Stud. Univ. Babeş-Bolyai Math. 69(2024), No. 2, 267–281
DOI: 10.24193/subbmath.2024.2.02

Better approximations for quasi-convex functions

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Abstract. In this paper, by using Hölder-İşcan, Hölder integral inequality and an general identity for differentiable functions we can get new estimates on generalization of Hadamard, Ostrowski and Simpson type integral inequalities for functions whose derivatives in absolute value at certain power are quasi-convex functions. It is proved that the result obtained Hölder-İşcan integral inequality is better than the result obtained Hölder inequality.

Mathematics Subject Classification (2010): 26A51, 26D10, 26D15.

Keywords: Hölder-İşcan inequality, Hermite-Hadamard inequality, Simpson and Ostrowski type inequality, midpoint and trapezoid type inequality, quasi-convex functions.

1. Introduction

A function $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is said to be convex if the inequality

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y)$$

valid for all $x, y \in I$ and $t \in [0,1]$. If this inequality reverses, then f is said to be concave on interval $I \neq \varnothing$.

Integral inequalities have played an important role in the development of all branches of Mathematics and the other sciences. The inequalities discovered by Hermite and Hadamard for convex functions are very important in the literature. The classical Hermite-Hadamard integral inequality provides estimates of the mean value of a continuous convex function $f : [a,b] \to \mathbb{R}$. Firstly, let's recall the Hermite-Hadamard integral inequality. In addition, readers can refer to the [8, 9, 10, 11, 14, 16, 12, 13, 17, 18, 19] articles and the references therein for more detailed information on both convexity and the different classes of convexity.

Received 17 February 2022; Accepted 03 April 2023.

Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex function defined on the interval I of real numbers and $a, b \in I$ with $a < b$. The following inequality

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}. \tag{1.1}$$

holds. This double inequality is known in the literature as Hermite-Hadamard integral inequality for convex functions [1, 4]. Note that some of the classical inequalities for means can be derived from (1.1) for appropriate particular selections of the mapping f. Both inequalities hold in the reversed direction if the function f is concave.

Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a mapping differentiable in I°, the interior of I, and let $a, b \in I^\circ$ with $a < b$. If $|f'(x)| \leq M$, $x \in [a, b]$, then we the following inequality holds

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)dt \right| \leq \frac{M}{b-a}\left[\frac{(x-a)^2 + (b-x)^2}{2}\right]$$

for all $x \in [a, b]$. This result is known in the literature as the Ostrowski inequality [3].

The following inequality is well known in the literature as Simpson's inequality

.

Let $f : [a, b] \to \mathbb{R}$ be a four times continuously differentiable mapping on (a, b) and $\left\|f^{(4)}\right\|_\infty = \sup\limits_{x \in (a,b)} \left|f^{(4)}(x)\right| < \infty$. Then the following inequality holds:

$$\left| \frac{1}{3}\left[\frac{f(a)+f(b)}{2} + 2f\left(\frac{a+b}{2}\right)\right] - \frac{1}{b-a}\int_a^b f(x)dx \right| \leq \frac{1}{2880}\left\|f^{(4)}\right\|_\infty (b-a)^4.$$

In recent years many authors have studied error estimations for Simpson's inequality; for refinements, counterparts, generalizations and new Simpson's type inequalities, see [20, 21] and therein.

Definition 1.1 ([2]). A function $f : [a, b] \to \mathbb{R}$ is said quasi-convex on $[a, b]$ if

$$f(tx + (1-t)y) \leq \max\{f(x), f(y)\}$$

for any $x, y \in [a, b]$ and $t \in [0, 1]$.

Lemma 1.2 ([5]). *Let the function $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I° such that $f' \in L[a, b]$, where $a, b \in I$ with $a < b$ and $\theta, \lambda \in [0, 1]$. Then the following equality holds:*

$$\begin{aligned}
&(1-\theta)(\lambda f(a) + (1-\lambda) f(b)) + \theta f((1-\lambda) a + \lambda b) - \frac{1}{b-a}\int_a^b f(x)dx \\
=\; &(b-a)\left[-\lambda^2 \int_0^1 (t-\theta) f'(ta + (1-t)[(1-\lambda) a + \lambda b]) dt \right. \\
&\left. + (1-\lambda)^2 \int_0^1 (t-\theta) f'(tb + (1-t)[(1-\lambda) a + \lambda b]) dt \right].
\end{aligned}$$

In [6], İşcan gave the following theorems for quasi-convex functions.

Theorem 1.3. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I such that $f' \in L[a,b]$, where $a, b \in I^\circ$ with $a < b$ and $\alpha, \lambda \in [0,1]$. If $|f'|^q$ is quasi-convex on $[a,b]$, $q > 1$, then the following inequality holds:*

$$\begin{aligned} &\left|(1-\theta)\left(\lambda f(a)+(1-\lambda)f(b)\right)+\theta f\left((1-\lambda)a+\lambda b\right)-\frac{1}{b-a}\int_a^b f(x)dx\right| \\ \leq\ & (b-a)\left(\frac{\theta^{p+1}+(1-\theta)^{p+1}}{p+1}\right)^{\frac{1}{p}}\left[\lambda^2\left(\sup\left\{\left|f'(a)\right|^q,\left|f'(C)\right|^q\right\}\right)^{\frac{1}{q}}\right. \\ & \left.+(1-\lambda)^2\left(\sup\left\{\left|f'(b)\right|^q,\left|f'(C)\right|^q\right\}\right)^{\frac{1}{q}}\right] \qquad (1.2) \end{aligned}$$

where $C = (1-\lambda)a + \lambda b$ and $\frac{1}{p} + \frac{1}{q} = 1$.

