ we get that f^{-1} exists and is an element of the resolvent family $\{\mathcal{J}_\lambda\}_{\lambda\ge 0}$ with $\lambda=1$.

We conclude these observations with a simple consequence of the classical maximum principle (or, alternatively, the classical Schwarz Lemma) to deduce the following relations.

Lemma 2.16. *Let Δ be the open unit disk in $\mathbb{C}$ and let $h\in \mathrm{Hol}(\Delta,\mathbb{C})$ with $h\,(0)=0$. Assume that*

$$\sup_{z\in\Delta} \mathrm{Re}\, h\,(z)\,\overline{z}=N<\infty. \tag{2.2}$$

Then

$$L=\mathrm{Re}\, h'\,(0)\quad\le N.$$

Theorem 2.17. [91] *Let h be holomorphic in the open unit disk with $h\,(0)=0$, and let*

$$N=\sup_{z\in\Delta} h\,(z)\,\overline{z}<\infty.$$

Then h is a locally semi-complete vector field if and only if the following condition holds:

$$L=h'\,(0)<\min\{0,N\}\,.$$

In this case h is semi-complete on each disk of radius $r\in\left(0,\dfrac{-L}{2N-L}\right)$.

3. Semi-complete vector fields and semigroups

Let $\mathbb{X}$ be a complex Banach space and let $\mathbb{X}^*$ denote the dual of the Banach space $\mathbb{X}$ and let $\langle z,z^*\rangle$ denote the duality pairing of $z^*\in\mathbb{X}^*$ and $z\in\mathbb{X}$. For each $z\in\mathbb{X}$ the set $J(z)$ defined by

$$J(z)=\left\{z^*\in\mathbb{X}^*:\ \langle z,z^*\rangle=\|z\|^2=\|z^*\|^2\right\} \tag{3.1}$$

is not empty by virtue of the Hahn-Banach theorem and is a closed and convex bounded subset of $\mathbb{X}^*$.

The mapping $J:z\mapsto z^*$ is in general multi-valued, however it is single-valued if $\mathbb{X}^*$ is strictly convex.

Let D be a domain in $\mathbb{X}$ and let $\mathrm{Hol}(D,\mathbb{X})$ be the set of all holomorphic mappings on D with values in $\mathbb{X}$.

Definition 3.1. A mapping $f\in Hol(D,\mathbb{X})$ is said to be a semi-complete vector field on D if the Cauchy problem

$$\begin{cases} \frac{\partial u(t,z)}{\partial t}+f(u(t,z))=0\\ u(0,z)=z \end{cases} \tag{3.2}$$

has a unique solution $u=u(t,z)\in D$ for all $z\in D$ and $t\ge 0$.

Furthermore, one can show (see [82] and [86]) that the function $u(t,z)$ satisfies the following partial differential equation

$$\frac{\partial u(t,z)}{\partial t}+\frac{\partial u(t,z)}{\partial z}f(z)=0, \quad z\in D.$$

Definition 3.2. A mapping f is a *complete* vector field if the solution of (3.2) exists for all $t\in\mathbb{R}$ and $z\in D$.

In other words, f is complete if both f and $-f$ are semi-complete.

Note also that f is complete if and only if this solution $\{u(t,\cdot)\}$ of (3.2) is a group (with respect to the parameter $t\in(-\infty,\infty)$) of automorphisms of D.

The set of semi-complete vector fields on D will be denoted by $\mathcal{G}(D)$. The set of complete vector fields is denoted by $\mathcal{G}_{aut}(D)$.

Various presentations of semi-complete and complete vector fields on the open unit ball $\mathcal{B}$ in $\mathbb{C}^n$, general Hilbert and Banach spaces can be found in [1], [2], [3], [86], [82], [35] and [37].

It is well known that in the case where $D=\Delta=\{z\in\mathbb{C}:|z|<1\}$, a semi-complete vector field is complete if and only if it admits the representation

$$f(z)=a-\overline{a}z^2+ibz$$

for some complex number a and real b (see, for example, [86] and [35]).

If a family $\{F_t=u(t,\cdot)\}$, $t\geq 0$, $(t\in R)$ forms a semigroup (group) of holomorphic self-mappings of Δ, it follows from the remarkable result of E. Berkson and H. Porta [9] that the limit

$$f(z)=\lim_{t\to 0^+}\frac{1}{t}\left(z-F_t(z)\right)$$

exists and defines a semi-complete (complete) vector field f on Δ.Clearly $f\in\mathrm{Hol}(\Delta,\mathbb{C})$ and determines the holomorphic generator of $\{F_t\}$ via the above formula.

In general, if D is a convex domain in $\mathbb{X}$ and the latter limit exists one can identify the set $\mathcal{G}(D)$ of semi-complete vector fields with the set of all holomorphic generators on D. The set $\mathcal{G}(D)$ is a real cone in $\mathrm{Hol}(D,\mathbb{C})$, while the set $\mathcal{G}_{aut}(D)$ of all group generators on D is a real Banach algebra (see [2] and [82]).

We observe also, that for some $z_0\in D$, the equality $F_t(z_0)=z_0$ holds for all $t\geq 0$ if and only if $f(z_0)=0$.

Definition 3.3. Let D be a domain in $\mathbb{X}$ and let $h\in\mathrm{Hol}(D,\mathbb{X})$. One says that h satisfies the **range condition** on D if for each $\lambda\geq 0$ the following condition holds $(I-\lambda h)(D)\supset D$ and the equation

$$z-\lambda h(z)=w \tag{3.3}$$

has a unique solution

$$z=\mathcal{J}_\lambda(w)\left(=(I-\lambda h)^{-1}(w)\right) \tag{3.4}$$

holomorphic in $w\in D$.

In this case the family $\{\mathcal{J}_\lambda\}_{\lambda\geq 0}\in\mathrm{Hol}(D)$ is called the *resolvent family* of h on D. Obviously, the inverse function $(I-h)^{-1}$ is an element of the resolvent family with $\lambda=1$.

Theorem 3.4. [82] *Let D be a bounded convex domain in $\mathbb{X}$ and let $f \in \mathrm{Hol}(D, \mathbb{X})$. The mapping f defines a semi-complete vector field on D if and only if it satisfies the range condition of Definition* 3.3.

