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# Ostrowski type inequalities via $\psi - (\alpha, \beta, \gamma, \delta)-$convex function

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**Abstract.** In this paper, we are introducing very first time the class of $\psi - (\alpha, \beta, \gamma, \delta)-$convex function in mixed kind, which is the generalization of many classes of convex functions. We would like to state well-known Ostrowski inequality via Montgomery identity for $\psi-(\alpha, \beta, \gamma, \delta)-$convex function in mixed kind. In addition, we establish some Ostrowski type inequalities for the class of functions whose derivatives in absolute values at certain powers are $\psi - (\alpha, \beta, \gamma, \delta)$-convex functions in mixed kind by using different techniques including Hölder's inequality and power mean inequality. Also, various established results would be captured as special cases. Moreover, some applications in terms of special means would also be given.

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**Keywords:** Ostrowski inequality, Montgomery identity, convex functions, special means.

## 1. Introduction

In almost every field of science, inequalities play a significant role. Although it is a very vast disciplineour focus is mainly on Ostrowski-type inequalities. In 1938, Ostrowski established the following interesting integral inequality for differentiable mappings with bounded derivatives [15]. This inequality is well known in the literature as Ostrowski inequality.

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**Theorem 1.1.** *Let $f : [a,b] \to \mathbb{R}$ be differentiable function on $(a,b)$ with the property that $|f'(t)| \leq M$ $\forall t \in (a,b)$. Then*

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)dt \right| \leq M(b-a)\left[\frac{1}{4} + \left(\frac{x - \frac{a+b}{2}}{b-a}\right)^2\right], \tag{1.1}$$

*$\forall x \in (a,b)$. The constant $\frac{1}{4}$ is the best possible in the kind that it cannot be replaced by a smaller quantity.*

Ostrowski inequality has applications in numerical integration, probability and optimization theory, statistics, information, and integral operator theory. Until now, a large number of research papers and books have been written on generalizations of Ostrowski inequalities and their numerous applications in [7]-[11]. Now we would like to present the Montgomery identity:

**Theorem 1.2.** [7] *Let $a < b$, $f \in AC[a,b]$ and $f' \in L_1[a,b]$, then the Montgomery identity holds:*

$$f(x) = \frac{1}{b-a}\int_a^b f(t)dt + \frac{1}{b-a}\int_a^b P_1(x,t)f'(t)dt,$$

*where $P_1(x,t)$ is the Peano Kernel defined by:*

$$P_1(x,t) = \begin{cases} t-a, & \text{if } t \in [a,x], \\ t-b, & \text{if } t \in (x,b], \end{cases}$$

$\forall x \in [a,b]$.

From literature, we recall and introduce some definitions for various convex functions.

**Definition 1.3.** [3] The $\tau : I \subset \mathbb{R} \to \mathbb{R}$ is said to be convex function, if

$$\tau(tx + (1-t)y) \leq t\tau(x) + (1-t)\tau(y),$$

$\forall x, y \in I, t \in [0,1]$.

We recall here definition of $P-$convex function from [3]:

**Definition 1.4.** Let $\tau : I \subset \mathbb{R} \to \mathbb{R}$ is a $P-$convex, if $\tau(x) \geq 0$ and

$$\tau(tx + (1-t)y) \leq \tau(x) + \tau(y),$$

$\forall x, y \in I$ and $t \in [0,1]$.

Here we also have definition of quasi$-$convex (for detailed discussion see [3].

**Definition 1.5.** The $\tau : I \subset \mathbb{R} \to \mathbb{R}$ is known as quasi$-$convex, if

$$\tau(tx + (1-t)y) \leq \max\{\tau(x), \tau(y)\}$$

$\forall x, y \in I$, $t \in [0,1]$.

Now we present definition of $s-$convex functions in the first kind as follows which are extracted from [14]:

**Definition 1.6.** [4] Let $s \in (0,1]$. The $\tau : I \subset [0,\infty) \to [0,\infty)$ is said to be $s-$convex in the $1^{st}$ kind, if

$$\tau\left(tx+(1-t)y\right) \leq t^s\tau(x)+(1-t^s)\tau(y),$$

$\forall x,y \in I, t \in [0,1]$.

**Remark 1.7.** If $s \to 0$, we get refinement of quasi-convexity (see Definition 1.5).

For second kind convexity we recall definition from [14].

**Definition 1.8.** Let $s \in (0,1]$. The $\tau : I \subset [0,\infty) \to [0,\infty)$ is said to be $s-$convex in the $2^{nd}$ kind, if

$$\tau\left(tx+(1-t)y\right) \leq t^s\tau(x)+(1-t)^s\tau(y),$$

$\forall x,y \in I, t \in [0,1]$.