Theorem 1.4. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I such that $f' \in L[a,b]$, where $a, b \in I^\circ$ with $a < b$ and $\alpha, \lambda \in [0,1]$. If $|f'|^q$ is quasi-convex on $[a,b]$, $q \geq 1$, then the following inequality holds:*

$$\begin{aligned} &\left|(1-\theta)\left(\lambda f(a)+(1-\lambda)f(b)\right)+\theta f\left((1-\lambda)a+\lambda b\right)-\frac{1}{b-a}\int_a^b f(x)dx\right| \\ \leq\ & (b-a)\left(\theta^2-\theta+\frac{1}{2}\right)\left[\lambda^2\left(\sup\left\{\left|f'(a)\right|^q,\left|f'(C)\right|^q\right\}\right)^{\frac{1}{q}}\right. \\ & \left.+(1-\lambda)^2\left(\sup\left\{\left|f'(b)\right|^q,\left|f'(C)\right|^q\right\}\right)^{\frac{1}{q}}\right] \qquad (1.3) \end{aligned}$$

where $C = (1-\lambda)a + \lambda b$.

A refinement of Hölder integral inequality better approach than Hölder integral inequality can be given as follows:

Theorem 1.5 (Hölder-İşcan Integral Inequality [7]**).** *Let $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$. If f and g are real functions defined on interval $[a,b]$ and if $|f|^p$, $|g|^q$ are integrable functions on $[a,b]$ then*

$$\begin{aligned} \int_a^b |f(x)g(x)|\,dx \leq\ & \frac{1}{b-a}\left\{\left(\int_a^b (b-x)\,|f(x)|^p\,dx\right)^{\frac{1}{p}}\left(\int_a^b (b-x)\,|g(x)|^q\,dx\right)^{\frac{1}{q}}\right. \\ & \left.+\left(\int_a^b (x-a)\,|f(x)|^p\,dx\right)^{\frac{1}{p}}\left(\int_a^b (x-a)\,|g(x)|^q\,dx\right)^{\frac{1}{q}}\right\} \end{aligned}$$

An refinement of power-mean integral inequality as a result of the Hölder-İşcan integral inequality can be given as follows:

Theorem 1.6 (Improved power-mean integral inequality [15]**).** *Let $q \geq 1$. If f and g are real functions defined on interval $[a,b]$ and if $|f|$, $|g|^q$ are integrable functions on*

$[a,b]$ *then*

$$\begin{aligned}
&\int_a^b |f(x)g(x)|\,dx \\
\leq\ & \frac{1}{b-a}\left\{\left(\int_a^b (b-x)\,|f(x)|\,dx\right)^{1-\frac{1}{q}}\left(\int_a^b (b-x)\,|f(x)|\,|g(x)|^q\,dx\right)^{\frac{1}{q}}\right. \\
&\left.+\left(\int_a^b (x-a)\,|f(x)|\,dx\right)^{1-\frac{1}{q}}\left(\int_a^b (x-a)\,|f(x)|\,|g(x)|^q\,dx\right)^{\frac{1}{q}}\right\}
\end{aligned}$$

Our aim is to obtain the general integral inequalities giving the Hermite-Hadamard, Ostrowsky and Simpson type inequalities for the quasi-convex function in the special case using the Hölder, Hölder-İşcan integral inequalities and above lemma.

Throught this paper, we will use the following notation for shortness

$$M_1 = \left(\max\left\{|f'(A_\lambda)|^q, |f'(a)|^q\right\}\right)^{1/q} = \max\left\{|f'(A_\lambda)|, |f'(a)|\right\} \quad (1.4)$$

$$M_2 = \left(\max\left\{|f'(A_\lambda)|^q, |f'(b)|^q\right\}\right)^{1/q} = \max\left\{|f'(A_\lambda)|, |f'(b)|\right\}, \quad (1.5)$$

where $A_\lambda = (1-\lambda)\,a + \lambda b$.

2. Main results

Using Lemma 1.2 we shall give another result for quasi-convex functions as follows.