For the one-dimensional case we list the following geometric properties of the resolvent established in [39] and [40] among others:

- Any resolvent J_λ is a *hyperbolically convex* self-mapping of Δ and, consequently, is a star-like function of order $\frac{1}{2}$ (see definition 2.11 and sources in [75], [69], [70], [72], [71] *and* [93]).
- Any resolvent J_λ satisfies $\mathrm{Re}\,\frac{J_\lambda}{z} > \frac{1}{2(1+\lambda f'(0))}$. Consequently, J_λ is a generator on Δ and, moreover, the semigroup generated by J_λ converges to 0 uniformly on Δ with exponential squeezing coefficient $\kappa = 1/[2(1+\lambda f'(0))]$.
- If a generator f itself is a star-like function of order $\alpha > \frac{1}{2}$, then any element J_λ, $\lambda \geq 0$, of the resolvent family extends to a $(\sin \pi\alpha)$-quasiconformal mapping of $\mathbb{C}$.

Quantitative characteristics of semi-complete vector fields can be formulated as follows.

Theorem 3.5. *Let $\mathcal{B}$ be the open unit ball in $\mathbb{X}$ and let $f \in Hol\,(\mathcal{B}, \mathbb{X})$. Then $f \in \mathcal{G}(\mathcal{B})$ if and only if one of the following conditions hold:*

(i) Abate's inequality [1]*:*

$$\mathrm{Re}\left[2\left\langle f(z), z^*\right\rangle + \left\langle f'(z)z, z^*\right\rangle\left(1 - \|z\|^2\right)\right] \geq 0.$$

(ii) Aharonov-Reich-Elin-Shoikhet's [2] *criterion:*

$$\mathrm{Re}\,\left\langle f(z), z^*\right\rangle \geq \mathrm{Re}\,\left\langle f(0), z^*\right\rangle\left(1 - \|z\|^2\right),\ z \in \mathcal{B}.$$

Note that condition (i) originally was establish by Abate in [1] for the finite dimensional Euclidian ball. For the general Banach space it was shown in [2] (see also [82]) by using the reduction to the unit disk Δ in the complex plane $\mathbb{C}$.

Let $\mathcal{B}$ be the open unit ball in $\mathbb{X}$. Following G. Kohr let us denote by $\mathbf{D}$ the invariant differential operator on $\mathrm{Hol}(\mathcal{B}, \mathbb{X})$ defined by

$$\mathbf{D}f(z) = (1 - \|z\|^2)f'(z).$$

Remark 3.6. For the one-dimensional case operator $\mathbf{D}$ has the property that $\mathbf{D}f\,(z) = (f \circ T)'\,(0)$, where T is the automorphism on Δ given by $T\,(w) = \frac{z+w}{1+\bar{z}w}$, $w \in \Delta$. Thus, if $\mathbf{D}$ is the invariant differential operator on $\mathrm{Hol}(\Delta, \mathbb{C})$ defined by the above formula, condition (i) of Theorem 3.5 can be written as

$$\mathrm{Re}\left[2f(z)\overline{z} + \mathbf{D}f(z)\,|z|^2\right] \geq 0.$$

So, in this case the set $\mathcal{G}(\Delta)$ can be described by the last inequality given in Remark 3.6.

Another very useful representation of the class $\mathcal{G}(\Delta)$ was obtained by Berkson and Porta in [9].

Theorem 3.7. *If Δ is the open unit disk in the complex plane $\mathbb{C}$, then $f \in \mathcal{G}(\Delta)$ if and only if*

$$f(z) = (z - \tau)(1 - z\overline{\tau})p(z) \tag{3.5}$$

for some $\tau \in \overline{\Delta}$ and $p \in Hol\,(\Delta, \mathbb{C})$ with $\operatorname{Re} p(z) \geq 0$, $z \in \Delta$.

*The equivalence of conditions (i)-(ii) and (*3.5*) by using direct complex analysis methods was shown in* [2].

In addition, we notice that since presentation (3.5) is unique it follows that $f \in \mathcal{G}(\Delta)$ *must have at most one null point in* Δ.

This fact is no longer true for the higher dimensional case. If, in particular, $\mathbb{X}$ is a reflexive Banach space, then the null point set of $f \in \mathcal{G}(\mathcal{B})$ is a holomorphic retract of $\mathcal{B}$, whence a connected analytic submanifold of D (see [82] and references therein). In particular, for $\mathbb{X} = \mathbb{H}$ being a complex Hilbert space, the null point set of a semi-complete vector field is an affine submanifold of $\mathbb{X}$. In any case, it follows by the uniqueness of the solution of the Cauchy problem (3.2) that the null point set of $f \in \mathcal{G}(\mathcal{B})$ coincides with the common fixed point set of the generated semigroup $S = \{u(t, \cdot)\}_{t=0}^{\infty}$. In particular, $f \in \mathcal{G}(\mathcal{B})$ has a unique null point $\tau \in \mathcal{B}$ if and only if $\tau\,(= u(t, \tau))$ is a unique fixed point of $u(t, \cdot)$ for at least one, hence, for all $t > 0$.

This point τ is referred to be the *Denjoy-Wolff point* for the semigroup $S = \{u(t, \cdot)\}_{t \geq 0}$ generated by f, if

$$\lim_{t \to \infty} u(t, x) = \tau, \text{ for each } x \in \mathcal{B}.$$

It is known (see, for example, [82] and reference therein) that $\tau \in \mathcal{B}$ is the Denjoy-Wolff point of S if and only if the *spectrum $\sigma\,(A)$ of the linear operator $A = f'\,(0)$ lies in the open right-half plane.*

Proposition 3.8. [82]*Let D be a bounded convex domain in $\mathbb{X}$ and let $f \in \mathcal{G}(D)$. Then the null point set of f in D is a connected analytic submanifold of D.*

Now by $\mathcal{N}\,(\mathcal{B})$ we denote the class$\{f \in \mathcal{G}(\mathcal{B}) : f(0) = 0,\ \operatorname{Re}\sigma\,(f'\,(0)) > 0\}$. In other words, $\mathcal{N}\,(\mathcal{B})$ consists of those semi-complete vector fields which generate the semigroups with the Denjoy-Wolff point at the origin. This class of generators is closely related to the class $S^*\,(\mathcal{B})$ of star-like mappings or, more generally, the class $Sp\,(\mathcal{B})$ of spiral-like mappings on $\mathcal{B}$ (see, for example, [87] and [37]).

Namely, $h \in Sp(\mathcal{B})$ if and only if it is locally biholomorphic and satisfies the differential equation

$$Ah(x) = h'(x) \cdot f(x),$$

where $f \in \mathcal{N}\,(\mathcal{B})$ and $A = f'\,(0)$. In particular, h is star-like if and only if operator A can be chosen $A = I$ - the identity operator on $\mathbb{X}$ (see, for details the books [47], [86], [50], [82] and [37]). For the geometric description of the convex hull of the set $S^*\,(\cdot)$ see a pioneer work [21] (see also a recent works [26] and [42]).