**Remark 1.9.** Further if $s \to 0$, we easily get $P-$convexity (see Definition 1.4).

**Definition 1.10.** [14] Let $(\alpha,\beta) \in (0,1]^2$. The $\tau : I \subset [0,\infty) \to [0,\infty)$ is said to be $(\alpha,\beta)-$convex in the $1^{st}$ kind, if

$$\tau\left(tx+(1-t)y\right) \leq t^\alpha\tau(x)+(1-t^\beta)\tau(y),$$

$\forall x,y \in I, t \in [0,1]$.

**Definition 1.11.** [14] Let $(\alpha,\beta) \in (0,1]^2$. The $\tau : I \subset [0,\infty) \to [0,\infty)$ is said to be $(\alpha,\beta)-$convex in the $2^{nd}$ kind, if

$$\tau\left(tx+(1-t)y\right) \leq t^\alpha\tau(x)+(1-t)^\beta\tau(y),$$

$\forall x,y \in I, t \in [0,1]$.

**Definition 1.12.** [14] The $\tau : I \subset \mathbb{R} \to \mathbb{R}$ is a Godunova-Levin convex, if $\tau(x) \geq 0$ and

$$\tau\left(tx+(1-t)y\right) \leq \frac{1}{t}\tau(x)+\frac{1}{1-t}\tau(y),$$

$\forall x,y \in I$ and $t \in (0,1)$.

**Definition 1.13.** [14] The $\tau : I \subset \mathbb{R} \to [0,\infty)$ is of Godunova-Levin $s-$convex, with $s \in (0,1]$, if

$$\tau\left(tx+(1-t)y\right) \leq \frac{1}{t^s}\tau(x)+\frac{1}{(1-t)^s}\tau(y),$$

$\forall t \in (0,1)$ and $x,y \in I$.

**Definition 1.14.** [14] Let $h : J \subseteq \mathbb{R} \to [0,\infty)$ with $h \neq 0$. The $\tau : I \subseteq \mathbb{R} \to [0,\infty)$ is an $h-$convex, if $\forall x,y \in I$, we have

$$\tau\left(tx+(1-t)y\right) \leq h(t)\tau(x)+h(1-t)\tau(y),$$

$\forall t \in (0,1)$.

**Definition 1.15.** [3] The $\tau : I \subset \mathbb{R} \to \mathbb{R}$ is said to be $MT-$convex, if $\tau(x) \geq 0$, and

$$\tau\left(tx + (1-t)y\right) \leq \frac{\sqrt{t}}{2\sqrt{1-t}}\tau(x) + \frac{\sqrt{1-t}}{2\sqrt{t}}\tau(y),$$

$\forall x, y \in I, t \in (0,1)$.

Let $[a,b] \subseteq (0,+\infty)$, we may define special means as follows:

(a) The arithmetic mean

$$A = A(a,b) := \frac{a+b}{2};$$

(b) The geometric mean

$$G = G(a,b) := \sqrt{ab};$$

(c) The harmonic mean

$$H = H(a,b) := \frac{2}{\frac{1}{a}+\frac{1}{b}};$$

(d) The logarithmic mean

$$L = L(a,b) := \begin{cases} a & \text{if } a = b \\ \frac{b-a}{\ln b - \ln a}, & \text{if } a \neq b \end{cases};$$

(e) The identric mean

$$I = I(a,b) := \begin{cases} a & \text{if } a = b \\ \frac{1}{e}\left(\frac{b^b}{a^a}\right)^{\frac{1}{b-a}}, & \text{if } a \neq b. \end{cases};$$

(f) The $p-$logarithmic mean

$$L_p = L_p(a,b) := \begin{cases} a & \text{if } a = b \\ \left[\frac{b^{p+1} - a^{p+1}}{(p+1)(b-a)}\right]^{\frac{1}{p}}, & \text{if } a \neq b. \end{cases};$$

where $p \in \mathbb{R} \setminus \{0,-1\}$.

We make use of the beta function of Euler type, which is for $x, y > 0$ defined as

$$B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}dt = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)},$$

where $\Gamma(x) = \int_0^\infty e^{-u}u^{x-1}du$.

The main aim of our study is to generalize the Ostrowski inequality (1.1) for $\psi-(\alpha,\beta,\gamma,\delta)-$convex in mixed kind, which is given in Section 2. Moreover, we establish some Ostrowski type inequalities for the class of functions whose derivatives in absolute values at certain powers are $\psi-(\alpha,\beta,\gamma,\delta)$-convex functions in mixed kind by using different techniques including Hölder's inequality and power means inequality. Also, we give special cases of our results. The application of midpoint inequalities in the special means, some particular cases of these inequalities are given in Section 3. The last section gives us a conclusion with some remarks and future ideas.