Theorem 2.1. *Let $f : I \subseteq [1,\infty) \to \mathbb{R}$ be a differentiable mapping on I° such that $f' \in L[a,b]$, where $a,b \in I^\circ$ with $a < b$ and $\theta,\lambda \in [0,1]$. If $|f'|^q$ is quasi-convex function on the interval $[a,b]$, $q > 1$, then the following inequality holds:*

$$\begin{aligned}
&\left|(1-\theta)\left(\lambda f(a) + (1-\lambda)\,f(b)\right) + \theta f((1-\lambda)\,a + \lambda b) - \frac{1}{b-a}\int_a^b f(x)dx\right| \\
\leq\ & \frac{b-a}{2}2^{\frac{1}{p}}\left[\lambda^2 M_1 + (1-\lambda)^2 M_2\right]\left[N_1^{\frac{1}{p}}(\theta,p) + N_2^{\frac{1}{p}}(\theta,p)\right], \quad (2.1)
\end{aligned}$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. Suppose that $A_\lambda = (1-\lambda)\,a + \lambda b$. From Lemma 1.2, Hölder-İşcan integral inequality and the quasi-convexity of the function $|f'|^q$, we have

$$\begin{aligned}
&\left|(1-\theta)\left(\lambda f(a) + (1-\lambda)\,f(b)\right) + \theta f(A_\lambda) - \frac{1}{b-a}\int_a^b f(x)dx\right| \\
\leq\ & (b-a)\left[\lambda^2\int_0^1 |t-\theta|\,|f'(ta + (1-t)\,A_\lambda)|\,dt\right. \\
&\left.+ (1-\lambda)^2\int_0^1 |t-\theta|\,|f'(tb + (1-t)\,A_\lambda)|\,dt\right]
\end{aligned}$$

$$
\begin{aligned}
&\leq (b-a)\,\lambda^2\left\{\left(\int_0^1 (1-t)\,|t-\theta|^p\,dt\right)^{\frac{1}{p}}\left(\int_0^1 (1-t)\,|f'(ta+(1-t)\,A_\lambda)|^q\,dt\right)^{\frac{1}{q}}\right.\\
&\quad\left.+\left(\int_0^1 t\,|t-\theta|^p\,dt\right)^{\frac{1}{p}}\left(\int_0^1 t\,|f'(ta+(1-t)\,A_\lambda)|^q\,dt\right)^{\frac{1}{q}}\right\}\\
&\quad+(b-a)\,(1-\lambda)^2\left\{\left(\int_0^1 (1-t)\,|t-\theta|^p\,dt\right)^{\frac{1}{p}}\right.\\
&\quad\times\left(\int_0^1 (1-t)\,|f'(tb+(1-t)\,A_\lambda)|^q\,dt\right)^{\frac{1}{q}}\\
&\quad\left.+\left(\int_0^1 t\,|t-\theta|^p\,dt\right)^{\frac{1}{p}}\left(\int_0^1 t\,|f'(tb+(1-t)\,A_\lambda)|^q\,dt\right)^{\frac{1}{q}}\right\}\\
&\leq (b-a)\,\lambda^2\left\{\left[M_1\left(\frac{1}{2}\right)^{\frac{1}{q}}N_1^{\frac{1}{p}}(\theta)+M_1\left(\frac{1}{2}\right)^{\frac{1}{q}}N_2^{\frac{1}{p}}(\theta)\right]\right.\\
&\quad\left.+(1-\lambda)^2\left[M_2\left(\frac{1}{2}\right)^{\frac{1}{q}}N_1^{\frac{1}{p}}(\theta)+M_2\left(\frac{1}{2}\right)^{\frac{1}{q}}N_2^{\frac{1}{p}}(\theta)\right]\right\}\\
&= \frac{b-a}{2}2^{1-\frac{1}{q}}\left[\lambda^2M_1+(1-\lambda)^2M_2\right]\left[N_1^{\frac{1}{p}}(\theta)+N_2^{\frac{1}{p}}(\theta)\right]\\
&= \frac{b-a}{2}2^{\frac{1}{p}}\left[\lambda^2M_1+(1-\lambda)^2M_2\right]\left[N_1^{\frac{1}{p}}(\theta)+N_2^{\frac{1}{p}}(\theta)\right].
\end{aligned}
$$

By simple computation

$$
\begin{aligned}
N_1(\theta,p) \quad &: \;= \int_0^1 (1-t)\,|t-\theta|^p\,dt && (2.2)\\
&= (1-\theta)\,\frac{\theta^{p+1}+(1-\theta)^{p+1}}{p+1}+\frac{\theta^{p+2}-(1-\theta)^{p+2}}{p+2}\\
N_2(\theta,p) \quad &: \;= \int_0^1 t\,|t-\theta|^p\,dt && (2.3)\\
&= \theta\frac{\theta^{p+1}+(1-\theta)^{p+1}}{p+1}+\frac{(1-\theta)^{p+2}-\theta^{p+2}}{p+2}.
\end{aligned}
$$

Thus, we obtain the inequality (2.1). This completes the proof. □

Remark 2.2. The inequality (2.1) gives better results than the inequality (1.2). Let us show that

$$
\frac{b-a}{2}2^{\frac{1}{p}}\left[\lambda^2M_1+(1-\lambda)^2M_2\right]\left[N_1^{\frac{1}{p}}(\theta,p)+N_2^{\frac{1}{p}}(\theta,p)\right]
$$

$$
\leq (b-a)\left(\frac{\theta^{p+1}+(1-\theta)^{p+1}}{p+1}\right)^{\frac{1}{p}}
$$

$$\times\left[\lambda^2\left(\sup\left\{|f'(a)|^q,|f'(C)|^q\right\}\right)^{\frac{1}{q}}+(1-\lambda)^2\left(\sup\left\{|f'(b)|^q,|f'(C)|^q\right\}\right)\right]^{\frac{1}{q}}.$$