For the finite dimensional case $\mathbb{X} = \mathbb{C}^n$ the subclass $\mathcal{M}\,(\mathcal{B}) = \{f \in \mathcal{N}(\mathcal{B}) : f(0) = 0,\ f'\,(0) = I\}$ of $\mathcal{N}\,(\mathcal{B})$ was studied by *Gabriela Kohr* (see [60], [61], [50] and references therein). In particular, the following result was presented in [50].

Theorem 3.9. *If $\mathbb{X} = \mathbb{C}^n$, then the set $\mathcal{M}\,(\mathcal{B})$ is compact.*

To motivate our further discussion we note that for the one-dimensional case characterizations of the class $\mathcal{N}(\mathcal{B})$ can be written as:

(a') Re $\left[2f(z)\overline{z}+\mathbf{D}f(z)\left|z\right|^2\right]\geq 0$ and $f(0)=0$

or

(b') Re $f(z)\overline{z}\geq 0,\ z\in\Delta\ \left(or\ \operatorname{Re}\frac{f(z)}{z}\geq 0,\ z\neq 0\right)$.

Surprisingly, it turns out, that formally much weaker condition than condition (a'), namely,

$$\operatorname{Re} f'(z)\geq 0,\quad f(0)=0, \tag{3.6}$$

also implies condition (b), that is the property of f to be a semi-complete vector field on Δ. The class of functions satisfying (3.6) is a well-known class consisting of univalent functions due to the Noshiro-Warshawskii Theorem (see [50], [37]). We mention *interalia* the following result.

Theorem 3.10. [90] *Let* $f\in Hol(\mathcal{B},\mathbb{C})$, $f(0)=f'(0)-I=0$ *satisfy the generalized Noshiro-Warshawskii condition:*

$$\operatorname{Re}\left\langle f'(x)x,x^*\right\rangle\geq 0,\ x\in\mathcal{B}. \tag{3.7}$$

Then f is a strongly semi-complete vector field satisfying

$$\operatorname{Re}\left\langle f(x),x^*\right\rangle\geq(2\log 2-1)\left\|x\right\|^2>0,\ \ x\in\mathcal{B}. \tag{3.8}$$

This inspires us to consider some more general classes of holomorphic mappings defined by a convex combination of conditions (a) and (b) of Theorem 3.5.

The following question (G. Kohr) naturally rises from Theorem 3.5 and Remark 3.6.

Whether the condition,

$$\operatorname{Re}\left[\alpha\left\langle f(x),x^*\right\rangle+\left\langle f'(x)x,x^*\right\rangle(1-\left\|x\right\|^2)\right]\geq 0,$$
$$x\in\mathcal{B},\ f(0)=0,\text{and }\alpha\geq 0 \tag{3.9}$$

also characterizes the class $\mathcal{N}(\mathcal{B})$?

The answer is affirmative for all $\alpha\geq 2$. At the same time, condition (3.9) is sufficient, but is not necessary [14].

Following condition (3.9) at the end of this section we consider some special subclasses of $\mathcal{N}(\mathcal{B})$ which define the so-called parametric filtration of the class $\mathcal{N}(\mathcal{B})$ ([14] and [39]).

For $0\leq t\leq 1$ we denote $\mathcal{G}_t(\mathcal{B})$ the class which consists of functions $f\in\operatorname{Hol}(\mathcal{B},\mathbb{X})$ such that $f(0)=0$ and

$$\operatorname{Re}\left[t\left\langle f(x),x^*\right\rangle+(1-t)\left\langle f'(x)x,x^*\right\rangle(1-\left\|x\right\|^2)\right]\geq 0.$$

Theorem 3.11. *For each* $0\leq s\leq t\leq 1$ *the following inclusions hold*

$$\mathcal{G}_s(\mathcal{B})\subseteq\mathcal{G}_t(\mathcal{B})\subseteq\mathcal{N}(\mathcal{B}). \tag{3.10}$$

Moreover,

(i) For all $\frac{2}{3}\leq s\leq t\leq 1$ *the following equality holds*

$$\mathcal{G}_s(\mathcal{B})=\mathcal{G}_t(\mathcal{B})=\mathcal{N}(\mathcal{B}).$$

(ii) For each $0\leq s<t\leq\frac{2}{3}$ *the inclusion* $\mathcal{G}_s(\mathcal{B})\subset\mathcal{G}_t(\mathcal{B})$ *is strong.*

4. Null points of holomorphic generators in the disk

First we summarize some preliminary properties of continuous semigroups and their generators, which follow from the Berkson–Porta representation (i) of Theorem 3.5.

Consider a semigroup $S = \{F_t\}_{t\geq 0} \subset \mathrm{Hol}(\Delta)$ generated by $f \in \mathcal{G}(\Delta)$ and make the following observations.

- ⋄ If the point τ in [9] is an interior null point of Δ and f does not vanish identically on Δ, then τ is the unique null point of f in Δ, and (due to the uniqueness of the solution to the Cauchy problem (3.2)), τ is a common fixed point of S, i.e.,

$$F_t(\tau) = \tau \quad \text{for all} \quad t \geq 0. \tag{4.1}$$

- ⋄ If $\tau \in \partial\Delta$, then it is a fixed point of F_t for each $t \geq 0$ in the sense that

$$\lim_{r\to 1^-} F_t(r\tau) = \tau. \tag{4.2}$$

In general, if S is not the trivial semigroup of the identity mappings and does not contain an elliptic automorphism of Δ, then the point $\tau \in \overline{\Delta}$ in (4.2) is an attractive fixed point of the semigroup S, i.e.,

$$\lim_{t\to\infty} F_t(z) = \tau \quad \text{for all} \quad z \in \Delta. \tag{4.3}$$

The last assertion is a continuous analog of the *Denjoy–Wolff Theorem.*

Definition 4.1. The point τ in (4.3) is called the **Denjoy–Wolff point** of the semigroup $S = \{F_t\}_{t\geq 0}$.

To proceed we need the following notions.

Definition 4.2. One says that a function $f \in \mathrm{Hol}(\Delta, \mathbb{C})$ has the **angular limit** L at a point $\tau \in \partial\Delta$ denoted by $L := \angle \lim\limits_{z\to\tau} f(z)$ if $f(z) \to L$ as $z \to \tau$ in each nontangential approach region

$$\Gamma(\tau, k) = \left\{ z \in \Delta : \frac{|z-\tau|}{1-|z|} < k \right\}, \quad k > 1.$$

Definition 4.3. If L in definition 4.2 is finite and the angular limit (finite or infinite)

$$M := \angle \lim_{z\to\tau} \frac{f(z) - L}{z - \tau}$$

exists, then M is said to be the **angular derivative** of f at τ. We denote it by $f'(\tau)$.

