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A two-steps fixed-point method for the simplicial cone constrained convex quadratic optimization

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Abstract. In this paper, we deal with the resolution of the simplicial cone constrained convex quadratic optimization (abbreviated SCQO). It is known that the optimality conditions of SCQO is only a standard linear complementarity problem (LCP). Under a suitable condition, the solution of LCP is equivalent to find the solution of an absolute value equations AVE. For its numerical solution, we propose an efficient two-steps fixed point iterative method for solving the AVE. Moreover, we show that this method converges globally linear to the unique solution of the AVE and which is in turn an optimal solution of SCQO. Some numerical results are reported to demonstrate the efficiency of the proposed algorithm.

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1. Introduction

Consider the simplicial cone constrained convex quadratic optimization SCQO:

$$\min_x \left[f(x) = \frac{1}{2} x^T Q x + x^T b + c \right] \text{ s.t. } x \in \mathbb{S} \tag{1.1}$$

where $Q \in \mathbb{R}^{n\times n}$ is a symmetric positive definite matrix, $b \in \mathbb{R}^n, c \in \mathbb{R}^n$, and

$$\mathbb{S} = \left\{ Ax \mid x \in \mathbb{R}^n_+ \right\}$$

is the simplicial cone associated with the nonsingular matrix $A \in \mathbb{R}^{n\times n}$. The importance of quadratic programming lies in its theoretical properties, its applications in

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different scientific fields and several disciplines such as economics, finance, telecommunications, medicine and the engineering sciences. Another great advantage of the quadratic case is that we can transform several real and academic problems (polynomial minimization problems, least squares problems in numerical analysis, etc.) into an equivalent quadratic problem without loss of generality. Simplicial cone constrained convex quadratic programming is equivalent to the problem of projecting the point onto a simplicial cone (see [5, 6, 9]), with its KKT optimality conditions consisting a linear complementarity problem (see [8, 19]). From this optimality conditions, under suitable conditions, the convex quadratic programming under a simplicial cone constraints is equivalent to finding the unique solution of the following absolute value equation:

$$(A^TQA + I)x + (A^TQA - I)|x| = -A^Tb. \tag{1.2}$$

This equation is a special case of the general absolute value equations AVE of the type:

$$\bar{A}x - \bar{B}|x| = \bar{b}$$

where $\bar{A}, \bar{B}$ are given $(n \times n)$ real square matrices and $\bar{b} \in \mathbb{R}^n$. The AVE was first introduced by Rohn [18] and investigated in more general context in Mangasarian (see [16]). Other studies for the AVE can be found in [1, 3, 2, 7, 10, 12, 13, 15, 17]. Besides some numerical methods are used to solve it. In particular, Mangasarian in [14] proposed a semi-smooth Newton's method for solving the AVE, and under suitable conditions he showed the finite and linear convergence to a solution of the AVE. However, other numerical approaches focus on reformulating the AVE as an horizontal linear complementarity problems (HLCP) (see [4]), where they introduce an infeasible path-following interior-point method for solving the AVE by using is equivalent reformulations as an HLCP. In this paper, we propose a new two-steps fixed point iterative method for solving the AVE (1.2) which is introduced in [11], and under a new mild assumption we show that this method is always well-defined and the generated sequence converges globally and linearly to the unique solution of the AVE from any starting initial point. Finally, numerical results are provided to illustrate the efficiency of this algorithm to solving the SCQO.

Our paper is organized as follows. In section 2, some notations and basic results used in the paper are presented. In section 3, the reformulation of problem (1.1) as an absolute value equation AVE and the unique solvability of AVE is studied. Any solution of the AVE generates a solution of our convex quadratic programming problem SCQO. In section 4, a description and a convergence property of the two-steps fixed point iterative method for solving the AVE are stated. In section 5, numerical results are presented. We end this paper with a conclusion and some remarks in section 6.

