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Janowski subclasses of starlike mappings

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Dedicated to the memory of Professor Gabriela Kohr

Abstract. In this paper, two subclasses of biholomorphic starlike mappings named Janowski starlike and Janowski almost starlike with complex parameters are introduced and studied. We determine M such that holomorphic mappings f which satisfy the condition $\|Df(z) - I\| \le M$, $z \in B^n$, are Janowski starlike, respectively Janowski almost starlike. We also derive sufficient conditions for normalized holomorphic mappings (expressed in terms of their coefficient bounds) to belong to one of the subclasses of mappings mentioned above.

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1. Introduction and preliminaries

The main reason for studying the properties of subclasses of biholomorphic mappings in several variables is the fact that many of the classical results regarding the class of univalent functions in one complex variable cannot be extended (without imposing supplementary restrictions) to higher dimensions.

In this paper we generalize the Janowski starlike and almost starlike classes of biholomorphic mappings studied in [4].

In [7], W. Janowski introduced the following class of univalent normalized functions defined on the unit disk U of the complex plane.

If $A, B \in \mathbb{R}$, $-1 \le B < A < 1 \le 1$, then

$$S^*[A,B] = \left\{ f \in \mathcal{H}(U),\ f(0) = 0,\ f'(0) = 1,\ \frac{zf'(z)}{f(z)} \prec \frac{1+Az}{1+Bz} \right\}.$$

Closely related to Janowski starlike class of functions is the following class of univalent functions [17, 18].

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If $a, b \in \mathbb{R}$, $a \geq b$ then

$$S^*(a,b) = \left\{ f \in \mathcal{H}(U),\ f(0) = 0,\ f'(0) = 1,\ \left| \frac{zf'(z)}{f(z)} - a \right| < b \right\}.$$

Various results concerning the classes $S^*[A,B]$ and $S^*(a,b)$ can be found in [7, 17, 18] (and the references therein).

Later, in [10] the authors introduced and studied the class $S^*[A,B]$ for some complex parameters A and B, $A \neq B$ which satisfy one of the following two conditions:

$$|A| \leq 1, |B| < 1 \text{ and } \Re(1 - A\overline{B}) \geq |A - B| \tag{1.1}$$

$$|A| \leq 1, |B| = 1 \text{ and } 1 - A\overline{B} > 0. \tag{1.2}$$

Many results related to the class $S^*[A,B]$ for $A, B \in \mathbb{C}$ may be found in [1, 10].

The main goal of the present paper is to generalize the Janowski mappings studied in [4] by introducing the n-dimensional version of the class $S^*[A,B]$ with complex parameters A and B that satisfy equivalent conditions to (1.1) and (1.2).

To this end, we recall some notions of function theory in several complex variables that will be used throughout the paper.

We denote by $\mathbb{C}^n$ the space of n complex variables $z = (z_1, z_2, \ldots, z_n)$ with the Euclidean inner product $\langle z, w \rangle = \sum_{j=1}^n z_j \overline{w_j}$ and the Euclidean norm $\|z\| = \langle z, z \rangle^{1/2}$. The open unit ball $\{z \in \mathbb{C}^n :\ \|z\| < 1\}$ is denoted by B^n. By $\mathcal{L}(\mathbb{C}^n)$ we denote the the space of linear continuous operators from $\mathbb{C}^n$ into $\mathbb{C}^n$ with the standard operator norm, $\|A\| = \sup\{\|A(z)\| :\ \|z\| = 1\}$. $I_n = I$ is the identity in $\mathcal{L}(\mathbb{C}^n)$.

Let $\mathcal{H}(B^n)$ be the set of holomorphic mappings from B^n into $\mathbb{C}^n$. If $f \in \mathcal{H}(B^n)$ we say that f is normalized if $f(0) = 0$ and the complex Jacobian matrix of f at $z = 0$, $Df(0)$, is the identity operator I. If $f \in \mathcal{H}(B^n)$ is a normalized mapping, then

$$f(z) = z + \sum_{k=2}^{\infty} A_k(z^k),\ z \in B^n,$$

where $A_k(w^k) = \frac{1}{k!} D^k f(0)(w^k)$, and $D^k f(0)(w^k)$ is the k^{th} order Fréchet derivative of f at $z = 0$.