It is known (see [76] p. 79) that this angular derivative exists finitely if and only if the angular limit $\angle \lim\limits_{z\to\tau} f'(z)$ exists finitely,hence $f'(\tau) = \angle \lim\limits_{z\to\tau} f'(z)$.

Remark 4.4. By using the Riesz–Herglotz representation (see, for example, [47]) of functions with a positive real part, one can show (see [37] and [15]) that if $\tau \in \partial\Delta$ is the boundary Denjoy–Wolff point of the semigroup $S = \{F_t\}_{t\geq 0}$ generated by $f \in \mathcal{G}(\Delta)$, then the angular derivative $f'(\tau)$ exists finitely and is a real nonnegative number. Moreover, $f'(\tau) = \lim_{r\to 1^-} \frac{f(r\tau)\overline{\tau}}{r-1}$. Some inequalities for angular derivatives related to interpolation problems are given in [11]. The second angular derivative and parabolic iteration were studied in [26].

Thus, every non-trivial semigroup $S = \{F_t\}_{t\geq 0}$ on Δ which does not contain an elliptic automorphism of Δ, falls in one of three mutually exclusive classes depending on the nature of its Denjoy–Wolff point τ. These classes can be described in terms of generators as follows: the Denjoy–Wolff point τ of S satisfies $f(\tau) = 0$ and the semigroup S must be of one of the following three types:

– **dilation type** if $\tau \in \Delta$ and $\operatorname{Re} f'(\tau) > 0$;
– **hyperbolic type** if $\tau \in \partial\Delta$ and $0 < f'(\tau) < \infty$;
– **parabolic type** if $\tau \in \partial\Delta$ and $f'(\tau) = 0$.

The real part $\operatorname{Re} f'(\tau)$ vanishes at an interior null point $\tau \in \Delta$ of a generator $f \in \mathcal{G}(\Delta)$ if and only if the semigroup S generated by f contains either the identity mappings or elliptic automorphisms of Δ.

Without loss of generality up to appropriate Möbius transformations of the unit disk we distinguish two cases: $\tau = 0$ and $\tau = 1$.

4.1. Interior null points

Let f be the generator of a one-parameter continuous semigroup $S = \{F_t\}_{t\geq 0}$ on Δ. Suppose that S is not trivial, does not contain elliptic automorphisms, and that $\tau \in \Delta$ is the interior null point of f. Without loss of generality we set $\tau = 0$. In this case $\tau = 0$ is the attractive fixed point of the semigroup, $\Re f'(\tau) > 0$, and the rate of convergence of the semigroup in terms of the Euclidean distance is completely determined by the following theorem (see, for example, [86]).

Theorem 4.5. *Let $f \in \mathcal{G}(\Delta)$ be such that $f(0) = 0$ and $\lambda := \operatorname{Re} f'(0) > 0$, and let $S = \{F_t\}_{t\geq 0}$ be the semigroup generated by f. Then there exists $c \in [0,1]$ such that for all $z \in \Delta$ and $t \geq 0$, the following estimates hold:*

$$(i)\ \ |z| \cdot \exp\left(-\lambda t \frac{1 + c|z|}{1 - c|z|}\right) \leq |F_t(z)| \leq |z| \cdot \exp\left(-\lambda t \frac{1 - c|z|}{1 + c|z|}\right);$$

$$(ii)\ \ \exp(-\lambda t)\frac{|z|}{(1 + c|z|)^2} \leq \frac{|F_t(z)|}{(1 - c\,|F_t(z)|)^2} \leq \exp(-\lambda t)\frac{|z|}{(1 - c|z|)^2}.$$

Inequality (ii) implies that for each $z \in \Delta$ the rate of convergence of the semigroup to its interior Denjoy–Wolff point is of exponential type.

Note that for $c = 1$ estimate (i) is due to Gurganus [51], while estimate (ii) was established by Poreda [77].

4.2. Boundary null points

Note that a generator $f \in \mathcal{G}(\Delta)$ may have more than one boundary null point, and for each such point $\zeta \in \partial\Delta$, the angular derivative $f'(\zeta)$ exists and is a real number or infinity (see [86], [33], [25] and [35]).

Definition 4.6. A point $\zeta \in \partial\Delta$ is called a **boundary regular null point** of $f \in \operatorname{Hol}(\Delta, \mathbb{C})$ if the angular (radial) derivative $f'(\zeta)$ exists finitely.

In fact, a boundary regular null point ζ of f is the attractive fixed point of the semigroup S generated by f if and only if $f'(\zeta) \geq 0$. If for a boundary null point $\zeta \in \partial\Delta$ of f, $f'(\zeta) < 0$ then ζ is a **repelling (or repulsive) fixed point** of S (see [36]-[32]).

For a point $\zeta \in \overline{\Delta}$, we define the class

$$\mathcal{G}[\zeta] := \{f \in \mathcal{G}(\Delta) : \ f(\zeta) = 0 \quad \text{and} \quad f'(\zeta) \quad \text{exists finitely}\}. \tag{4.4}$$

In other words, the class $\mathcal{G}[\zeta]$ is the subcone of $\mathcal{G}(\Delta)$ of all generators vanishing at the point ζ, and having a finite (angular) derivative at this point.

For a boundary point $\zeta \in \partial\Delta$, each element f of $\mathcal{G}[\zeta]$ has another useful parametric representation.

Theorem 4.7. **(***see* [86] *cf. also* [87] *and* [82]**)** *Let $\zeta \in \partial\Delta$. Then $f \in \mathcal{G}[\zeta]$ admits the representation*

$$f(z) = (z-\zeta)(1-z\overline{\zeta})p(z) + \frac{\lambda}{2}(\overline{\zeta}z^2 - \zeta), \tag{4.5}$$

where

$$\lambda = f'(\zeta), \quad \operatorname{Re} p\,(z) \geq 0 \quad \text{ and } \quad \angle \lim_{z\to\zeta} \left(1 - z\overline{\zeta}\right) p\,(z) = 0. \tag{4.6}$$

It is clear that the point ζ in (4.5) is the Denjoy–Wolff point of the corresponding semigroup if and only if $\lambda \geq 0$.