Using the equalities (2.2), (2.3) and the concavity of the function $h:[0,\infty)\to\mathbb{R}$, $h(x)=x^\lambda, 0<\lambda\le 1$, by sample calculation we obtain

$$\begin{aligned}
&\frac{b-a}{2}2^{\frac{1}{p}}\left[\lambda^2 M_1+(1-\lambda)^2 M_2\right]\left[N_1^{\frac{1}{p}}(\theta,p)+N_2^{\frac{1}{p}}(\theta,p)\right]\\
\le\ & (b-a)2^{\frac{1}{p}}\left[\lambda^2 M_1+(1-\lambda)^2 M_2\right]\left[\frac{N_1(\theta,p)+N_2(\theta,p)}{2}\right]^{\frac{1}{p}}\\
=\ & (b-a)\left[\lambda^2 M_1+(1-\lambda)^2 M_2\right]\left(\frac{\theta^{p+1}+(1-\theta)^{p+1}}{p+1}\right)^{\frac{1}{p}},
\end{aligned}$$

which is the required.

Corollary 2.3. *Under the assumptions of Theorem 2.1 with $\theta=1$, then we have the following generalized midpoint type inequality*

$$\begin{aligned}
&\left|f((1-\lambda)a+\lambda b)-\frac{1}{b-a}\int_a^b f(x)dx\right|\\
&\le\frac{b-a}{2}2^{\frac{1}{p}}\left[\lambda^2 M_1+(1-\lambda)^2 M_2\right]\left(\frac{1}{p+2}\right)^{\frac{1}{p}}\left[1+\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\right]. \qquad (2.4)
\end{aligned}$$

where $\frac{1}{p}+\frac{1}{q}=1$.

Corollary 2.4. *Under the assumptions of Theorem 2.1 with $\theta=0$, then we have the following generalized trapezoid type inequality*

$$\begin{aligned}
&\left|\lambda f(a)+(1-\lambda)f(b)-\frac{1}{b-a}\int_a^b f(x)dx\right|\\
&\le\frac{b-a}{2}2^{\frac{1}{p}}\left[\lambda^2 M_1+(1-\lambda)^2 M_2\right]\left(\frac{1}{p+2}\right)^{\frac{1}{p}}\left[1+\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\right],
\end{aligned}$$

where $\frac{1}{p}+\frac{1}{q}=1$.

Corollary 2.5. *Under the assumptions of Theorem 2.1 with $\theta=1$, if $|f'(x)|\le M$, $x\in[a,b]$, then we have the following Ostrowski type inequality*

$$\begin{aligned}
&\left|f(x)-\frac{1}{b-a}\int_a^b f(u)du\right|\\
&\le M\left(\frac{1}{2}\right)^{1-\frac{1}{p}}\left[\frac{(x-a)^2+(b-x)^2}{b-a}\right]\left(\frac{1}{p+2}\right)^{\frac{1}{p}}\left[1+\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\right] \qquad (2.5)
\end{aligned}$$

for each $x\in[a,b]$.

Proof. For each $x \in [a,b]$, there exist $\lambda_x \in [0,1]$ such that $x = (1-\lambda_x)\, a + \lambda_x b$. Hence we have $\lambda_x = \frac{x-a}{b-a}$ and $1-\lambda_x = \frac{b-x}{b-a}$. Therefore, for each $x \in [a,b]$, from the inequality (2.1) we obtain the inequality (2.5). □

Corollary 2.6. *Under the assumptions of Theorem 2.1 with $\lambda = \frac{1}{2}$ and $\theta = \frac{2}{3}$, then we have the following Simpson type inequality*

$$\begin{aligned}&\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)dx\right|\\ &\leq \frac{b-a}{4}2^{\frac{1}{p}}A\left(M_1,M_2\right)\left[N_1^{\frac{1}{p}}\left(\frac{2}{3},p\right)+N_2^{\frac{1}{p}}\left(\frac{2}{3},p\right)\right],\end{aligned}$$

where A is the arithmetic mean.

Corollary 2.7. *Under the assumptions of Theorem 2.1 with $\lambda = \frac{1}{2}$ and $\theta = 1$, then we have the following midpoint type inequality*

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{b-a}{4}2^{\frac{1}{p}}A\left(M_1,M_2\right)\left(\frac{1}{p+2}\right)^{\frac{1}{p}}\left[1+\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\right],$$

where A is the arithmetic mean.

Corollary 2.8. *Under the assumptions of Theorem 2.1 with $\lambda = \frac{1}{2}$ and $\theta = 0$, then we have the following trapezoid type inequality*

$$\left|\frac{f\left(a\right)+f\left(b\right)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{b-a}{4}2^{\frac{1}{p}}A\left(M_1,M_2\right)\left(\frac{1}{p+2}\right)^{\frac{1}{p}}\left[1+\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\right],$$

where A is the arithmetic mean.