## 2. Generalization of Ostrowski type inequalities

Convexity is a very simple and ordinary concept, due to its massive applications in industry and business, convexity has a great influence on our daily life. In the solution of many real-world problems, the concept of convexity is very decisive. The problems faced in constrained control and estimation are convex. Geometrically, a real-valued function is said to be convex if the line segment joining any two of its points lies on or above the graph of the function in Euclidean space. We are introducing the very first time the class of $(s,r)-$convex and $\psi-(\alpha,\beta,\gamma,\delta)-$convex function in mixed kind.

**Definition 2.1.** [12] Let $(s,r)\in(0,1]^2$. The $\tau: I\subset[0,\infty)\to[0,\infty)$ is said to be $(s,r)-$convex in mixed kind, if

$$\tau\left(tx+(1-t)y\right)\leq t^{rs}\tau(x)+(1-t^r)^s\tau(y),$$

$\forall x,y\in I, t\in[0,1]$.

**Definition 2.2.** [12] Let $(\alpha,\beta,\gamma,\delta)\in(0,1]^4$. The $\tau: I\subset[0,\infty)\to[0,\infty)$ is said to be $(\alpha,\beta,\gamma,\delta)-$convex in mixed kind, if

$$\tau\left(tx+(1-t)y\right)\leq t^{\alpha\gamma}\tau(x)+(1-t^\beta)^\delta\tau(y),$$

$\forall x,y\in I, t\in[0,1]$.

**Definition 2.3.** [12] Let $\psi:(0,1)\to(0,\infty)$, the $\tau: I\subset\mathbb{R}\to[0,\infty)$ is a $\psi-$convex, if $\forall x,y\in I$ we have

$$\tau\left(tx+(1-t)y\right)\leq t\psi(t)\tau(x)+(1-t)\psi(1-t)\tau(y),$$

$\forall t\in(0,1)$.

Introducing a new class of convex functions that generalizes numerous well-known and highly regarded classes of convex functions, providing a broader framework for analysis and application in mathematical and optimization contexts.

**Definition 2.4.** Let $(\alpha,\beta,\gamma,\delta)\in(0,1]^4$, and $\psi:(0,1)\to(0,\infty)$. The $\tau: I\subset[0,\infty)\to[0,\infty)$ is said to be $\psi-(\alpha,\beta,\gamma,\delta)-$convex in mixed kind, if

$$\tau\left(tx+(1-t)y\right)\leq t^{\alpha\gamma}\psi(t)\tau(x)+(1-t^\beta)^\delta\psi(1-t)\tau(y),\tag{2.1}$$

$\forall x,y\in I, t\in[0,1]$.

**Remark 2.5.** In Definition 2.4, we have the following cases.

1. If $\psi(t)=1$ in (2.1), we get $(\alpha,\beta,\gamma,\delta)-$convex in mixed kind.
2. If $\psi(t)=\gamma=\delta=1$ in (2.1), we get $(\alpha,\beta)-$convex in $1^{st}$ kind.
3. If $\psi(t)=\beta=\gamma=1$ in (2.1), we get $(\alpha,\beta)-$convex in $2^{nd}$ kind.
4. If $\psi(t)=1,\alpha=\delta=s,\beta=\gamma=r$, where $s,r\in(0,1]$ in (2.1), we get $(s,r)-$convex in mixed kind.

5. If $\alpha=\beta=s$ and $\psi(t)=\gamma=\delta=1$ where $s\in(0,1]$ in (2.1), we get $s-$convex in $1^{st}$ kind.
6. If $\alpha=\beta\to 0$, and $\psi(t)=\gamma=\delta=1$, in (2.1), we get refinement of quasi-convex.
7. If $\alpha=\delta=s$, $\psi(t)=\beta=\gamma=1$ where $s\in(0,1]$ or ($\alpha=\beta=\gamma=\delta=1, \psi(t)=t^{s-1}$ with $s\in(0,1]$) in (2.1), we get $s-$convex in $2^{nd}$ kind.
8. If $\alpha=\delta\to 0$, and $\psi(t)=\beta=\gamma=1$, or ($\alpha=\beta=\gamma=\delta=1$, and $\psi(t)=\frac{1}{t}$) in (2.1), we get $P-$convex.
9. If $\psi(t)=\alpha=\beta=\gamma=\delta=1$ in (2.1), gives us ordinary convex.
10. If $\alpha=\beta=\gamma=\delta=1$ in (2.1), gives us $\psi-$convex.
11. If $\alpha=\beta=\gamma=\delta=1, l(t)=t$, $h=l\psi$ in (2.1), we get $h-$convex.
12. If $\alpha=\beta=\gamma=\delta=1, \psi(t)=\frac{1}{t^{s+1}}$ with $s\in[0,1)$ in (2.1), then we get the class of Godunova-Levin $s-$convex.
13. If $\alpha=\beta=\gamma=\delta=1, \psi(t)=\frac{1}{t^2}$ in (2.1), then we get the concept of Godunova-Levin convex.
14. If $\alpha=\beta=\gamma=\delta=1, \psi(t)=\frac{1}{2\sqrt{t(1-t)}}$ in (2.1), then we get the concept of $MT-$convex.