In general, it turns out, that even for a boundary regular null point τ of f, which is not necessarily the Denjoy–Wolff point, but $f \in \mathcal{C}^3_A(\tau)$ (i.e., f has the third angular derivative at the point τ, its quadratic part, say g, is also a generator of a semigroup of linear-fractional transformations on Δ. Therefore, the natural question is: *which conditions provide $f = g$?*

Theorem 4.8. *Let $f \in \mathcal{G}[1]$ be of class $\mathcal{C}^3_A(1)$ and let $g(z) = f'(1)(z-1)+\frac{1}{2}f''(1)(z-1)^2$ be its quadratic part. Then*

(i) g is the generator of a semigroup of linear-fractional transformations on Δ;

(ii) $f'(1) - \operatorname{Re} f''(1) \geq 0$;

(iii) If $h := f - g$ belongs to the class $\mathcal{G}(\Delta)$ then $\operatorname{Re} f'''(1) \geq 0$. Moreover, $\operatorname{Re} f'''(1) = 0$ if and only if $f = g$.

In particular, $f(z) \equiv 0$ if and only if $f'(1) = f''(1) = f'''(1) = 0$.

Since for a self-mapping F of Δ the mapping $I - F$ defines a semi-complete vector field (generator) on Δ, the latter assertion is a generalization of the Burns-Krantz Theorem [23].

In this connection we also would like to mention that the classical Shwarz Lemma and Shawrz-Pick Lemma are the prototype of earlier rigidity results. Recently Filippo Bracci, Daniela Kraus and Oliver Roth [20] have continue the study and developments of this issue and established several versions for conformal pseudometrics on the unit disk including boundary versions of Ahlfors-Schwarz and Nehari-Schwarz Theorems, as well as for holomorphic self-mappings of strongly convex domains in $\mathbb{C}^n$.

The so-called "slice rigidity property" of holomorphic mappings Kobayashi-isometrically preserving complex geodesics have been given by Filippo Bracci, Łukasz Kosiński, Włodzimierz Zwonek [18] More precisely.

Let Δ be the unit disc in $\mathbb{C}$ and let $F : \Delta \to \mathbb{C}$ be a Riemann mapiing such that $F(\Delta) = \Delta$. Then it was presented a necessary and sufficient condition in terms of hyperbolic distance and horocycles which assures that a compactly divergent sequence $\{z_n\} \subset \Delta$ has the property that $\{f^{-1}(z_n)\}$ converges orthogonally to a point of Δ.

In this connection we also mentioned the papers [15] and [20].

In addition, verifying the proofs in [43] and [36] one shows that the point $w = 0$ is the boundary regular fixed point of the restriction of $\mathcal{J}_r$ on Ω whenever $r \in \left(-\frac{1}{f'(0)}, 0\right)$ with $\mathcal{J}_r'(0) = \frac{1}{1+rf'(0)}$.To illustrate Proposition 6.2 take Example 2 in [40] and consider the semigroup generator $f(z) = z(1-z)$. Its resolvent $\mathcal{J}_r$ is:

$$\mathcal{J}_r(w) = \frac{r+1-\sqrt{(r+1)^2-4rw}}{2r}$$

First we see that the angular limit $\angle \lim\limits_{w\to 1} \mathcal{J}_r(w) = 1$ for every $r < 0$ and $\mathcal{J}_r'(1) = \frac{1}{1-r}$ for every $r \in (-\infty, 1)$. In addition, $\mathcal{J}_r(0) = 0$ if $r \geq -1$ and $\mathcal{J}_r'(0) = \frac{1}{1+r}$ for $r > -1$. Finally, it can be seen by using the results in [43] that the maximal BFID corresponding to $\zeta = 1$ is $\Omega = \left\{z : \left|z - \frac{1}{2}\right| < \frac{1}{2}\right\}$.

5. Analytic extension of one-parameter semigroups

In this section we discuss the following problem.

Let $\{F_t\}_{t\geq 0}$ be the semigroup generated by an $f \in \mathcal{G}(\Delta)$. Whether there is a domain $\mathbb{Q}$ in the right half-plane such that the semigroup admits an analytic extension to $\mathbb{Q}$, preserving it's algebraic properties?

For the case when $f = A$ is a continuous linear operator on X an affirmative answer was given by E. Hille [29] see also the book of Hille-Phillips [56].

Proposition 5.1. *Let B be the infinetesimal generator of a semigroup of linear continuos operators $\{A_t\}$, $t \geqq 0$, on X such that $A_t X \subset D(B)$ −the domain of definition of B.Assume that there is a constant $N > 0$ with $t\,\|BA_t\| \leq N$, $0 \leq t < 1$.Then there is a holomorphic operator function $\{A_\varsigma\} : \varsigma \in D$, where $\mathbb{Q} = \left\{\varsigma : \operatorname{Re}\varsigma > 0,\ \ |\arg\varsigma| < \frac{1}{eN}\right\}$ and $A_{\varsigma_1}A_{\varsigma_2} = A_{\varsigma_1+\varsigma_2}$ whenewer ς_1 and ς_2 belong to D. In addition, the strong limit $\lim_{\varsigma\to 0} A_\varsigma x = x$, $x \in D$, whenever $|\arg\varsigma| \leq \frac{\varepsilon}{eN}$, $\varepsilon \in (0,1)$.*

Remark 5.2. By using the tools and methods of the theory composition operators (see, for example, [27], [92]) one can easily establish analogs of this result for nonlinear semigroup of holomorphic mappings [37].

The following fact is a key for our considerations in the sequel.

Theorem 5.3. [31] *Let $\alpha, \beta \in (0, \frac{\pi}{2})$. Then the semigroup $\{F_t\}_{t\geq 0}$ generated by f, $f(z) = zp(z)$, can be analytically extended to the sector $\{t \in \mathbb{C} :\ \arg t \in (-\alpha, \beta)\}$ for all z in the open unit disk Δ if and only if $-\frac{\pi}{2} + \alpha < \arg p(z) < \frac{\pi}{2} - \beta,\ z \in \mathbb{D}$.*

We deduce here another result from [31] which completes the material in previous sections.