Theorem 2.9. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I such that $f' \in L\,[a,b]$, where $a,b \in I^{\circ}$ with $a<b$ and $\alpha,\lambda \in [0,1]$. If $|f'|^q$ is quasi-convex on $[a,b]$, $q \geq 1$, then the following inequality holds:*

$$\begin{aligned}&\left|(1-\theta)\left(\lambda f(a)+(1-\lambda)f(b)\right)+\theta f\left((1-\lambda)a+\lambda b\right)-\frac{1}{b-a}\int_a^b f(x)dx\right|\\ \leq\ &(b-a)\left[\lambda^2 M_1+(1-\lambda)^2 M_2\right]\left[N_1\left(\theta\right)+N_2\left(\theta\right)\right] \qquad (2.6)\end{aligned}$$

where $C = (1-\lambda)a + \lambda b$.

Proof. Suppose that $A_\lambda = (1-\lambda)a + \lambda b$. From Lemma 1.2, improved power-mean integral inequality and the quasi-convexity of the function $|f'|^q$, we have

$$\begin{aligned}
& \left|(1-\theta)(\lambda f(a) + (1-\lambda) f(b)) + \theta f(A_\lambda) - \frac{1}{b-a}\int_a^b f(x)dx\right| \\
\leq\ & (b-a)\left[\lambda^2 \int_0^1 |t-\theta| \left|f'(ta + (1-t)A_\lambda)\right| dt \right. \\
& \left. + (1-\lambda)^2 \int_0^1 |t-\theta| \left|f'(tb + (1-t)A_\lambda)\right| dt\right] \\
\leq\ & (b-a)\lambda^2 \left\{ \left(\int_0^1 (1-t)|t-\theta| dt\right)^{1-\frac{1}{q}} \right. \\
& \times \left(\int_0^1 (1-t)|t-\theta| \left|f'(ta + (1-t)A_\lambda)\right|^q dt\right)^{\frac{1}{q}} \\
& \left. + \left(\int_0^1 t|t-\theta| dt\right)^{1-\frac{1}{q}} \left(\int_0^1 t|t-\theta| \left|f'(ta + (1-t)A_\lambda)\right|^q dt\right)^{\frac{1}{q}} \right\} \\
& + (b-a)(1-\lambda)^2 \left\{ \left(\int_0^1 (1-t)|t-\theta| dt\right)^{1-\frac{1}{q}} \right. \\
& \times \left(\int_0^1 (1-t)|t-\theta| \left|f'(tb + (1-t)A_\lambda)\right|^q dt\right)^{\frac{1}{q}} \\
& \left. + \left(\int_0^1 t|t-\theta| dt\right)^{1-\frac{1}{q}} \left(\int_0^1 t|t-\theta| \left|f'(tb + (1-t)A_\lambda)\right|^q dt\right)^{\frac{1}{q}} \right\} \\
\leq\ & (b-a)\lambda^2 M_1 \left[N_1(\theta) + N_2(\theta)\right] + (b-a)(1-\lambda)^2 M_2 \left[N_1(\theta) + N_2(\theta)\right] \\
=\ & (b-a)\left[\lambda^2 M_1 + (1-\lambda)^2 M_2\right]\left[N_1(\theta) + N_2(\theta)\right]
\end{aligned}$$

where

$$\begin{aligned}
N_1(\theta) &:= \int_0^1 (1-t)|t-\theta| dt = (1-\theta)\frac{\theta^2 + (1-\theta)^2}{2} + \frac{\theta^3 - (1-\theta)^3}{3} \\
N_2(\theta) &:= \int_0^1 t|t-\theta| dt = \theta\frac{\theta^2 + (1-\theta)^2}{2} + \frac{(1-\theta)^3 - \theta^3}{3}.
\end{aligned}$$

□

Remark 2.10. The inequality (2.6) coincides with the the inequality (1.3).

Using Lemma 1.2 we shall give another result for quasi convex functions as follows using the Hölder and Hölder-İşcan integral inequality. After, we will compare the results obtained with Hölder and Hölder-İşcan inequalities.

Theorem 2.11. *Let $f : I \subseteq [1,\infty) \to \mathbb{R}$ be a differentiable mapping on I° such that $f' \in L[a,b]$, where $a, b \in I^\circ$ with $a < b$ and $\theta, \lambda \in [0,1]$. If $|f'|^q$ is quasi convex* *function on* $[a,b]$, $q>1$, *then the following inequality holds:*

$$\begin{aligned}&\left|(1-\theta)\left(\lambda f(a)+(1-\lambda)f(b)\right)+\theta f((1-\lambda)a+\lambda b)-\frac{1}{b-a}\int_a^b f(x)dx\right|\\ \leq\ & (b-a)\left[\lambda^2 M_1+(1-\lambda)^2 M_2\right]\left(\frac{\theta^{q+1}+(1-\theta)^{q+1}}{q+1}\right)^{\frac{1}{q}} \qquad (2.7)\end{aligned}$$

where $\frac{1}{p}+\frac{1}{q}=1$.