**Theorem 2.6.** *Suppose all the assumptions of Theorem* 1.2 *hold. If* $\tau:[a,b]\subseteq\mathbb{R}\to\mathbb{R}$ *is* $\psi-(\alpha,\beta,\gamma,\delta)-$*convex in mixed kind, then*

$$\begin{aligned}
&\tau\left(f(x)-\frac{1}{b-a}\int_a^b f(t)dt\right)\\
&\leq\left(\frac{x-a}{b-a}\right)^{\alpha\gamma}\psi\left(\frac{x-a}{b-a}\right)\left[\frac{1}{x-a}\int_a^x\tau\left[(t-a)f'(t)\right]dt\right]\\
&+\left(1-\left(\frac{x-a}{b-a}\right)^{\beta}\right)^{\delta}\psi\left(\frac{b-x}{b-a}\right)\left[\frac{1}{b-x}\int_x^b\tau\left[(t-b)f'(t)\right]dt\right],
\end{aligned}\tag{2.2}$$

$\forall x\in[a,b]$.

*Proof.* Utilizing the Theorem 1.2, we get

$$\begin{aligned}
f(x)-\frac{1}{b-a}\int_a^b f(t)dt &= \left(\frac{x-a}{b-a}\right)\left[\frac{1}{x-a}\int_a^x(t-a)f'(t)dt\right]\\
&+\left(1-\left(\frac{x-a}{b-a}\right)\right)\left[\frac{1}{b-x}\int_x^b(t-b)f'(t)dt\right],
\end{aligned}$$

using the $\psi-(\alpha,\beta,\gamma,\delta)-$convexity in mixed kind of $\tau:[a,b]\subseteq\mathbb{R}\to\mathbb{R}$, we have

$$\begin{aligned}
&\tau\left(f(x)-\frac{1}{b-a}\int_a^b f(t)dt\right)\\
&\leq\left(\frac{x-a}{b-a}\right)^{\alpha\gamma}\psi\left(\frac{x-a}{b-a}\right)\tau\left[\frac{1}{x-a}\int_a^x(t-a)f'(t)dt\right]\\
&+\left(1-\left(\frac{x-a}{b-a}\right)^{\beta}\right)^{\delta}\psi\left(\frac{b-x}{b-a}\right)\tau\left[\frac{1}{b-x}\int_x^b(t-b)f'(t)dt\right],
\end{aligned}$$

$\forall x \in [a,b]$, which is an inequality of interest in itself as well. If we use Jensen's integral inequality we get (2.2). □

**Corollary 2.7.** *In Theorem* 2.6, *one can see the following.*

1. *If* $\psi(t) = 1$, *in* (2.2), *then functional generalization of Ostrowski inequality for* $(\alpha,\beta,\gamma,\delta)-$*convex functions in mixed kind:*

$$\begin{aligned}&\tau\left(f(x)-\frac{1}{b-a}\int_a^b f(t)dt\right)\\&\leq\left(\frac{x-a}{b-a}\right)^{\alpha\gamma}\left[\frac{1}{x-a}\int_a^x \tau\left[(t-a)f'(t)\right]dt\right]\\&+\left(1-\left(\frac{x-a}{b-a}\right)^{\beta}\right)^{\delta}\left[\frac{1}{b-x}\int_x^b \tau\left[(t-b)f'(t)\right]dt\right]\end{aligned}$$

2. *If* $\psi(t) = \gamma = \delta = 1$, *and* $\alpha,\beta \in (0,1]$ *in* (2.2), *then functional generalization of Ostrowski inequality for* $(\alpha,\beta)-$*convex functions in* $1^{st}$ *kind:*

$$\begin{aligned}&\tau\left(f(x)-\frac{1}{b-a}\int_a^b f(t)dt\right)\\&\leq\left(\frac{x-a}{b-a}\right)^{\alpha}\left[\frac{1}{x-a}\int_a^x \tau[(t-a)f'(t)]dt\right]\\&+\left(1-\left(\frac{x-a}{b-a}\right)^{\beta}\right)\left[\frac{1}{b-x}\int_x^b \tau[(t-b)f'(t)]dt\right].\end{aligned}$$