Theorem 5.4. *Let f be a semi-complete vector field and let $\mathcal{J}_r$ be its resolvent with $r \geq \frac{6}{\operatorname{Re} q}$. Denote $\gamma_r := \frac{1-A(r\operatorname{Re} q)}{1+A(r\operatorname{Re} q)}$, where A is defined by*

$$A(r) := \frac{6r(1+r)}{(1+r)^3 - 3(5r-1)} \tag{5.1}$$

Then for the semigroup $\{\Phi_{t,r}\}_{t\geq 0}$ generated by J_r the following assertions hold:

(i) *for fixed $t > 0$ the net $\{\Phi_{t,r}\}$ converges to 0 as $r \to \infty$, uniformly on the open unit disk with the exponential squeezing coefficient*

$$\kappa(r) := \frac{\left(\operatorname{Re}(1+rq)^{\frac{1}{\gamma_r}}\right)^{\gamma_r}}{2^{1-\gamma_r}|1+rq|^2}.$$

Theorem 5.5. *For every fixed z in the open unit disk, and $r > 0$ the mapping $\Phi_{t,r}(z)$ can be analytically extended in the parameter t to the sector*

$$\left\{t \in \mathbb{C} : |\arg t - \arg(1+rq)| < \frac{\pi\gamma_r}{2}\right\}.$$

6. Backward flow invariant domains

To proceed we quote partially the result proved in [43] (see also [37]).

Lemma 6.1. *A function $f \in \mathcal{N}$ has a boundary regular null point $\zeta \in \partial\Delta$ if and only if there is a simply connected domain $\Omega \subset \Delta$ such that f generates a one-parameter group $S = \{F_t\}_{-\infty<t<\infty}$ of hyperbolic automorphisms on Ω such that the points $z = 0$ and $z = \zeta$ belong to $\partial\Omega$ and are boundary regular fixed points of S on $\partial\Omega$. Moreover, $f'(\zeta)$ is a real negative number.*

We call such a domain *backward flow invariant domain (or shortly BFID)*. Note that in general a BFID Ω is not unique for a point $\zeta \in \partial\Delta$, but there is a unique BFID Ω (called the maximal BFID) with the above properties such that Ω has a corner of opening π at the point ζ (see [76]). Other characterizations of backward flow invariant domains can be found in [43], [36], [35] and [13].

An interesting phenomenon occurs when we consider the resolvent family only on BFID. Namely,

Proposition 6.2. *Let $f \in \mathcal{N}(\Delta)$ have a boundary regular null point $\zeta \in \partial\Delta$ and Ω is a BFID in Δ corresponding to ζ. If Ω is convex, then the restriction of the resolvent family $\mathcal{J}_r$ on Ω can be continuously extended in the parameter $r \in (-\infty, 0)$ such that ζ is a boundary fixed point of $\mathcal{J}_r$ for every $r < 0$. Moreover, $\lim\limits_{r\to-\infty} \mathcal{J}_r(w) = \zeta$ whenever $w \in \Omega$.*

Lemma 6.3. *A function $f \in \mathcal{N}(\Delta)$ has a boundary regular null point $\zeta \in \partial\Delta$ if and only if there is a simply connected domain $\Omega \subset \Delta$ such that f generates a one-parameter group $S = \{F_t\}_{-\infty<t<\infty}$ of hyperbolic automorphisms on Ω such that the points $z = 0$ and $z = \zeta$ belong to $\partial\Omega$ and are boundary regular fixed points of S on $\partial\Omega$. Moreover, $f'(\zeta)$ is a real negative number.*

It follows from the Scwarz Lemma that $F_t'^{-tf'(0)} < 1$ while $F_t'^{-tf'(\zeta)} > 1$.

Thus the point $w = 0$ is a boundary regular fixed point of the restriction of $\mathcal{J}_r$ on Ω whenever $r \in \left(-\frac{1}{f'(0)}, 0\right)$ with $\mathcal{J}_r'(0) = \frac{1}{1+rf'(0)}$.

To illustrate Proposition 6.2 and the latter fact, return now to the semigroup generator $f(z) = z(1-z)$ and its resolvent

$$\mathcal{J}_r(w) = \frac{r+1-\sqrt{(r+1)^2-4rw}}{2r}$$

that were considered in [39] and [40]. First we see that the angular limit $\angle \lim\limits_{w\to 1} \mathcal{J}_r(w) = 1$ for every $r < 0$ and $\mathcal{J}_r'(1) = \frac{1}{1-r}$ for every $r \in (-\infty, 1)$. In addition, $\mathcal{J}_r(0) = 0$ if $r \geq -1$ and $\mathcal{J}_r'(0) = \frac{1}{1+r}$ for $r > -1$. Finally, it can be shown by using the results in [43] that the maximal BFID corresponding to $\zeta = 1$ is $\Omega = \left\{z : \left|z - \frac{1}{2}\right| < \frac{1}{2}\right\}$.

$$\operatorname{Re} \frac{w\mathcal{J}_r'(w)}{\mathcal{J}_r(w)} = \operatorname{Re} \frac{1}{1 - w\varphi'\left(\mathcal{J}_r\right)} > \frac{1}{2}\,.$$

7. Inverse Löwner chains

Theorem 3.11 tells us that $\Omega_r = \mathcal{J}_r(\Delta),\ 0 \leq r < \infty$, is a decreasing family of domains in the unit disk Δ (in this connection see also [20]).

One can thus introduce some aspects of Inverse Löwner theory which lead to a deeper geometric understandings of the structure of the family of nonlinear resolvents for $f \in \mathcal{N}(\Delta)$.

Definition 7.1. A mapping $p : \Delta \times [0, +\infty) \to \mathbb{C}$ is called a *Herglotz function* of divergence type if the following three conditions are satisfied:

(a) $p_t(z) = p(z,t)$ is analytic in $z \in \Delta$ and measurable in $t \geq 0$,
(b) $\operatorname{Re} p(z,t) > 0$ $(z \in \Delta,\ t \geq 0)$,
(c) $p(0,t)$ is locally integrable in $t \geq 0$ and

$$\int_0^\infty \operatorname{Re} p(0,t)dt = +\infty.$$

Note that the term *Herglotz function of order d* is used in [12] to mean the function $p(z,t)$ with the divergence condition being replaced by $L^d([0,\infty))$-convergence in the above definition.

The following result was proved by Becker [8].

Theorem 7.2. *Let $p(z,t)$ be a Herglotz function of divergence type. Then there exists a unique solution $f_t(z) = f(z,t)$, (which is analytic and univalent in $|z| < 1$ for each $t \in [0, +\infty)$ and locally absolutely continuous in $0 \leq t < \infty$ for each $z \in \Delta$) to the differential equation*

$$\dot{f}(z,t) = zf'(z,t)p(z,t) \quad (z \in \Delta,\ t \geq 0) \tag{7.1}$$

with the normalization conditions $f_0(0) = 0$ and $f_0'(0) = 1$. Moreover, the solution satisfies $f_s \prec f_t$ for $0 \leq s \leq t$.