A holomorphic mapping $f : B^n \longrightarrow \mathbb{C}^n$ is said to be biholomorphic if the inverse f^{-1} exists and is biholomorphic on the open set $f(B^n)$. Any holomorphic and injective mapping on B^n is biholomorphic on B^n. Let $S(B^n)$ be the set of normalized biholomorphic mappings on B^n. If $f \in \mathcal{H}(B^n)$, we say that f is locally biholomorphic on B^n if $\det Df(z) \neq 0$, $z \in B^n$. Let $\mathcal{L}S(B^n)$ be the set of normalized locally biholomorphic mappings on B^n.

A locally biholomorphic mapping f with $f(0) = 0$ is starlike [19] if and only if

$$\text{Re } \langle (Df(z))^{-1} f(z), z \rangle > 0,\ z \in B^n \setminus \{0\}.$$

We denote by $S^*(B^n)$ the set of biholomorphic normalized starlike mappings.

We present next the notions of starlikeness of order $\alpha \in [0,1)$ (see [2, 9]) and almost starlikeness of order $\alpha \in [0,1)$ (see [8, 6]). We denote by $S^*_\alpha(B^n)$ the class of mappings that are starlike of order α on B^n and by $\mathcal{A}S^*_\alpha(B^n)$ the class of almost starlike mappings of order α on B^n.

Definition 1.1. Let $\alpha \in [0,1)$.

$$S^*_\alpha(B^n) = \left\{ f \in \mathcal{L}S(B^n) : \mathrm{Re}\ \frac{\|z\|^2}{\langle (Df(z))^{-1}f(z), z\rangle} > \alpha, z \in B^n \setminus \{0\} \right\}$$

$$\mathcal{A}S^*_\alpha(B^n) = \left\{ f \in \mathcal{L}S(B^n) : \mathrm{Re}\ \frac{\langle (Df(z))^{-1}f(z), z\rangle}{\|z\|^2} > \alpha, z \in B^n \setminus \{0\} \right\}.$$

It is obvious that $S^*_0(B^n) = \mathcal{A}S^*_0(B^n) = S^*(B^n)$.

In [4] we introduced the following classes of starlike mappings on B^n.

Definition 1.2. Let $a, b \in \mathbb{R}$ such that $|a-1| < b \leq a$.

$$S^*(a,b,B^n) = \left\{ f \in \mathcal{L}S(B^n) :\ \left| \frac{\|z\|^2}{\langle (Df(z))^{-1}f(z), z\rangle} - a \right| < b,\ z \in B^n \setminus \{0\} \right\}$$

$$\mathcal{A}S^*(a,b,B^n) = \left\{ f \in \mathcal{L}S(B^n) :\ \left| \frac{\langle (Df(z))^{-1}f(z), z\rangle}{\|z\|^2} - a \right| < b,\ z \in B^n \setminus \{0\} \right\}.$$

If in the previous definition we take $a = b$, then $S^*(a,a,B^n) = \mathcal{A}S^*_{1/2a}(B^n)$ and $\mathcal{A}S^*(a,a,B^n) = S^*_{1/2a}(B^n)$.

Various results concerning the classes $S^*(a,b,B^n)$ and $\mathcal{A}S^*(a,b,B^n)$ can be found in [4, 11, 14, 15].

The following set of normalized mappings is the generalization to n-complex variables of the well-known Carathéodory class of one variable holomorphic functions with positive real part on the unit disk of complex plane.

$$\mathcal{M} = \{h \in \mathcal{H}(B^n) :\ h(0) = 0,\ Dh(0) = I,\ \mathrm{Re}\ \langle h(z), z\rangle > 0,\ z \in B^n \setminus \{0\}\}.$$

The class $\mathcal{M}$, introduced in [16] plays a fundamental role in the study of the Loewner differential equation (see for example [3, 5, 6] and the references therein). Also, it is closely related to certain subclasses of biholomorphic mappings on B, such as starlike mappings, mappings with parametric representation [5], etc.