3. *If* $\psi(t) = \beta = \gamma = 1$, *and* $\alpha,\delta \in (0,1]$ *in* (2.2), *then functional generalization of Ostrowski inequality for* $(\alpha,\delta)-$*convex functions in* $2^{nd}$ *kind:*

$$\begin{aligned}&\tau\left(f(x)-\frac{1}{b-a}\int_a^b f(t)dt\right)\\&\leq\left(\frac{x-a}{b-a}\right)^{\alpha}\left[\frac{1}{x-a}\int_a^x \tau[(t-a)f'(t)]dt\right]\\&+\left(1-\left(\frac{x-a}{b-a}\right)\right)^{\delta}\left[\frac{1}{b-x}\int_x^b \tau[(t-b)f'(t)]dt\right].\end{aligned}$$

4. *If* $\psi(t) = 1, \alpha = \delta = s$, *and* $\beta = \gamma = r$, *where* $s, r \in (0,1]$ *in* (2.2), *then functional generalization of Ostrowski inequality for* $(s,r)-$*convex functions in mixed kind:*

$$\begin{aligned}&\tau\left(f(x)-\frac{1}{b-a}\int_a^b f(t)dt\right)\\&\leq\left(\frac{x-a}{b-a}\right)^{rs}\left[\frac{1}{x-a}\int_a^x \tau[(t-a)f'(t)]dt\right]\\&+\left(1-\left(\frac{x-a}{b-a}\right)^{r}\right)^{s}\left[\frac{1}{b-x}\int_x^b \tau[(t-b)f'(t)]dt\right].\end{aligned}$$

5. *If* $\alpha = \beta = s$ *and* $\psi(t) = \gamma = \delta = 1$, *where* $s \in (0, 1]$ *in* (2.2), *then functional generalization of Ostrowski inequality for* $s-$*convex functions in* $1^{st}$ *kind:*

$$\begin{aligned}
&\tau\left(f(x) - \frac{1}{b-a}\int_a^b f(t)dt\right) \\
&\leq \left(\frac{x-a}{b-a}\right)^s \left[\frac{1}{x-a}\int_a^x \tau[(t-a)f'(t)]dt\right] \\
&+\left(1-\left(\frac{x-a}{b-a}\right)^s\right)\left[\frac{1}{b-x}\int_x^b \tau[(t-b)f'(t)]dt\right].
\end{aligned}$$

6. *If* $\alpha = \beta \to 0$ *and* $\psi(t) = \gamma = \delta = 1$ *in* (2.2), *then functional generalization of Ostrowski inequality for quasi*$-$*convex functions:*

$$\tau\left(f(x) - \frac{1}{b-a}\int_a^b f(t)dt\right) \leq \frac{1}{x-a}\int_a^x \tau[(t-a)f'(t)]dt.$$

7. *If* $\alpha = \delta = s$, *and* $\psi(t) = \beta = \gamma = 1$, *where* $s \in [0, 1]$ *in* (2.2), *then functional generalization of Ostrowski inequality for* $s-$*convex functions in* $2^{nd}$ *kind:*

$$\begin{aligned}
&\tau\left(f(x) - \frac{1}{b-a}\int_a^b f(t)dt\right) \\
&\leq \left(\frac{x-a}{b-a}\right)^s \left[\frac{1}{x-a}\int_a^x \tau[(t-a)f'(t)]dt\right] \\
&+\left(\frac{b-x}{b-a}\right)^s\left[\frac{1}{b-x}\int_x^b \tau[(t-b)f'(t)]dt\right].
\end{aligned}$$

8. *If* $\alpha = \delta \to 0$ *and* $\psi(t) = \beta = \gamma = 1$ *in* (2.2), *then functional generalization of Ostrowski inequality for* $P-$*convex functions:*

$$\begin{aligned}
&\tau\left(f(x) - \frac{1}{b-a}\int_a^b f(t)dt\right) \\
&\leq \frac{1}{x-a}\int_a^x \tau[(t-a)f'(t)]dt + \frac{1}{b-x}\int_x^b \tau[(t-b)f'(t)]dt.
\end{aligned}$$

9. *If* $\psi(t) = \alpha = \beta = \gamma = \delta = 1$ *in* (2.2), *then functional generalization of Ostrowski inequality for convex functions which is inequality* (2.1) *of Theorem* 7 *in* [8].