Here we have written

$$\dot{f}(z,t) = \frac{\partial}{\partial t} f(z,t), \qquad f'(z,t) = \frac{\partial}{\partial z} f(z,t).$$

for the partial derivatives of $f(.,.)$. Observe that the uniqueness assertion is no longer valid if we drop the univalence condition on f_t. For instance, one can consider the function $\tilde{f}(z,t) = \Phi(f(z,t))$ which satisfies (7.1) as well as $\tilde{f}(0,0) = 0$ and $\tilde{f}'(0,0) = 1$ when Φ is an entire function with $\Phi(0) = 0$ and $\Phi'(0) = 1$.

We now give a definition belonging to Betker [10].

Definition 7.3. A family of analytic functions $g_t(z) = g(z,t)$ $(0 \le t < \infty)$ on the unit disk Δ is called an *inverse Löwner chain* if the following conditions are satisfied:

(i) $g_t : \Delta \to \mathbb{C}$ is univalent for each $t \ge 0$,
(ii) $g_t \prec g_s$ whenever $0 \le s \le t$,
(iii) $b(t) = g_t'(0)$ is locally absolutely continuous in $t \ge 0$ and $b(t) \to 0$ as $t \to \infty$.

Note that condition (ii) means that $g_t(\Delta) \subset g_s(\Delta)$ and $g_t(0) = g_s(0)$ for $0 \le s \le t$. The following lemma gives us sufficient conditions for $g(z,t)$ to be an inverse Löwner chain.

Lemma 7.4. *Let $g_t(z) = g(z,t)$ be a family of analytic functions on Δ for $0 \le t < \infty$ with the following properties:*

1) *g_t is univalent on Δ for each $t \ge 0$,*
2) *$g(0,s) = g(0,t)$ for $0 \le s \le t$,*
3) *$g(z,0) = z$ for $z \in \Delta$,*
4) *$g(z,t)$ is locally absolutely continuous in $t \ge 0$ for each $z \in \Delta$,*
5) *the differential equation*

$$\dot{g}(z,t) = -zg'(z,t)p(z,t) \quad (z \in \Delta,\ t \ge 0) \tag{7.2}$$

holds for a Herglotz function $p(z,t)$ of divergence type. Then $g_t(\Delta) \subset g_s(\Delta)$ for $0 \le s \le t$.

Corollary 7.5. *Under the assumptions of Lemma 7.4, we suppose, in addition, that the inequality*

$$\left|\arg p(z,t)\right| < \frac{\pi\alpha}{2}, \quad z \in \Delta,\ t \ge 0, \tag{7.3}$$

holds for a constant $0 < \alpha < 1$. Then the conformal mapping g_t on Δ extends to a k-quasiconformal mapping of $\mathbb{C}$ for each $t \ge 0$, where $k = \sin(\pi\alpha/2)$.

Here for a constant $0 \le k < 1$, a mapping $f : \mathbb{C} \to \mathbb{C}$ is called k-quasiconformal if f is a homeomorphism in the Sobolev class $W^{1,2}_{\mathrm{loc}}(\mathbb{C})$ and if it satisfies $|\partial_{\bar{z}} f| \le k |\partial_z f|$ almost everywhere on $\mathbb{C}$.

Proposition 7.6. *The family $\mathcal{J}_r(w) = \mathcal{J}(w,r), r \ge 0,$ is an inverse Löwner chain with the Herglotz function $p(w,r)$ of divergence type. In particular, $\mathcal{J}_r(\Delta) \subset \mathcal{J}_s(\Delta)$ for $0 \le s \le r$.*

Remark 7.7. The condition (7.3) is known to be equivalent to that the semigroup $\left\{F_t\right\}_{t \ge 0}$ in $\mathrm{Hol}(\Delta)$ generated by $f(z)$ can be analytically extended to the sector $\{t \in \mathbb{C} : |\arg t| < \pi(1-\alpha)/2\}$ in the parameter t (see [41]).

By virtue of Corollary 7.5, it is enough to prove the following assertion.

Corollary 7.8. *Suppose that a holomorphic function $f : \Delta \to \Delta$ with $f(0) = 0$, $f'(0) > 0$ is star-like of order α with $\frac{1}{2} < \alpha < 1$. Then its nonlinear resolvent $\mathcal{J}_r : \Delta \to \Delta$ extends to a k-quasiconformal mapping of $\mathbb{C}$ for every $r \ge 0$, where $k = \sin(\pi\alpha)$.*

8. Rigidity properties of holomorphic generators

Let Δ be the open unit disk in the complex plane $\mathbb{C}$. By $Hol\,(\Delta, \mathbb{C})$ we denote the family of all holomorphic functions on Δ. For the special case when $F \in Hol\,(\Delta, \mathbb{C})$ is a self-mapping of Δ we will simply write $F \in Hol\,(\Delta)$.

The famous rigidity theorem of D. M. Burns and S. G. Krantz [23] (which can be considered as a boundary version of the second part of the classical Schwarz Lemma) asserts:

♦ *Let $F \in Hol(\Delta, \Delta)$ be such that*

$$F\,(z) = 1 + (z-1) + O\left((z-1)^4\right),$$

as $z \to 1$. Then $F\,(z) \equiv z$ on Δ.

It was also mentioned in [23] that the exponent 4 is sharp, and it follows from the proof of the theorem that $O\left((z-1)^4\right)$ can be replaced by $o\left((z-1)^3\right)$. There are many generalized versions of this result in different settings for one dimensional, finite dimensional and infinite dimensional situations (see, for example,[87] [88], [90], [5],[6],[11], [33],[38] and [32] and references therein). Similar results appeared earlier in the literature of conformal mappings with the additional hypothesis that F is univalent (and often the function F is assumed to be quite smooth - even analytic - in a neighborhood of the point $z = 1$). The theorem presented in [23] has no such hypothesis. The exponent 4 is sharp: simple geometric arguments show that the function

$$F(z) = z + \frac{1}{10}(z-1)^3$$

satisfies the conditions of the theorem with 4 replaced by 3. Note also that it follows from the proof that $O((z-1)^4)$ can be replaced by $o((z-1)^3)$.

The Burns–Krantz Theorem was improved in 1995 by Thomas L. Kriete and Barbara D. MacCluer [67], who replaced F with its real part and considered the radial limit in $o((z-1)^3)$ instead of the unrestricted limit. Here is a more precise statement of their result.