2. Preliminaries

Let $\mathbb{R}^n$ be the Euclidean space provided with the usual scalar product $\langle x, y\rangle = x^Ty$ where x and y are two vectors of $\mathbb{R}^n$ and x^T is the transpose of x. The nonegative orthant of $\mathbb{R}^n$ is denoted by $\mathbb{R}^n_+$. For $x \in \mathbb{R}^n_+$, we write $x \geq 0$, and means that $x_i \geq 0$, $\forall i$. For $x = (x_1, x_2, \ldots, x_n)^T \in \mathbb{R}^n$, we denote by:

$$x_i^+ := \max(0, x_i),\ x_i^- := \max(0, -x_i),\ |x| := (|x_1|, \ldots, |x_n|)^T.$$

It is easy from the definitions of x^+ and x^-, to conclude that:

$$x = x^+ - x^-,\ x^+ \in \mathbb{R}^n_+,\ x^- \in \mathbb{R}^n_+, \langle x^+, x^- \rangle = 0, |x| = x^+ + x^-, \forall x \in \mathbb{R}^n.$$

For $x \in \mathbb{R}^n$, $\mathrm{sign}(x)$ denotes a vector with components equal to 1, 0, -1 depending on whether the corresponding component of x is positive, zero or negative. We denote by $\mathbb{R}^{n\times n}$ the vector space of real square matrices of order n, the identity matrix is denoted by I_n. If $x \in \mathbb{R}^n$ then $X = Diag(x)$ denotes the $n \times n$ diagonal matrix with $X_{ii} = x_i$, $\forall i = 1, \ldots, n$. Let $A \in \mathbb{R}^{n\times n}$, its spectral matrix norm is denoted by $\|A\| := \max\{\|Ax\| : x \in \mathbb{R}^n, \|x\| = 1\}$, where $\|x\|$ denotes the Euclidean norm, this definition implies:

$$\|Ax\| \leq \|A\| \|x\|,\ \|AB\| \leq \|A\| \|B\|,\ \forall A, B \in \mathbb{R}^{n\times n}.$$

For a matrix M, $\rho(M)$ denote its spectral radius. In addition, if M is a real symmetric matrix, $\rho(M) = \|M\|$. Finally,

Lemma 2.1. *For all $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^n$, we have:*

$$\||x| - |y|\| \leq \|x - y\|.$$

Proof. For detailed proof see Lemma 5 [15]. □

3. The SCQO as an absolute value equation

Recall that the SCQO problem is given by:

$$\min_x \left[f(x) = \frac{1}{2} x^T Q x + x^T b + c \right] \text{ s.t. } x \in \mathbb{S}.$$

Starting from the definition of simplicial cones $\mathbb{S}$ associated with the nonsingular matrix A, the problem (1.1) can be formulated as a quadratic programming problem under positive constraints:

$$\min_y \left[f(y) = \frac{1}{2} y^T A^T Q A y + y^T A^T b + c \right] \text{ s.t. } y \in \mathbb{R}^n_+. \tag{3.1}$$

As the problem (3.1) is convex and the constraints are positive then the optimality conditions of K.K.T are necessary and sufficient and we have, $y \in \mathbb{R}^n_+$ is an optimal solution of problem (3.1) if and only if there exists $z \in \mathbb{R}^n_+$ such that:

$$z - A^T Q A y = A^T b,\ z^T y = 0,\ y \geq 0, z \geq 0, \tag{3.2}$$

which is a standard linear complementarity problem (see [8]).
Next, letting $z = |s| - s$ and $y = |s| + s$, then the LCP (3.2) is reformulated as the following absolute value equations (AVE) of type

$$\bar{A}s + \bar{B}|s| = \bar{b}, \tag{3.3}$$

where

$$\bar{A} = A^TQA + I,\ \bar{B} = A^TQA - I,\ \bar{b} = -A^Tb.$$

Hence, solving the problem (1.1) is equivalent to solving the AVE (3.3). The following result is needed to guarantee the unique solvability of the AVE.

Theorem 3.1 (Theorem 8 [2]**).** *Assume that $\bar{A}$ is invertible and the matrices $\bar{A}, \bar{B}$ satisfy the following condition, $\left\|\bar{A}^{-1}\bar{B}\right\| < 1$, then the AVE (3.3) has a unique solution for any $\bar{b} \in \mathbb{R}^n$.*

For our case since $\bar{A} = A^TQA + I$ and $\bar{B} = A^TQA - I$ where Q is symmetric positive definite and A is invertible, the condition $\left\|\bar{A}^{-1}\bar{B}\right\| < 1$ of Theorem 3.1, is satisfied. We check this result through the following lemma.