Some subclasses of the class $\mathcal{M}$ are presented next.

Let $g \in \mathcal{H}(U)$ be an univalent function, such that $g(0) = 1$ and $\mathrm{Re}\ g(\zeta) > 0$ on U.

Let $\mathcal{M}_g$ be the class of holomorphic mappings given by

$$\mathcal{M}_g = \left\{ h \in \mathcal{H}(B^n) : h(0)=0,\ Dh(0)=I,\ \frac{1}{\|z\|^2}\langle h(z), z\rangle \in g(U),\ z \in B \setminus \{0\} \right\}.$$

It is clear that $\mathcal{M}_g \subseteq \mathcal{M}$ and for $g(\zeta) = \frac{1-\zeta}{1+\zeta}$ it follows that $\mathcal{M}_g = \mathcal{M}$. Particular choices of the function g provide various subclasses of class $\mathcal{M}$.

If g is a univalent function with $g(0) = 1$ and positive real part on U we denote by $S^*_g(B^n)$ the subset of $S^*(B^n)$ consisting of the normalized locally biholomorphic mappings f such that $(Df(z))^{-1}f(z) \in \mathcal{M}_g$.

In this paper, our main concern is the case when the function $g \in \mathcal{H}(U)$ has positive real part on U and is of the following particular form: $g(\zeta) = \frac{1+A\zeta}{1+B\zeta}$, with $A, B \in \mathbb{C}$, $A \neq B$.

2. Janowski subclasses of starlike mappings

In this section we introduce and study two subclasses of biholomorphic mappings named Janowski starlike and Janowski almost starlike with complex parameters.

Let $a \in \mathbb{C}, b \in \mathbb{R}$ such that $|a-1| < b \leq \Re a$. We denote by $S^*(a, b, B^n)$ the class of Janowski starlike mappings on B^n and by $\mathcal{A}S^*(a, b, B^n)$ the class of Janowski almost starlike mappings on B^n.

Definition 2.1. Let $a \in \mathbb{C}, b \in \mathbb{R}$ such that $|a-1| < b \leq \Re a$.

$$S^*(a, b, B^n) = \left\{ f \in \mathcal{L}S(B^n) : \ \left| \frac{\|z\|^2}{\langle (Df(z))^{-1} f(z), z \rangle} - a \right| < b, \ z \in B^n \setminus \{0\} \right\}$$

$$\mathcal{A}S^*(a, b, B^n) = \left\{ f \in \mathcal{L}S(B^n) : \ \left| \frac{\langle (Df(z))^{-1} f(z), z \rangle}{\|z\|^2} - a \right| < b, \ z \in B^n \setminus \{0\} \right\}.$$

In the next remark we present the relationships between the two subclasses of starlike mappings introduced above.

Remark 2.2. Let $a \in \mathbb{C}, b \in \mathbb{R}$ such that $|a-1| < b \leq \Re a$.

Then the following assertions are true:

(i) $S^*(a, b, B^n) = \mathcal{A}S^*\left(\frac{\overline{a}}{|a|^2 - b^2}, \frac{b}{|a|^2 - b^2}, B^n\right)$, if $b < |a|$;

(ii) $S^*(a, a, B^n) = \mathcal{A}S^*_{\frac{1}{2a}}(B^n)$, if $b = |a|$;

(iii) $\mathcal{A}S^*(a, b, B^n) = S^*\left(\frac{\overline{a}}{|a|^2 - b^2}, \frac{b}{|a|^2 - b^2}, B^n\right)$, if $b < |a|$;

(iv) $\mathcal{A}S^*(a, a, B^n) = S^*_{\frac{1}{2a}}(B^n)$, if $b = |a|$.