10. *If* $\alpha = \beta = \gamma = \delta = 1$, *in* (2.2), *then functional generalization of Ostrowski inequality for* $\psi-$*convex functions:*

$$\begin{aligned}&\tau\left(f(x)-\frac{1}{b-a}\int_a^b f(t)dt\right)\\&\leq\frac{1}{b-a}\left[\psi\left(\frac{x-a}{b-a}\right)\int_a^x\tau\left[(t-a)\,f'(t)\right]dt\right.\\&\left.+\psi\left(\frac{b-x}{b-a}\right)\int_x^b\tau\left[(t-b)\,f'(t)\right]dt\right].\end{aligned}$$

11. *If* $\alpha = \beta = \gamma = \delta = 1, l(t) = t$, *and* $h = l\psi$ *in* (2.2), *then functional generalization of Ostrowski inequality for* $h-$*convex functions:*

$$\begin{aligned}&\tau\left(f(x)-\frac{1}{b-a}\int_a^b f(t)dt\right)\\&\leq h\left(\frac{x-a}{b-a}\right)\left[\frac{1}{x-a}\int_a^x\tau[(t-a)f'(t)]dt\right]\\&+h\left(\frac{b-x}{b-a}\right)\left[\frac{1}{b-x}\int_x^b\tau[(t-b)f'(t)]dt\right].\end{aligned}$$

12. *If* $\alpha = \beta = \gamma = \delta = 1, \psi(t) = \frac{1}{t^{s+1}}$ *with* $s \in [0,1]$ *in* (2.2), *then functional generalization of Ostrowski inequality for GL* $s-$*convex:*

$$\begin{aligned}&\tau\left(f(x)-\frac{1}{b-a}\int_a^b f(t)dt\right)\\&\leq(b-a)^s\left[\frac{1}{(x-a)^{s+1}}\int_a^x\tau[(t-a)f'(t)]dt\right.\\&\left.+\frac{1}{(b-x)^{s+1}}\int_x^b\tau[(t-b)f'(t)]dt\right].\end{aligned}$$

13. *If* $\alpha = \beta = \gamma = \delta = 1, \psi(t) = \frac{1}{t^2}$ *in* (2.2), *then functional generalization of Ostrowski inequality for GL convex:*

$$\begin{aligned}&\tau\left(f(x)-\frac{1}{b-a}\int_a^b f(t)dt\right)\\&\leq(b-a)\left[\frac{1}{(x-a)^2}\int_a^x\tau[(t-a)f'(t)]dt+\frac{1}{(b-x)^2}\int_x^b\tau[(t-b)f'(t)]dt\right].\end{aligned}$$

*14. If* $\alpha = \beta = \gamma = \delta = 1, \psi(t) = \frac{1}{2\sqrt{t(1-t)}}$ *in* (2.2), *then functional generalization of Ostrowski inequality for* $MT-$*convex:*

$$\tau\left(f(x) - \frac{1}{b-a}\int_a^b f(t)dt\right)$$
$$\leq \frac{1}{2\sqrt{(x-a)(b-x)}}\left[\int_a^x \tau[(t-a)f'(t)]dt + \int_x^b \tau[(t-b)f'(t)]dt\right].$$

In order to prove our main results, we need the following lemma that has been obtained in [16].

**Lemma 2.8.** *Let* $f : [a,b] \to \mathbb{R}$ *be an absolutely continuous mapping on* $(a,b)$ *with* $a < b$. *If* $f' \in L_1[a,b]$, *then* $\forall x \in (a,b)$

$$f(x) - \frac{1}{b-a}\int_a^b f(t)dt = \frac{(x-a)^2}{b-a}\int_0^1 tf'(tx+(1-t)a)dt - \frac{(b-x)^2}{b-a}\int_0^1 tf'(tx+(1-t)b)dt.$$

**Theorem 2.9.** *Let* $a < b, f \in AC[a,b], f' \in L_1[a,b]$, *and* $|f'|$ *is* $\psi - (\alpha,\beta,\gamma,\delta)-$*convex function with* $|f'(x)| \leq M$, *then* $\forall x \in (a,b)$

$$\left|f(x) - \frac{1}{b-a}\int_a^b f(t)dt\right|$$
$$\leq M\left(\int_0^1 \left(t^{\alpha\gamma+1}\psi(t) + t(1-t^\beta)^\delta\psi(1-t)\right)dt\right)\ \kappa_a^b(x), \tag{2.3}$$

*where* $\kappa_a^b(x) = \frac{(x-a)^2+(b-x)^2}{b-a}$.