Lemma 3.2. *Let $\bar{A} = A^TQA + I$ and $\bar{B} = A^TQA - I$ such that A is invertible matrix and Q is symmetric positive definite. Then the matrix $\bar{A}$ is invertible and $\left\|\bar{A}^{-1}\bar{B}\right\| < 1$.*

Proof. Because Q is symmetric positive definite and A is invertible, then the matrix A^TQA is symmetric positive definite, hence $\bar{A}$ is symmetric positive definite too, which implies that $\bar{A}$ is invertible. Next, since A^TQA is symmetric positive definite, then A^TQA has positive real eigenvalues denoted by $\lambda_i(A^TQA) := \lambda_i > 0, \forall i = 1, ..., n$. In addition, it is known that the eigenvalues of $\bar{A}$ and $\bar{B}$, are given by $\lambda_i + 1 > 0$ and $\lambda_i - 1$, respectively. Because, $\bar{A}$ and $\bar{B}$ are real symmetric matrices, we then have,

$$\begin{aligned}\left\|\bar{A}^{-1}\bar{B}\right\| &\leq \left\|\bar{A}^{-1}\right\|\left\|\bar{B}\right\| = \rho(\bar{A}^{-1})\rho(\bar{B}) \\ &= \max_i\left(\left|\frac{\lambda_i - 1}{\lambda_i + 1}\right|\right).\end{aligned}$$

As $\lambda_i > 0$, then

$$\left|\frac{\lambda_i - 1}{\lambda_i + 1}\right| < 1.$$

So $\left\|\bar{A}^{-1}\bar{B}\right\| < 1$. This gives the required result. □

Proposition 3.3. *If s^* is the solution of the AVE (3.3) then $(y^*, z^*) = (|s^*| + s^*, |s^*| - s^*)$ is the solution of the LCP (3.2). Consequently, Ay^* is the optimal solution of problem (1.1).*

Proof. Let s^* be the unique solution of the AVE, then

$$\bar{A}s^* + \bar{B}\,|s^*| = \bar{b}.$$

So

$$\begin{aligned}\left(A^TQA + I\right)s^* + (A^TQA - I)\,|s^*| &= -A^Tb \\ \Leftrightarrow A^TQA(|s^*| + s^*) + |s^*| - s^* &= -A^Tb \\ \Leftrightarrow z^* - A^TQAy^* &= A^Tb.\end{aligned}$$

Now, since $y^* = |s^*| + s^* = 2(s^*)^+, z^* = |s^*| - s^* = 2(s^*)^-$, then we have $y^* \geq 0$, $z^* \geq 0$ and $z^{*T}y^* = 0$, hence, the pair (y^*, z^*) is a solution of LCP (3.2). Finally, we deduce that Ay^* is an optimal solution of the SCQO problem. This completes the proof. □

4. Two-steps Picard's fixed point iterative method for SCQO

In this section, we derive a new fixed-point iterative approach for solving the equation (3.3). Let $t = |s|$ then, the AVE (3.3) is equivalent to the following system:

$$\begin{cases} \bar{A}s + \bar{B}t = \bar{b} \\ -|s| + t = 0. \end{cases} \tag{4.1}$$

The latter can be expressed as follows:

$$\begin{pmatrix} \bar{A} & \bar{B} \\ -D(s) & I \end{pmatrix} \begin{pmatrix} s \\ t \end{pmatrix} = \begin{pmatrix} \bar{b} \\ 0 \end{pmatrix}, \tag{4.2}$$

where $D(s)$:=Diag (sign (s)), $s \in \mathbb{R}^n$. Note that the system (4.2) is nonlinear, it is generally impossible to obtain an exact solution. We will therefore be satisfied with an approximated solution. Since the matrix $\bar{A}$ is invertible hence from (4.1) we can obtain the following fixed point equation:

$$\begin{cases} s^* = \bar{A}^{-1}\left(-\bar{B}t^* + \bar{b}\right) \\ t^* = (1-r)\, t^* + r\, |s^*| \end{cases} \tag{4.3}$$

where $r > 0$, is a suitable parameter that we shall specified it later. According to the fixed-point equation, we generate a sequence $(s^{(k)}, t^{(k)})$ converging to the solution of AVE. So the new fixed-point iteration is given by:

$$\begin{cases} s^{(k+1)} = \bar{A}^{-1}\left(-\bar{B}t^{(k)} + \bar{b}\right) \\ t^{(k+1)} = (1-r)\, t^{(k)} + r\left|s^{(k+1)}\right|, k = 0, 1, \ldots \end{cases} \tag{4.4}$$

The details of our algorithm for solving the AVE (3.3) is described in Figure 1.