Proof. The assertions (i) and (iii) are immediate consequences of the fact that the disk of center a and radius b is mapped by the function $1/\zeta$ onto the disk of center $\frac{\overline{a}}{|a|^2 - b^2}$ and radius $\frac{b}{|a|^2 - b^2}$.

The assertions (ii) and (iv) are immediate consequences of the fact that the disk of center a and radius a is mapped by the function $1/\zeta$ onto the half-plane $\left\{\zeta \mid \Re\zeta > \frac{1}{2a}\right\}$. □

The following remark (see also [10]) presents the conditions satisfied by the complex parameters A, B, $A \neq B$ such that $g(\zeta) = \frac{1+A\zeta}{1+B\zeta}$, $\zeta \in U$, is a holomorphic function with positive real part on U.

Remark 2.3. Let $A, B \in \mathbb{C}$, $A \neq B$ and let $g \in \mathcal{H}(U)$ be the function defined by $g(\zeta) = \frac{1+A\zeta}{1+B\zeta}$. If g has positive real part on U, then the complex parameters A and B satisfy one of the following conditions:

$$|B| < 1 \text{ and } \Re(1 - A\overline{B}) \geq |A - B| \tag{2.1}$$

$$|B| = 1 \text{ and } -1 \leq A\overline{B} < 1 \tag{2.2}$$

Proof. The fact that $g \in \mathcal{H}(U)$ immediately implies that $|B| \leq 1$.

The function g maps the unit disk U either onto an open disk (when $|B| < 1$) or onto a half-plane (when $|B| = 1$). It remains for us to determine the conditions satisfied by A and B such that the image $g(U)$ to be situated in the right half-plane.

When $|B| < 1$, g maps the unit disk U onto the open unit disk given by

$$\left| g(\zeta) - \frac{1 - A\overline{B}}{1 - |B|^2} \right| < \frac{|A - B|}{1 - |B|^2},\ \zeta \in U.$$

The above disk is in the right half-plane if

$$\Re g(\zeta) > \frac{\Re(1 - A\overline{B}) - |A - B|}{1 - |B|^2} \geq 0,\ \zeta \in U,$$

hence (2.1) is fulfilled.

When $|B| = 1$, g maps the unit disk U onto a half-plane, which has to be situated in the right half-plane, hence the image of the unit circle is a vertical line. Therefore, $\Re g(\overline{B}) = \Re g(i\overline{B})$, wherefrom we obtain that $A\overline{B} \in \mathbb{R}$ and $g(\overline{B}) = \frac{1+A\overline{B}}{2}$. In this case $\Re g(\zeta) > 0, \zeta \in U$, if and only if $1 = \Re g(0) > \frac{1+A\overline{B}}{2} \geq 0$, hence (2.2) is fulfilled. □

We next determine the function g such that $\mathcal{A}S^*(a, b, B^n) = S_g^*(B^n)$.

Remark 2.4. (i) Let $a \in \mathbb{C}$ and $b \in \mathbb{R}$ such that $|a - 1| < b \leq \Re a$. Then

$$\mathcal{A}S^*(a, b, B^n) = S_g^*(B^n), \text{where } g(\zeta) = \frac{1 + \frac{a - |a|^2 + b^2}{b}\zeta}{1 + \frac{1 - \overline{a}}{b}\zeta},\ \zeta \in U \tag{2.3}$$

(ii) Let $A, B \in \mathbb{C}$, $A \neq B$, and let $g : U \longrightarrow \mathbb{C}$, $g(\zeta) = \frac{1+A\zeta}{1+B\zeta}$ be a holomorphic function with positive real part on U. Then:

$$S_g^*(B^n) = \mathcal{A}S^*\left(\frac{1 - A\overline{B}}{1 - |B|^2}, \frac{|A - B|}{1 - |B|^2}, B^n\right) \text{ if } |B| < 1 \text{ and } \Re(1 - A\overline{B}) \geq |A - B| \tag{2.4}$$

$$S_g^*(B^n) = \mathcal{A}S^*_{\frac{1+A\overline{B}}{2}}(B^n) \text{ if } |B| = 1 \text{ and } -1 < A\overline{B} < 1.$$