*Proof.* From the Lemma 2.8 we have

$$\left|f(x) - \frac{1}{b-a}\int_a^b f(t)dt\right| \leq \frac{(x-a)^2}{b-a}\int_0^1 t\left|f'(tx+(1-t)a)\right|dt + \frac{(b-x)^2}{b-a}\int_0^1 t\left|f'(tx+(1-t)b)\right|dt. \tag{2.4}$$

Since $|f'|$ is $\psi - (\alpha,\beta,\gamma,\delta)-$convex and $|f'(x)| \leq M$, we get

$$\int_0^1 t\left|f'(tx+(1-t)a)\right|dt \leq M\int_0^1 t\left(t^{\alpha\gamma}\psi(t) + (1-t^\beta)^\delta\psi(1-t)\right)dt, \tag{2.5}$$

and similarly

$$\int_0^1 t\left|f'(tx+(1-t)b)\right|dt \leq M\int_0^1 t\left(t^{\alpha\gamma}\psi(t) + (1-t^\beta)^\delta\psi(1-t)\right)dt. \tag{2.6}$$

By using inequalities (2.5) and (2.6) in (2.4), we get (2.3). □

**Corollary 2.10.** *In Theorem* 2.9, *one can see the following.*

1. *If $\psi(t)=1$, in (2.3), then Ostrowski inequality for $(\alpha,\beta,\gamma,\delta)-$convex functions in mixed kind:*

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)dt\right|\leq M\left(\frac{1}{\alpha\gamma+2}+\frac{B\left(\frac{2}{\beta},\delta+1\right)}{\beta}\right)\ \kappa_a^b(x).$$

2. *If $\psi(t)=\gamma=\delta=1$, $\alpha\in[0,1]$ and $\beta\in(0,1]$, in (2.3), then Ostrowski inequality for $(\alpha,\beta)-$convex functions in $1^{st}$ kind:*

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)dt\right|\leq M\left(\frac{1}{\alpha+2}+\frac{B\left(\frac{2}{\beta},2\right)}{\beta}\right)\ \kappa_a^b(x).$$

3. *If $\psi(t)=\beta=\gamma=1$, $\alpha\in[0,1]$ and $\delta\in[0,1]$, in (2.3), then Ostrowski inequality for $(\alpha,\delta)-$convex functions in $2^{nd}$ kind:*

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)dt\right|\leq M\left(\frac{1}{\alpha+2}+\frac{1}{(\delta+1)(\delta+2)}\right)\ \kappa_a^b(x).$$

4. *If $\psi(t)=1,\alpha=\delta=s$, $\beta=\gamma=r$, where $s\in[0,1]$ and $r\in(0,1]$ in (2.3), then Ostrowski inequality for $(s,r)-$convex functions in mixed kind:*

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)dt\right|\leq M\left(\frac{1}{rs+2}+\frac{B\left(\frac{2}{r},s+1\right)}{r}\right)\ \kappa_a^b(x).$$

5. *If $\alpha=\beta=s$ and $\psi(t)=\gamma=\delta=1$, where $s\in(0,1]$ in (2.3), then Ostrowski inequality for $s-$convex functions in $1^{st}$ kind:*

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)dt\right|\leq M\left(\frac{1}{s+2}+\frac{B\left(\frac{2}{s},2\right)}{s}\right)\ \kappa_a^b(x).$$

6. *If $\alpha=\delta\to 0$ and $\psi(t)=\beta=\gamma=1$ in (2.3), then Ostrowski inequality for $P-$convex functions:*

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)dt\right|\leq M\ \kappa_a^b(x).$$

7. *If $\psi(t)=\beta=\gamma=1$, $\alpha=\delta=s$ where $s\in[0,1]$, then (2.3) reduces to the inequality (2.1) of Theorem 2 in [1].*
8. *If $\psi(t)=\alpha=\beta=\gamma=\delta=1$, then (2.3) reduces to the inequality (1.1).*
9. *If $\alpha=\beta=\gamma=\delta=1$ in (2.3), then Ostrowski inequality for $\psi-$convex:*

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)dt\right|\leq M\left(\int_0^1\left(t^2\psi(t)+t(1-t)\psi(1-t)\right)dt\right)\ \kappa_a^b(x).$$

10. *If $\alpha=\beta=\gamma=\delta=1,l(t)=t$, then if $h=l\psi$, in (2.3), then Ostrowski inequality for $h-$convex:*

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)dt\right|\leq M\left(\int_0^1\left(th(t)+th(1-t)\right)dt\right)\ \kappa_a^b(x).$$

11. *If* $\alpha=\beta=\gamma=\delta=1, \psi(t)=t^{-(s+1)}$ *in* (2.3), *then Ostrowski inequality for GL* $s-$*convex:*

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)dt\right| \leq M\left(\frac{1}{1-s}\right)\ \kappa_a^b(x).$$