4.1. Algorithm

Input
An accuracy parameter $\epsilon > 0$;
a parameter r such that $0 < r < \frac{2}{\left\|\bar{A}^{-1}\bar{B}\right\|+1}$;
an initial starting point $t^0 \in \mathbb{R}^n$;
compute $t^1 = (1-r)t^0 + r|s^1|$, $s^1 = \bar{A}^{-1}\left(-\bar{B}t^{(0)} + \bar{b}\right)$; $k := 0$;
While $\frac{\left\|t^{k+1}-t^k\right\|}{\left\|\bar{b}\right\|} \geq \epsilon$ **do**
begin
compute : $\begin{cases} t^{(k+1)} = (1-r)\, t^{(k)} + r\left|s^{(k+1)}\right|, \\ s^{(k+1)} = \bar{A}^{-1}\left(-\bar{B}t^{(k)} + \bar{b}\right) \end{cases}$
$k := k + 1$;
end
end

Fig. 1. Algorithm. 4.1

In this section, we give detailed proof for the convergence of Algorithm 4.1.

Theorem 4.1. *Let $Q \in \mathbb{R}^{n\times n}$ be a symmetric positive definite matrix, $b \in \mathbb{R}^n$ and $A \in \mathbb{R}^{n\times n}$ is an invertible matrix then the sequence $(s^{(k)}, t^{(k)})$ generated by the iterative* *algorithm (4.4) to solve the problem (4.1) is well-defined for any starting point* $t^0 \in \mathbb{R}^n$. *In addition, if*

$$0 < r < \frac{2}{\left\|\bar{A}^{-1}\bar{B}\right\| + 1},$$

then the sequence $(s^{(k)}, t^{(k)})$ *converges linearly to the solution* (s^*, t^*) *of the nonlinear equation (4.1). Consequently,* $A(|s^*| + s^*)$ *is the solution of the problem (1.1).*

Proof. First, we check that the sequence $(s^{(k)}, t^{(k)})$ is well-defined, it suffices to show that the matrix $\bar{A} = A^T QA + I$ is invertible. This claim was proven by Lemma 3.2. Next, using formula (4.4) and Lemma 2.1, we have, on one hand that

$$\begin{aligned} \left\|t^{k+1} - t^*\right\| &= \left\|(1-r)\,t^k + r\left|s^{(k+1)}\right| - (1-r)\,t^* + r\left|s^*\right|\right\| \\ &= \left\|(1-r)\left(t^k - t^*\right) + r(\left|s^{(k+1)}\right| - \left|s^*\right|)\right\| \\ &\leq |1-r|\left\|(t^k - t^*)\right\| + r\left\|s^{k+1} - s^*\right\|. \end{aligned}$$

So

$$\begin{aligned} \left\|s^{k+1} - s^*\right\| &= \left\|\bar{A}^{-1}\left(-\bar{B}t^k + \bar{b}\right) - \bar{A}^{-1}\left(-\bar{B}t^* + \bar{b}\right)\right\| \\ &= \left\|-\bar{A}^{-1}\bar{B}(t^k - t^*)\right\| \leq \left\|\bar{A}^{-1}\bar{B}\right\|\left\|(t^k - t^*)\right\|. \end{aligned}$$

Therefore

$$\left\|t^{k+1} - t^*\right\| \leq (|1-r| + r\left\|\bar{A}^{-1}\bar{B}\right\|)\left\|(t^k - t^*)\right\|.$$

On the other hand,

$$\begin{aligned} \left\|s^{k+1} - s^*\right\| &\leq \left\|-\bar{A}^{-1}\bar{B}(t^k - t^*)\right\| \\ &\leq \left\|-\bar{A}^{-1}\bar{B}(t^k - (1-r)t^{k-1} + (1-r)t^{k-1} - t^*)\right\| \\ &\leq \left\|-\bar{A}^{-1}\bar{B}(r\left|s^k\right| + (1-r)t^{k-1} - t^*)\right\|. \end{aligned}$$

As $|s^*| = t^*$, we find

$$\begin{aligned} \left\|s^{k+1} - s^*\right\| &\leq \left\|-\bar{A}^{-1}\bar{B}(r\left|s^k\right| - r\left|s^*\right|) - (1-r)\bar{A}^{-1}\bar{B}(t^{k-1} - t^*)\right\| \\ &\leq r\left\|\bar{A}^{-1}\bar{B}\right\|\left\|s^k - s^*\right\| + |1-r|\left\|s^k - s^*\right\| \\ &\leq (|1-r| + r\left\|\bar{A}^{-1}\bar{B}\right\|)\left\|s^k - s^*\right\|. \end{aligned}$$