Proof. To prove (2.3) we have to determine $A, B \in \mathbb{C}$, $A \neq B$, $|B| < 1$, $\Re(1 - A\overline{B}) \geq |A - B|$ such that $a = \frac{1 - A\overline{B}}{1 - |B|^2}$ and $b = \frac{|A-B|}{1-|B|^2}$.
Straightforward computations lead to the following values:

$$B = \frac{1 - \overline{a}}{b}\frac{|A - B|}{\overline{A} - \overline{B}} = \frac{1 - \overline{a}}{b}e^{i\phi},\ A = \frac{b^2 + a - |a|^2}{b}e^{i\phi},\ \phi = \arg(A - B).$$

Since the image of U under the function g is invariant to the rotations of the unit disk, without loss of generality we can assume that $B = \frac{1-\overline{a}}{b}$ and hence $A = \frac{b^2+a-|a|^2}{b}$. By direct computations we obtain that

$$1 - A\overline{B} = \frac{\overline{a}(b^2 - |a - 1|^2)}{b^2} \text{ and } B = \frac{b^2 - |a - 1|^2}{b}.$$

Therefore the desired inequality $\Re(1 - A\overline{B}) \geq |A - B|$ is equivalent to $\Re\overline{a} \geq b$, which is true. The inequality $|B| < 1$ results immediately from $|a - 1| < b$.
The equalities in (2.4) easily follow from the fact that the unit disk is mapped by the function $g(\zeta) = \frac{1+A\zeta}{1+B\zeta}$ onto the disk centered at $a = \frac{1-A\overline{B}}{1-|B|^2}$ with radius $b = \frac{|A-B|}{1-|B|^2}$ when $|B| < 1$, respectively onto the half-plane $\left\{\zeta \mid \mathrm{Re}\ \zeta > \frac{1+A\overline{B}}{2}\right\}$ when $|B| = 1$. □

We next determine the function g such that $S^*(a,b,B^n) = S_g^*(B^n)$.

Remark 2.5. (i) Let $a \in \mathbb{C}$ and $b \in \mathbb{R}$ such that $|a-1| < b \leq \Re a$. Then

$$S^*(a,b,B^n) = S_g^*(B^n), \text{where } g(\zeta) = \frac{1+\frac{\overline{a}-1}{b}\zeta}{1+\frac{|a|^2-a-b^2}{b}\zeta}, \zeta \in U. \tag{2.5}$$

(ii) Let $A, B \in \mathbb{C}$, $A \neq B$, and let $g : U \longrightarrow \mathbb{C}$, $g(\zeta) = \frac{1+A\zeta}{1+B\zeta}$ be a holomorphic function with positive real part on U. Then:

$$S_g^*(B^n) = S^*\left(\frac{1-A\overline{B}}{1-|A|^2}, \frac{|A-B|}{1-|A|^2}, B^n\right) \text{ for } |A| < 1 \text{ and } \Re(1-A\overline{B}) \geq |A-B| \tag{2.6}$$

$$S_g^*(B^n) = S^*_{\frac{1+A\overline{B}}{2}}(B^n) \text{ if } |A| = 1 \text{ and } -1 < A\overline{B} < 1.$$

Proof. The equality in (2.5) is an immediate consequence of (2.3) and assertion (iii) from Remark 2.2. The equalities in (2.6) can be justified by using similar arguments to those presented in the proof of (ii), Remark 2.4. $\square$

3. Sufficient conditions for Janowski starlikeness

In this section we obtain sufficient conditions for normalized holomorphic mappings to belong to $S^*(a,b,B^n)$, respectively $\mathcal{A}S^*(a,b,B^n)$, where $a \in \mathbb{C}, b \in \mathbb{R}$, $|a-1| < b \leq \Re a$.