Proof. Using Lemma 1.2, Hölder integral inequality and quasi convexity of the function $|f'|^q$, we have

$$\begin{aligned}&\left|(1-\theta)\left(\lambda f(a)+(1-\lambda)f(b)\right)+\theta f(A_\lambda)-\frac{1}{b-a}\int_a^b f(x)dx\right|\\ \leq\ & (b-a)\lambda^2\left[\int_0^1|t-\theta|\left|f'\left(ta+(1-t)A_\lambda\right)\right|dt\right.\\ & \left.+(1-\lambda)^2\int_0^1|t-\theta|\left|f'\left(tb+(1-t)A_\lambda\right)\right|dt\right]\\ \leq\ & (b-a)\lambda^2\left(\int_0^1|t-\theta|^q\left|f'\left(ta+(1-t)A_\lambda\right)\right|^q dt\right)^{\frac{1}{q}}\\ & +(b-a)(1-\lambda)^2\left(\int_0^1|t-\theta|^q\left|f'\left(tb+(1-t)A_\lambda\right)\right|^q dt\right)^{\frac{1}{q}}\\ \leq\ & (b-a)\left[\lambda^2 M_1+(1-\lambda)^2 M_2\right]\left(\int_0^1|t-\theta|^q dt\right)^{\frac{1}{q}}\\ =\ & (b-a)\left[\lambda^2 M_1+(1-\lambda)^2 M_2\right]\left(\frac{\theta^{q+1}+(1-\theta)^{q+1}}{q+1}\right)^{\frac{1}{q}},\end{aligned}$$

where

$$\int_0^1|t-\theta|^q dt=\frac{\theta^{q+1}+(1-\theta)^{q+1}}{q+1}.$$

□

Theorem 2.12. *Let* $f : I \subseteq [1,\infty)\to\mathbb{R}$ *be a differentiable mapping on* I° *such that* $f'\in L[a,b]$, *where* $a,b\in I^\circ$ *with* $a<b$ *and* $\theta,\lambda\in[0,1]$. *If* $|f'|^q$ *is quasi convex function on* $[a,b]$, $q>1$, *then the following inequality holds:*

$$\begin{aligned}&\left|(1-\theta)\left(\lambda f(a)+(1-\lambda)f(b)\right)+\theta f((1-\lambda)a+\lambda b)-\frac{1}{b-a}\int_a^b f(x)dx\right|\\ \leq\ & (b-a)\left(\frac{1}{2}\right)^{\frac{1}{p}}\left(\lambda^2 M_1+(1-\lambda)^2 M_2\right)\left[C^{\frac{1}{q}}(\theta,q)+D^{\frac{1}{q}}(\theta,q)\right] \qquad (2.8)\end{aligned}$$

where $\frac{1}{p}+\frac{1}{q}=1$.

Proof. From Lemma 1.2 and by Hölder-İşcan integral inequality, we have

$$\begin{aligned}
&\left|(1-\theta)\left(\lambda f(a)+(1-\lambda)f(b)\right)+\theta f(A_\lambda)-\frac{1}{b-a}\int_a^b f(x)dx\right| \qquad (2.9)\\
\leq\ & (b-a)\lambda^2\left[\int_0^1 |t-\theta|\left|f'\left(ta+(1-t)A_\lambda\right)\right|dt\right.\\
&\left.+(1-\lambda)^2\int_0^1 |t-\theta|\left|f'\left(tb+(1-t)A_\lambda\right)\right|dt\right]\\
\leq\ & (b-a)\lambda^2\left\{\left(\int_0^1(1-t)dt\right)^{\frac{1}{p}}\left(\int_0^1(1-t)|t-\theta|^q\left|f'\left(ta+(1-t)A_\lambda\right)\right|^q dt\right)^{\frac{1}{q}}\right.\\
&\left.+\left(\int_0^1 tdt\right)^{\frac{1}{p}}\left(\int_0^1 t|t-\theta|^q\left|f'\left(ta+(1-t)A_\lambda\right)\right|^q dt\right)^{\frac{1}{q}}\right\}\\
&+(b-a)(1-\lambda)^2\left\{\left(\int_0^1(1-t)dt\right)^{\frac{1}{p}}\right.\\
&\times\left(\int_0^1(1-t)|t-\theta|^q\left|f'\left(tb+(1-t)A_\lambda\right)\right|^q dt\right)^{\frac{1}{q}}\\
&\left.+\left(\int_0^1 tdt\right)^{\frac{1}{p}}\left(\int_0^1 t|t-\theta|^q\left|f'\left(tb+(1-t)A_\lambda\right)\right|^q dt\right)^{\frac{1}{q}}\right\}\\
\leq\ & (b-a)\left(\frac{1}{2}\right)^{\frac{1}{p}}\left(\lambda^2 M_1+(1-\lambda)^2 M_2\right)\left[C^{\frac{1}{q}}(\theta,q)+D^{\frac{1}{q}}(\theta,q)\right].
\end{aligned}$$

Since $|f'|^q$ is quasi convex function on interval $[a,b]$, the following inequalities holds.

$$\int_0^1\left|f'\left(ta+(1-t)A_\lambda\right)\right|^q dt \leq \max\left\{|f'(a)|^q,|f'(A_\lambda)|^q\right\}=M_1 \qquad (2.10)$$

$$\int_0^1\left|f'\left(tb+(1-t)A_\lambda\right)\right|^q dt \leq \max\left\{|f'(b)|^q,|f'(A_\lambda)|^q\right\}=M_2 \qquad (2.11)$$