The sequence $(s^{(k)}, t^{(k)})$ is convergent if the following condition

$$|1-r| + r\left\|\bar{A}^{-1}\bar{B}\right\| < 1$$

holds. For that we distinguish two cases.
Case 1. If $0 < r \leq 1$, then

$$\begin{aligned} |1-r| + r\left\|\bar{A}^{-1}\bar{B}\right\| < 1 \quad &\Leftrightarrow \quad 1 - r + r\left\|\bar{A}^{-1}\bar{B}\right\| < 1 \\ &\Leftrightarrow \quad r(\left\|\bar{A}^{-1}\bar{B}\right\| - 1) < 0. \end{aligned}$$

Since $\left\|\bar{A}^{-1}\bar{B}\right\| < 1$ then,

$$r(\left\|\bar{A}^{-1}\bar{B}\right\| - 1) < 0, \forall\, 0 < r \leq 1.$$

Case 2. If $r \geq 1$, then

$$\begin{aligned} |1-r| + r\left\|\bar{A}^{-1}\bar{B}\right\| < 1 \quad &\Leftrightarrow \quad -1 + r + r\left\|\bar{A}^{-1}\bar{B}\right\| < 1 \\ &\Leftrightarrow \quad r < \frac{2}{\left\|\bar{A}^{-1}\bar{B}\right\| + 1}. \end{aligned}$$

Finally, regrouping the two cases, this gives the required result. □

5. Numerical results

In this section, we present numerical results for Algorithm 4.1 by using $\epsilon = 10^{-6}$ and $r = 0.9$. The algorithm has been applied on three examples of SCQO problem. The iterations have been carry out by **MATLAB** R2016a and run on a personal pc with 1.40 GHZ AMD E1-2500 APU Radeon(TM) HD Graphic, 8 GB memory and Windows 10 operating system. The starting point and the unique solution by t^0 and s^*, respectively. The stopping criterion used in our algorithm is the relative residue, i.e.,

$$RES := \frac{\left\|t^{k+1} - t^k\right\|}{\left\|\bar{b}\right\|} \leq 10^{-6}$$

In view of the influence of the initial point on the convergence of our algorithm, different values are used. For each problem, the hypotheses of Theorem 4.1 are checked. In the tables below, the symbols "It" and "CPU" denote the number of iterations produced by the algorithm and the elapsed times, respectively.

Problem 1. Consider the SCQO problem where Q, A and b are given by :

$$Q = \begin{bmatrix} 2 & 1 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix}, A = \begin{bmatrix} 3 & 0 & 0 & 0 & 0 \\ 0.5 & 3 & 0 & 0 & 0 \\ -1 & 0.5 & 3 & 0 & 0 \\ -1 & -1 & 0.5 & 3 & 0 \\ -1 & -1 & -1 & 0.5 & 3 \end{bmatrix}$$

and $b = [-3,\ 1,\ -10,\ -12,\ -2]^T$.

The starting point in this example is taken as:

$$t^0 = [0,\ -1,\ -1,\ 2,\ 1]^T.$$

After 21 iterations, the unique solution s^* of AVE is:

$$s^* = [0.2071,\ -7.6143,\ 0.5262,\ 0.7886,\ -2.2308]^T,$$

and

$$y^* = |s^*| + s^* = [0.4142,\ 0,\ 1.0525,\ 1.5771,\ 0]^T.$$

Therefore, the unique solution of Problem (1.1), is given by:

$$x^* = Ay^* = [1.2426,\ 0.2071,\ 2.7433,\ 4.8435,\ -0.6781]^T.$$

Problem 2. Let the matrices Q, A and the vector b of this example are given by:

$$Q=(q_{ij})=\begin{cases} 4, & \text{for} \quad i=j, \\ \frac{1}{2}, & \text{for} \quad |i-j|=1, i=1,2,\ldots n-2, \\ 1, & \text{for} \quad \begin{cases} j=i-2, i=1,2,\ldots n, \\ i=j-2, j=1,2,\ldots n, \end{cases} \\ 0, & \text{otherwise} \end{cases}$$

$$A=(a_{ij})=\begin{cases} -2, & \text{for} \quad i=j, \\ 4, & \text{for} \quad j=i-1, i=2,\ldots n, \\ -1, & \text{for} \quad i=j-1, j=2,\ldots n, \\ \frac{1}{2}, & \text{for} \quad j>i+2, i=1,2,\ldots n, \\ \frac{1}{5}, & \text{for} \quad i>j+1, j=1,2,\ldots n, \end{cases}$$

and

$$b=-2\bar{A}^{-1}(\bar{A}+\bar{B})e.$$

The computational results with different size of n are shown in Table 1. For the initialization of Problem 2, we take different values of t^0.