Theorem 3.1. *Let $f(z) = z + \sum_{k=2}^{\infty} A_k(z^k)$ be a holomorphic mapping on B^n and let $g : U \to \mathbb{C}$ be the function defined by $g(\zeta) = \frac{1+A\zeta}{1+B\zeta}$, $\zeta \in U$, where $A, B \in \mathbb{C}$, $A \neq B, |B| < 1$ and $\Re(1-A\overline{B}) \geq |A-B|$. If*

$$\|Df(z) - I\| \leq M, \quad z \in B^n \tag{3.1}$$

where M is defined by

$$M = \frac{2|A-B|(1-|B|)}{2|1-A\overline{B}| + 2|A-B| + (1-|B|^2)} \tag{3.2}$$

then $f \in S_g^*(B^n)$.

Proof. In order that f to be in $S_g^*(B^n)$ it is sufficient to show that

$$\left\| \frac{1-|B|^2}{|A-B|}(Df(z))^{-1}f(z) - \frac{1-A\overline{B}}{|A-B|}z \right\| < \|z\|, \forall z \in B^n \setminus \{0\}.$$

If h is the holomorphic function defined by

$$h(z) = \frac{1-|B|^2}{|A-B|}(Df(z))^{-1}f(z) - \frac{1-A\overline{B}}{|A-B|}z, z \in B^n$$

then $h(0) = 0$, $\|Dh(0)\| = |B| < 1$ hence $\lim_{z\to 0} \frac{\|h(z)\|}{\|z\|} < 1$.
Therefore, it suffices to prove that $\|h(z)\| < \|z\|$ for all $z \in B^n \setminus \{0\}$.

If the previous inequality is not true, then there exists a point $z_0 \in B^n \setminus \{0\}$ such that $\|h(z_0)\| = \|z_0\|$. By denoting $w_0 = (Df(z_0))^{-1} f(z_0)$, after direct computations we obtain

$$\|w_0\| \leq \frac{|1 - A\overline{B}| + |A - B|}{1 - |B|^2} \|z_0\|. \tag{3.3}$$

Since $\|Df(z_0) - I\| \leq M$, we have first $\|Df(z_0)w_0 - w_0\| \leq M\|w_0\|$ and hence

$$\|f(z_0) - w_0\| \leq M\|w_0\| \tag{3.4}$$

Now we prove that

$$\|f(z) - z\| < \frac{M}{2}, \forall z \in B^n. \tag{3.5}$$

If not, then there exists a point $z_1 \in B^n \setminus \{0\}$ such that

$$\max_{\|z\| \leq \|z_1\|} \|f(z) - z\| = \|f(z_1) - z_1\| = \frac{M}{2} \tag{3.6}$$

According to Lemma1 [12], there exists a real number $t \geq 2$ such that

$$\langle Df(z_1)(z_1) - z_1, f(z_1) - z_1 \rangle = t\|f(z_1) - z_1\|^2.$$

In view of the relations (3.1) and (3.6), the previous equality implies

$$t\frac{M^2}{4} \leq \|Df(z_1)(z_1) - z_1\| \cdot \|f(z_1) - z_1\| \leq \frac{M^2}{2}\|z_1\| < \frac{M^2}{2}, \tag{3.7}$$

hence $t < 2$, which is a contradiction.
Therefore, the relation (3.5) is true and by applying the Schwarz's Lemma we obtain

$$\|f(z) - z\| \leq \frac{M}{2}\|z\|^2, \ \forall z \in B^n. \tag{3.8}$$

By using the relations (3.4) and (3.8) we obtain first that

$$M\|w_0\| \geq \|z_0 - w_0\| - \|f(z_0) - z_0\| > \|z_0 - w_0\| - \frac{M}{2}\|z_0\|,$$

and then

$$M < \frac{\|z_0 - w_0\|}{\frac{\|z_0\|}{2} + \|w_0\|}. \tag{3.9}$$

On the other hand, by using the fact that $\|h(z_0)\| = \|z_0\|$, (3.3) and (3.2) we get

$$\frac{\|z_0 - w_0\|}{\frac{\|z_0\|}{2} + \|w_0\|} = \frac{\left\| w_0 - \frac{|1-A\overline{B}|}{1-|B|^2} z_0 + \frac{|1-A\overline{B}|}{1-|B|^2} z_0 - z_0 \right\|}{\frac{\|z_0\|}{2} + \|w_0\|} \geq M$$