Theorem 8.1. *Let $F \in \mathrm{Hol}(\Delta)$ with radial limit $F(1) = 1$ and angular derivative $F'(1) = 1$. If*

$$\liminf_{r\to 1^-} \frac{\mathrm{Re}\,(F(r) - r)}{(1-r)^3} = 0,$$

then $F(z) \equiv z$ on Δ.

In [96], Roberto Tauraso and Fabio Vlacci investigated rigidity of holomorphic self-mappings of the unit disk Δ after imposing some conditions on the boundary Schwarzian derivative of F defined by

$$\mathcal{S}_F(z) := \frac{F'''(z)}{F'(z)} - \frac{3}{2}\left(\frac{F''(z)}{F'(z)}\right)^2, \quad z \in \partial\Delta.$$

It is known that the Schwarzian derivative carries global information about F: it vanishes identically if and only if F is a Möbius transformation. Initially, the original rigidity result of Burns and Krantz was extended in [96] from the identity mapping to a parabolic automorphism.

Theorem 8.2. *Let $F \in \mathrm{Hol}(\Delta) \bigcap C_A^3(1)$. If*

$$F(1) = 1, \quad F'(1) = 1, \quad \operatorname{Re} F''(1) = 0 \quad \textit{and} \quad \operatorname{Re} \mathcal{S}_F(1) = 0,$$

then F is the parabolic automorphism of Δ defined by

$$\frac{1+F(z)}{1-F(z)} = \frac{1+z}{1-z} + ib,$$

where $b = \operatorname{Im} F''(1)$.

In the particular case $F''(1) = F'''(1) = 0$, this reduces to the result of Burns and Krantz, i.e., $F(z) \equiv z$ on Δ.

In [26] (2010), Contreras, Díaz-Madrigal and Pommerenke supplemented Theorem 8.2 as follows.

Theorem 8.3. *(1) A non-trivial (i.e., $F \neq I$) holomorphic map $F \in \mathrm{Hol}(\Delta)$ is a parabolic automorphism if and only if there exists $\zeta \in \partial\Delta$ such that $F \in C_A^3(\zeta)$ and*

$$F(\zeta) = \zeta, \quad F'(\zeta) = 1, \quad \operatorname{Re}(\zeta F''(\zeta)) = 0 \quad \textit{and} \quad \mathcal{S}_F(\zeta) = 0.$$

(2) $F \in \mathrm{Hol}(\Delta)$ is a hyperbolic automorphism if and only if there exist $\zeta \in \partial\Delta$ and $\alpha \in (0,1)$ such that $F \in C_A^3(\zeta)$ and

$$F(\zeta) = \zeta, \quad F'(\zeta) = \alpha, \quad \operatorname{Re}(\zeta F''(\zeta)) = \alpha(\alpha-1) \quad \textit{and} \quad \mathcal{S}_F(\zeta) = 0.$$

The following boundary rigidity principles are given in [88]. In particular, some conditions on behavior of a holomorphic self-mapping F of Δ in a neighborhood of a boundary regular fixed point (not necessarily the Denjoy–Wolff point) under which F is a linear-fractional transformation have established.

It is known that if a mapping $F \in \mathrm{Hol}(\Delta)$ with the boundary regular fixed point $\tau = 1$ and $F'(1) =: \alpha$ is linear fractional, then for all $k > 0$, and for some $\beta \geq 0$ the following equality holds for all $z \in \Delta$,

$$\frac{|1-F(z)|^2}{1-|F(z)|^2} = \frac{\alpha\,|1-z|^2}{(1-|z|^2) + \alpha\beta\,|1-z|^2}\,. \tag{8.1}$$

Moreover, F is an automorphism of Δ (either hyperbolic, $\alpha \neq 1$, or parabolic, $\alpha = 1$) if and only if $\beta = 0$.

It turns out that, that under some smoothness conditions, equality (8.1) (and even some weaker condition) is also sufficient for $F \in \mathrm{Hol}(\Delta)$ to be linear fractional.

Theorem 8.4. *Let $F \in \mathrm{Hol}(\Delta) \cap C_A^3(1)$, $F(1) = 1$ and $F'(1) = \alpha$. Then F is a linear fractional transformation if and only if the following conditions hold:*

(i) $\dfrac{|1-F(z)|^2}{1-|F(z)|^2} \leq \dfrac{1}{a}\,, \quad z \in \Delta;$

(ii) the Schwarzian derivative $\mathcal{S}_F(1) = 0$.

So, if conditions (i) and (ii) are satisfied, then equality (8.1) holds for all $z \in \Delta$.

In this connection we also would like to mention other two directions in generalization of the Burns-Krantz Theorem presented in [88], [89],[90] and [87].

The first one is to establish some rigidity property for those functions the third derivative of which is not necessarily zero. In other words, we assume that

$$\angle \lim_{z\to\tau} \frac{F(z)-z}{(z-1)^2} = 0, \text{ but } \angle \lim_{z\to\tau} \frac{F(z)-z}{(z-1)^3} = k.$$

It turns out that number k is always nonnegative number and the value $F(0)$ lies always in the closed disk of radius k centred in k.Moreover, $F(0)$ lies on the circle-the boundary of this disk if and only if F has a special form which immediately becomes the identity mapping whenever $k = 0$.

For he second direction we have interested is to extend the mentioned above results for those functions which are not necessarily self-mappings of Δ, but satisfy the so-called property to be *pseudo-contractive* on the open unit disk. Despite the latter class is much wider that the class of self mappings of Δ, it preserves many properties of its fixed points as well as the rigidity property in the spirit of the Burns-Krantz Theorem.

Theorem 8.5. *Let $F \in Hol(\Delta, \mathbb{C})$ be pseudo-contractive on Δ, with*

$$F(1) = F'(1) = 1 \text{ and } \angle \lim_{z\to\tau} \frac{F(z)-z}{(z-1)^2} = 0.$$

Assume also that there is the angular limit

$$\angle \lim_{z\to\tau} \frac{F(z)-z}{(z-1)^3} = \mu$$

Then μ is a real nonnegative number with

$$|F(0) - \mu| \le \mu$$

Moreover, If $F(0) = 2\mu$, then

$$F(z) = z - \mu \frac{(z-1)^3}{1+z}.$$

In particular, $\mu = 0$ if and only if $F(z) = z$.

Rigidity principles related to interpolation problems can be found in [6], [5] and [11]. Boundary behavior of semigroups and rigidity at the boundary point is considered in [38].

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David Shoikhet
Department of Mathematics, ORT Braude College, 21982 Karmiel, Israel
and
Holon Institute of Technology, 58102 Holon Israel
e-mail: `davs@braude.ac.il`