n		$t^0=[0,\ldots,0]^T$	$t^0=[5,\ldots,5]^T$	$t^0=[-10,\ldots,-10]^T$
10	CPU	$0.04966s$	$0.04552s$	$0.10820s$
	It	17	11	65
	RES	$9.923e-07$	$8.542e-07$	$9.606e-07$
50	CPU	$0.03704s$	$0.03391s$	$0.36887s$
	It	4	3	42
	RES	$9.136e-07$	$9.025e-07$	$9.576e-07$
100	CPU	$0.07339s$	$0.05023s$	$0.26949s$
	It	3	3	22
	RES	$6.032e-07$	$1.508e-07$	$9.365e-07$
1000	CPU	$5.34351s$	$4.42180s$	$5.36816s$
	It	2	1	2
	RES	$5.992e-07$	$3.371e-07$	$7.501e-07$
2000	CPU	$34.17236s$	$33.23488s$	$39.15840s$
	It	1	1	2
	RES	$3.370e-07$	$8.427e-07$	$1.873e-07$

Table 1. Computational results of Problem 2.

An exact solution with different size of n is given by:

$$s^*=[2,\,2,\,\ldots,\,2]^T,$$

and

$$y^*=[4,\,4,\,\ldots,\,4]^T.$$

An exact solution of problem (1.1) is given by: $x^*=Ay^*$.

Problem 3. The bloc matrices Q, A and the vector b of this example are given by:

$$Q=\begin{bmatrix} Q_{11} & I_n \\ I_n & Q_{22} \end{bmatrix},\; A=\begin{bmatrix} A_{11} & I_n \\ B & A_{11} \end{bmatrix}$$

where

$$Q_{11} = (q_{11})_{ij} = \begin{cases} 6, & \text{for} \quad i=j, \\ -1, & \text{for} \quad |i-j|=1, i=1,2,\dots n, \\ 0, & \text{otherwise.} \end{cases}$$

$$Q_{22} = (q_{22})_{ij} = \begin{cases} 5, & \text{for} \quad i=j, \\ -2, & \text{for} \quad |i-j|=1, i=1,2,\dots n, \\ \frac{1}{4}, & \text{for} \quad \begin{cases} j=i+1, i=1,2,\dots n-1, \\ j=i-1, j=1,2,\dots n-1, \end{cases} \\ 0, & \text{otherwise.} \end{cases}$$

$$A_{11} = (a_{11})_{ij} = \begin{cases} -2, & \text{for} \quad i=j, \\ -1, & \text{for} \quad j=i-1, i=3,\dots n, \\ 3, & \text{for} \quad j>i, \quad i=3,\dots n, \\ 0.5, & \text{otherwise.} \end{cases}$$

$$B = (b_{ij}) = \begin{cases} -1, & \text{for} \quad i=j, \\ 0, & \text{for} \quad \begin{cases} j=i+1, i=1,2,\dots n-1, \\ j=i-1, j=1,2,\dots n, \end{cases} \\ \frac{1}{n}, & \text{for} \quad \begin{cases} j=i+2, i=1,2,\dots n-2, \\ j=i-2, j=4,\dots n, \end{cases} \\ 0, & \text{otherwise,} \end{cases}$$

and

$$b = [-8, \dots, -8]^T.$$

For the initialization, we take:

$$t^0 = [0, \dots, 0, -1, \dots, -1]^T.$$

The numerical results with different size of n are summarized in Table 2.

n	10	50	100	1000	2000
CPU	$0.14101s$	$1.0919s$	$1.11202s$	$8.37809s$	$35.76440s$
It	122	137	92	6	3
RES	$9.81e-07$	$9.98e-07$	$9.95e-07$	$7.436e-07$	$7.35e-07$

Table 2. Computational results of Problem 3.