Because the previous inequality contradicts (3.9), the assumption that there exists a point $z_0 \in B^n \setminus \{0\}$ such that $\|h(z_0)\| = \|z_0\|$ is false, hence $\|h(z)\| < \|z\|$ for all $z \in B^n \setminus \{0\}$. This completes the proof. □

Let $a \in \mathbb{C}$ and $b \in \mathbb{R}$ such that $|a - 1| < b \leq \Re a$.

By taking $A = \frac{a - |a|^2 + b^2}{b}$ and $B = \frac{1 - \overline{a}}{b}$ in Theorem 3.1 and by using Remark 2.4 we obtain the following sufficient condition for a holomorphic mapping to belong to $\mathcal{A}S^*(a, b, B^n)$

Theorem 3.2. *Let $f(z) = z + \sum_{k=2}^{\infty} A_k(z^k)$ be a holomorphic mapping on B^n and let $a \in \mathbb{C}, b \in \mathbb{R}$ such that $|a-1| < b \leq \Re a$. If*

$$\|Df(z) - I\| \leq \frac{2(b - |1-a|)}{2(|a|+b)+1}, \quad z \in B^n$$

then $f \in \mathcal{A}S^*(a, b, B)$.

If we take $A = \frac{\overline{a}-1}{b}$ and $B = \frac{|a|^2 - a - b^2}{b}$ in Theorem 3.1 and use Remark 2.5 we get the following sufficient condition for a holomorphic mapping to be in $S^*(a, b, B^n)$.

Theorem 3.3. *Let $f(z) = z + \sum_{k=2}^{\infty} A_k(z^k)$ be a holomorphic mapping on B^n and let $a \in \mathbb{C}, b \in \mathbb{R}$ such that $|a-1| < b \leq \Re a$ and $b < |a|$. If*

$$\|Df(z) - I\| \leq \frac{2(b - ||a|^2 - a + b^2|)}{(|a|+b)(|a|-b+2)}, \quad z \in B^n$$

then $f \in S^*(a, b, B^n)$.

Next theorems present sufficient conditions that are expressed in terms of coefficients bounds of the considered mappings.

Theorem 3.4. *Let $f(z) = z + \sum_{k=2}^{\infty} A_k(z^k)$ be a holomorphic mapping on B^n and let $g : U \to \mathbb{C}$ be the univalent function defined by $g(\zeta) = \dfrac{1 + A\zeta}{1 + B\zeta}$, $\zeta \in U$, where $A, B \in \mathbb{C}$, $A \neq B, |B| < 1$ and $\Re(1 - A\overline{B}) \geq |A - B|$. If*

$$\sum_{k=2}^{\infty} \left(\left| k \frac{1 - A\overline{B}}{|A-B|} - \frac{1 - |B|^2}{|A-B|} \right| + k \right) \|A_k\| \leq 1 - |B| \tag{3.10}$$

then $f \in S_g^*(B^n)$.