Here, by simple computation we obtain

$$\int_0^1(1-t)dt=\int_0^1 tdt=\frac{1}{2},$$

$$\begin{aligned}
C(\theta,q) &= \int_0^1(1-t)|t-\theta|^q dt\\
&= (1-\theta)\left[\frac{\theta^{q+1}+(1-\theta)^{q+1}}{q+1}\right]+\left[\frac{\theta^{q+2}-(1-\theta)^{q+2}}{q+2}\right] \qquad (2.12)
\end{aligned}$$

$$\begin{aligned} D(\theta, q) &= \int_0^1 t\,|t-\theta|^q\,dt \\ &= \theta\left[\frac{\theta^{q+1}+(1-\theta)^{q+1}}{q+1}\right]-\left[\frac{\theta^{q+2}-(1-\theta)^{q+2}}{q+2}\right]. \end{aligned} \tag{2.13}$$

Thus, using (2.10)-(2.13) in (2.9), we obtain the inequality (2.8). This completes the proof. □

Remark 2.13. The inequality (2.8) is better than the inequality (2.7). For this, we need to show that

$$\begin{aligned} &(b-a)\left(\frac{1}{2}\right)^{\frac{1}{p}}\left(\lambda^2 M_1+(1-\lambda)^2 M_2\right)\left[C^{\frac{1}{q}}(\theta, q)+D^{\frac{1}{q}}(\theta, q)\right] \\ \leq\ &(b-a)\left[\lambda^2 M_1+(1-\lambda)^2 M_2\right]\left(\frac{\theta^{q+1}+(1-\theta)^{q+1}}{q+1}\right)^{\frac{1}{q}}. \end{aligned}$$

Using the inequalities (2.12), (2.13) and concavity of $\psi:[0,\infty)\to\mathbb{R}, \psi(x)=x^s, 0<s\leq 1$, we have

$$\begin{aligned} &(b-a)\left(\frac{1}{2}\right)^{\frac{1}{p}}\left(\lambda^2 M_1+(1-\lambda)^2 M_2\right)\left[C^{\frac{1}{q}}(\theta, q)+D^{\frac{1}{q}}(\theta, q)\right] \\ \leq\ &(b-a)2^{\frac{1}{q}}\left(\lambda^2 M_1+(1-\lambda)^2 M_2\right)\left(\frac{C(\theta, q)+D(\theta, q)}{2}\right)^{\frac{1}{q}} \\ =\ &(b-a)2^{\frac{1}{q}}\left(\lambda^2 M_1+(1-\lambda)^2 M_2\right)\left(\frac{1}{2}\frac{\theta^{q+1}+(1-\theta)^{q+1}}{q+1}\right)^{\frac{1}{q}} \\ =\ &(b-a)\left(\lambda^2 M_1+(1-\lambda)^2 M_2\right)\left(\frac{\theta^{q+1}+(1-\theta)^{q+1}}{q+1}\right)^{\frac{1}{q}} \end{aligned}$$

which is the required.

Corollary 2.14. *Under the assumptions of Theorem 2.12 with $\theta=1$, then we have the following generalized midpoint type inequality*

$$\begin{aligned} &\left|f((1-\lambda)a+\lambda b)-\frac{1}{b-a}\int_a^b f(x)dx\right| \qquad (2.14) \\ &\leq (b-a)\left(\frac{1}{2}\right)^{\frac{1}{p}}\left(\lambda^2 M_1+(1-\lambda)^2 M_2\right)\left(\frac{1}{q+2}\right)^{\frac{1}{q}}\left[1+\left(\frac{1}{q+1}\right)^{\frac{1}{q}}\right]. \end{aligned}$$

where $A_\lambda=(1-\lambda)a+\lambda b$ and $\frac{1}{p}+\frac{1}{q}=1$.

Corollary 2.15. *Under the assumptions of Theorem 2.12 with $\theta = 0$, then we have the following generalized trapezoid type inequality*

$$\begin{aligned}
&\left|\lambda f(a) + (1-\lambda) f(b) - \frac{1}{b-a}\int_a^b f(x)dx\right| \\
&\leq (b-a)\left(\frac{1}{2}\right)^{\frac{1}{p}}\left(\lambda^2 M_1 + (1-\lambda)^2 M_2\right)\left(\frac{1}{q+2}\right)^{\frac{1}{q}}\left[1+\left(\frac{1}{q+1}\right)^{\frac{1}{q}}\right],
\end{aligned}$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

Corollary 2.16. *Under the assumptions of Theorem 2.12 with $\theta = 1$, if $|f'(x)| \leq M$, $x \in [a,b]$, then we have the following Ostrowski type inequality*

$$\left|f(x) - \frac{1}{b-a}\int_a^b f(u)du\right| \leq (b-a) M \left(\frac{1}{2}\right)^{\frac{1}{p}}\left(\frac{1}{q+2}\right)^{\frac{1}{q}}\left[1+\left(\frac{1}{q+1}\right)^{\frac{1}{q}}\right]$$

for each $x \in [a,b]$.