For example, if $n = 10$ then,

$$s^* = [0.0984,\ 0.0042, 0.409,\ -1.782,\ 1.6329,\ 0.3763,\ 0.4448, 0.881, 0.7929, 2.7841]^T,$$

and

$$y^* = |s^*| + s^* = [0.1967, 0.0082, 0.8179, 0, 3.2658, 0.7525, 0.8896, 1.7619, 1.586, 5.568]^T.$$

An exact solution of problem (1.1) is given by:

$$x^* = Ay^* = [2.3928, 2.475, 2.3819, 2.206, 2.1049, 2.0303, 3.038, 3.3649, 3.2356, 2.3168]^T.$$

6. Conclusion and remarks

In this paper, a convex quadratic programming problem under simplicial cone constraints were studied, and via its optimality conditions is reduced to finding the unique solution of an absolute value equation AVE. For solving this AVE we applied a new two-steps Picard's iterative fixed point iteration. In particular, the sufficient conditions for the convergence of our algorithm are studied. The obtained numerical results deduced from the testing examples illustrate that the suggested algorithm is efficient and valid to solve the SCQO problems.

References

[1] Abdellah, L., Haddou, M., Migot, T., *Solving absolute value equations using complementarity and smoothing functions*, J. Comput. Appl. Math., **327**(2018),196-207.

[2] Achache, M., *On the unique solvability and numerical study of absolute value equations*, J. Numer. Anal. Approx. Theory, **48**(2019), no. 2, 112-121.

[3] Achache, M., Anane, N., *On the unique solvability and Picard's iterative method for absolute value equations*, Bulletin of Transilvania, Series III: Mathematics and Computer Sciences, (63)(2021), no. 1, 13-26.

[4] Achache, M., Hazzam, N., *Solving absolute value equations via linear complementarity and interior-point methods*, Journal of Nonlinear Functional Analysis, (2018), 1-10.

[5] Barrios, J.G., Ferreira, O.P., Németh, S.Z., *A semi-smooth Newton method for solving convex quadratic programming problem under simplicial cone constraints*, arXiv: 1503.0275 v1 [math. OC] (2015).

[6] Barrios, J.G., Ferreira, O.P., Németh, S.Z., *Projection onto simplicial cones by Picard's method*, Linear Algebra and Its Applications, **480**(2015), 27-43.

[7] Corradi, G., *A quasi-Newton method for non-smooth equations*, Int. J. Comput. Math., **82(5)**(2005), 573-581.

[8] Cottle, R.W., Pang, J.S., Stone, R.E., *The Linear Complementarity Problem*, Academic Press, New-York, 1992.

[9] Ferreira, O.P., Németh, S.Z., *Projection onto simplicial cones by a semi-smooth Newton method*, Optimization Letters, **9**(4)(2015), 731-741.

[10] Hu, S.L., Huang, Z.H., Zhang, Q., *A generalized Newton method for absolute value equations associated with second order cones*, J. Comput. Appl. Math., **235**(2011), 1490-1501.

[11] Ke, Y., *The new iteration algorithm for absolute value equation*, Applied Mathematics Letters, **99**(2020).

[12] Ketabchi, S., Moosaei, H., *An efficient method for optimal correcting of absolute value equations by minimal changes in the right hand side*, Comput. Math. Appl., **64**(2012), 1882-1885.

[13] Li, C.X., *A Modified generalized Newton method for absolute value equations*, J. Optim. Theory Appl., (2016), 1055-1059.

[14] Mangasarian, O.L., *A generalized Newton method for absolute value equations*, Optimization Letters, **3**(2009), 101-108.

[15] Mangasarian, O.L., *Linear complementarity as absolute value equation solution*, Optimization Letters, **8**(2014), 1529-1534.

[16] Mangasarian, O.L., Meyer, R.R., *Absolute value equations*, Linear Algebra and Its Applications, **419**(2006), 359-367.

[17] Noor, M.A., Iqbal, J., Khattri, S., Al-Said, E., *On a iterative method for solving absolute value equation*, Optimization Letters, **6**(2012), 1027-1033.

[18] Rhon, J., *A theorem of the alternatives for the equation* $Ax+B|x|=b$, Linear Multilinear A, **52**(2004), 421-426.

[19] Ujvàri, M., *On the projection onto a finitely generated cone*, Preprint WP 2007-5, MTASZTAKI, Laboratory of Operations Research and Decision Systems, Budapest, 2007.

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