Proof. From the inequality (3.10) it follows that

$$\sum_{k=2}^{\infty} k\|A_k\| \leq \frac{2}{2\frac{|1-A\overline{B}|}{|A-B|} - \frac{1-|B|^2}{|A-B|} + 2} \sum_{k=2}^{\infty} \left(\left| k \frac{1 - A\overline{B}}{|A-B|} - \frac{1 - |B|^2}{|A-B|} \right| + k \right) \|A_k\| < 1.$$

By direct computation of Fréchet derivatives of f we obtain

$$\|Df(z) - I\| = \left\| \sum_{k=2}^{\infty} k A_k(z^{k-1}, \cdot) \right\| \leq \sum_{k=2}^{\infty} k\|A_k\| < 1, \; z \in B^n.$$

Hence we obtain that $Df(z) = I - (I - Df(z))$ is an invertible linear operator and

$$\|(Df(z))^{-1}\| \leq \frac{1}{1 - \|I - Df(z)\|} \leq \frac{1}{1 - \sum_{k=2}^{\infty} k\|A_k\|}, \; z \in B^n. \tag{3.11}$$

For every $z \in B^n \setminus \{0\}$, we have

$$\left\| \frac{1 - |B|^2}{|A-B|} f(z) - \frac{1 - A\overline{B}}{|A-B|} Df(z)(z) \right\|$$
$$< \|z\| \left(|B| + \sum_{k=2}^{\infty} \left| k \frac{|1 - A\overline{B}|}{|A-B|} - \frac{1 - |B|^2}{|A-B|} \right| \|A_k\| \right).$$

By using the inequality (3.11) and the previous inequality we obtain

$$\frac{1}{\|z\|^2}\left|\left\langle \frac{1-|B|^2}{|A-B|}(Df(z))^{-1}f(z), z\right\rangle - \frac{1-A\overline{B}}{|A-B|}\|z\|^2\right|$$

$$< \frac{|B| + \sum_{k=2}^{\infty}\left|k\frac{|1-A\overline{B}|}{|A-B|} - \frac{1-|B|^2}{|A-B|}\right|\|A_k\|}{1-\sum_{k=2}^{\infty}\|A_k\|k} \leq 1,$$

where for the last inequality we have used the relation (3.10).
In conclusion

$$\left|\left\langle \frac{(Df(z))^{-1}f(z)}{\|z\|^2}, z\right\rangle - \frac{1-A\overline{B}}{1-|B|^2}\right| < \frac{|A-B|}{1-|B|^2},\ z \in B^n \setminus \{0\},$$

which implies that $f \in S_g^*(B^n)$ as desired.

□

Let $a \in \mathbb{C}$ and $b \in \mathbb{R}$ such that $|a-1| < b \leq \Re a$.

By taking $A = \frac{a-|a|^2+b^2}{b}$ and $B = \frac{1-\overline{a}}{b}$ in Theorem 3.4 we obtain the following sufficient condition for a holomorphic mapping to be in $\mathcal{A}S^*(a, b, B^n)$.

Theorem 3.5. *Let $f(z) = z + \sum_{k=2}^{\infty} A_k(z^k)$ be a holomorphic mapping on B^n and let $a \in \mathbb{C}, b \in \mathbb{R}$ such that $|a-1| < b \leq \Re a$. If*

$$\sum_{k=2}^{\infty}(|ka-1| + kb)\|A_k\| \leq b - |1-a|$$

then $f \in \mathcal{A}S^*(a, b, B)$.

If in the previous theorem we take $a = b = \frac{1}{2\alpha}$, $0 < \alpha < 1$, we get Theorem 2.2 [13].

If we take $A = \frac{\overline{a}-1}{b}$ and $B = \frac{|a|^2-a-b^2}{b}$ in Theorem 3.4 we get the following sufficient condition for a holomorphic mapping to be in $S^*(a, b, B^n)$.

Theorem 3.6. *Let $f(z) = z + \sum_{k=2}^{\infty} A_k(z^k)$ be a holomorphic mapping on B^n and let $a \in \mathbb{C}, b \in \mathbb{R}$ such that $|a-1| < b \leq \Re a$ and $b < |a|$. If*

$$\sum_{k=2}^{\infty}\left(\left|k\overline{a} - (|a|^2 - b^2)\right| + kb\right)\|A_k\| \leq b - ||a|^2 - a + b^2|$$

then $f \in S^*(a, b, B^n)$.

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