Proof. For each $x \in [a,b]$, there exist $\lambda_x \in [0,1]$ such that $x = (1-\lambda_x) a + \lambda_x b$. Hence we have $\lambda_x = \frac{x-a}{b-a}$ and $1 - \lambda_x = \frac{b-x}{b-a}$. Therefore, for each $x \in [a,b]$, from the inequality (2.8) we obtain the desired inequality. $\square$

Corollary 2.17. *Under the assumptions of Theorem 2.12 with $\lambda = \frac{1}{2}$ and $\theta = \frac{2}{3}$, then we have the following Simpson type inequality*

$$\begin{aligned}
&\left|\frac{1}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right] - \frac{1}{b-a}\int_a^b f(x)dx\right| \\
&\leq \frac{b-a}{2}\left(\frac{1}{2}\right)^{\frac{1}{p}} A(M_1, M_2)\left[C^{\frac{1}{q}}\left(\frac{2}{3}, q\right) + D^{\frac{1}{q}}\left(\frac{2}{3}, q\right)\right]
\end{aligned}$$

where A is the arithmetic mean.

Corollary 2.18. *Under the assumptions of Theorem 2.12 with $\lambda = \frac{1}{2}$ and $\theta = 1$, then we have the following midpoint type inequality*

$$\begin{aligned}
&\left|f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)dx\right| \\
&\leq \frac{b-a}{2}\left(\frac{1}{2}\right)^{\frac{1}{p}} A(M_1, M_2)\left(\frac{1}{q+2}\right)^{\frac{1}{q}}\left[1+\left(\frac{1}{q+1}\right)^{\frac{1}{q}}\right],
\end{aligned}$$

where A is the arithmetic mean.

Corollary 2.19. *Under the assumptions of Theorem 2.12 with* $\lambda = \frac{1}{2}$ *and* $\theta = 0$, *then we have the following trapezoid type inequality*

$$\left| \frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx \right|$$
$$\leq \frac{b-a}{2}\left(\frac{1}{2}\right)^{\frac{1}{p}} A(M_1, M_2)\left(\frac{1}{q+2}\right)^{\frac{1}{q}}\left[1+\left(\frac{1}{q+1}\right)^{\frac{1}{q}}\right],$$

where A *is the arithmetic mean.*

3. Some applications for special means

Let us recall the following special means of arbitrary real numbers a, b with $a \neq b$ and $\alpha \in [0,1]$:

1. The weighted arithmetic mean

$$A_\alpha(a,b) := \alpha a + (1-\alpha)b, \quad a, b \in \mathbb{R}$$

2. The unweighted arithmetic mean

$$A(a,b) := \frac{a+b}{2}, \quad a, b \in \mathbb{R}$$

3. The weighted harmonic mean

$$H_\alpha(a,b) := \left(\frac{\alpha}{a} + \frac{1-\alpha}{b}\right)^{-1}, \quad a, b \in \mathbb{R}\backslash\{0\}$$

4. The unweighted harmonic mean

$$H(a,b) := \frac{2ab}{a+b}, \quad a, b \in \mathbb{R}\backslash\{0\}$$

5. The Logarithmic mean

$$L(a,b) := \frac{b-a}{\ln b - \ln a}, \quad a, b > 0,\ a \neq b$$

6. The n-logarithmic mean

$$L_n(a,b) := \left(\frac{b^n - a^n}{(n+1)(b-a)}\right)^{\frac{1}{n}}, \ n \in \mathbb{N},\ a, b \in \mathbb{R},\ a \neq b$$

Proposition 3.1. *Let* $a, b \in \mathbb{R}$ *with* $a < b$, *and* $n \in \mathbb{N}, n \geq 2$. *Then, for* $\theta, \lambda \in [0,1]$ *and* $q > 1$, *we have the following inequality:*

$$\left|(1-\theta)A_\lambda(a^n, b^n) + \theta A_\lambda^n(a,b) - L_n^n(a,b)\right|$$
$$\leq \frac{b-a}{2} 2^{\frac{1}{p}}\left[\lambda^2 M_1 + (1-\lambda)^2 M_2\right]\left[N_1^{\frac{1}{p}}(\theta,p) + N_2^{\frac{1}{p}}(\theta,p)\right],$$

where $M_1 = \max\left\{|a|^{n-1}, |A_\lambda(a,b)|^{n-1}\right\}$, $M_2 = \max\left\{|A_\lambda(a,b)|^{n-1}, |b|^{n-1}\right\}$.

Proof. The assertion follows from the Theorem 2.1, for $f(x) = x^n, x \in \mathbb{R}$. □

Proposition 3.2. *Let $a, b \in \mathbb{R}$ with $0 < a < b$, and $\theta, \lambda \in [0,1]$. Then, for $q > 1$, we have the following inequality:*

$$\begin{aligned}&\left|(1-\theta) H_{\lambda}^{-1}(a,b) + \theta A_{\lambda}^{-1}(a,b) - L^{-1}(a,b)\right| \\ &\leq \frac{b-a}{2} 2^{\frac{1}{p}} \left[\lambda^2 M_1 + (1-\lambda)^2 M_2\right] \left[N_1^{\frac{1}{p}}(\theta,p) + N_2^{\frac{1}{p}}(\theta,p)\right],\end{aligned}$$

where $M_1 = \max\left\{a^{-2}, A_{\lambda}^{-2}(a,b)\right\}$, $M_2 = \max\left\{A_{\lambda}^{-2}(a,b), b^{-2}\right\}$.

Proof. The assertion follows from the Theorem 2.1, for $f(x) = \frac{1}{x}, x \in (0,\infty)$. □

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