12. *If* $\alpha=\beta=\gamma=\delta=1, \psi(t)=\frac{1}{2\sqrt{t(1-t)}}$ *in* (2.3), *then Ostrowski inequality for* $MT-$*convex:*

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)dt\right| \leq \frac{M\pi}{4}\ \kappa_a^b(x).$$

**Theorem 2.11.** *Let* $a<b, f\in AC[a,b], f'\in L_1[a,b]$, *and* $|f'|^q$ *is* $\psi-(\alpha,\beta,\gamma,\delta)-$*convex function for* $q\geq 1$ *with* $|f'(x)|\leq M$, *then* $\forall x\in(a,b)$

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)dt\right|$$
$$\leq \frac{M}{(2)^{1-\frac{1}{q}}}\left(\int_0^1\left(t^{\alpha\gamma+1}\psi(t)+t(1-t^{\beta})^{\delta}\psi(1-t)\right)dt\right)^{\frac{1}{q}}\ \kappa_a^b(x). \tag{2.7}$$

*Proof.* From the Lemma 2.8 and power mean inequality, we have

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)dt\right| \leq \frac{(x-a)^2}{b-a}\left(\int_0^1 tdt\right)^{1-\frac{1}{q}}\left(\int_0^1 t\left|f'\left(tx+(1-t)a\right)\right|^q dt\right)^{\frac{1}{q}}$$
$$+\frac{(b-x)^2}{b-a}\left(\int_0^1 tdt\right)^{1-\frac{1}{q}}\left(\int_0^1 t\left|f'\left(tx+(1-t)b\right)\right|^q dt\right)^{\frac{1}{q}}. \tag{2.8}$$

Since $|f'|^q$ is $\psi-(\alpha,\beta,\gamma,\delta)-$convex and $|f'(x)|\leq M$, we get

$$\int_0^1 t\left|f'\left(tx+(1-t)a\right)\right|^q dt \leq M^q\int_0^1 t\left(t^{\alpha\gamma}\psi(t)+(1-t^{\beta})^{\delta}\psi(1-t)\right)dt, \tag{2.9}$$

and

$$\int_0^1 t\left|f'\left(tx+(1-t)b\right)\right|^q dt \leq M^q\int_0^1 t\left(t^{\alpha\gamma}\psi(t)+(1-t^{\beta})^{\delta}\psi(1-t)\right)dt. \tag{2.10}$$

Using the inequalities (2.8) – (2.10), we get (2.7). □

**Corollary 2.12.** *In Theorem* 2.11, *one can see the following.*

1. *If* $q=1$, *then we get Theorem* 2.9.
2. *If* $\psi(t)=1$, *in* (2.7), *then Ostrowski inequality for* $(\alpha,\beta,\gamma,\delta)-$*convex functions in mixed kind:*

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)dt\right| \leq \frac{M}{(2)^{1-\frac{1}{q}}}\left(\frac{1}{\alpha\gamma+2}+\frac{B\left(\frac{2}{\beta},\delta+1\right)}{\beta}\right)^{\frac{1}{q}}\ \kappa_a^b(x).$$

3. *If $\psi(t)=\gamma=\delta=1$, $\alpha\in[0,1]$ and $\beta\in(0,1]$, in* (2.7), *then Ostrowski inequality for $(\alpha,\beta)-$convex functions in $1^{st}$ kind:*

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)dt\right|\leq\frac{M}{(2)^{1-\frac{1}{q}}}\left(\frac{1}{\alpha+2}+\frac{B\left(\frac{2}{\beta},2\right)}{\beta}\right)^{\frac{1}{q}}\ \kappa_a^b(x).$$

4. *If $\psi(t)=\beta=\gamma=1$, $\alpha\in[0,1]$ and $\delta\in[0,1]$, in* (2.7), *then Ostrowski inequality for $(\alpha,\delta)-$convex functions in $2^{nd}$ kind:*

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)dt\right|\leq\frac{M}{(2)^{1-\frac{1}{q}}}\left(\frac{1}{(\alpha+2)}+\frac{1}{(\delta+1)(\delta+2)}\right)^{\frac{1}{q}}\ \kappa_a^b(x).$$

5. *If $\psi(t)=1,\alpha=\delta=s$, $\beta=\gamma=r$, where $s\in[0,1]$ and $r\in(0,1]$ in* (2.7), *then Ostrowski inequality for $(s,r)-$convex functions in mixed kind:*

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)dt\right|\leq\frac{M}{(2)^{1-\frac{1}{q}}}\left(\frac{1}{rs+2}+\frac{B\left(\frac{2}{r},s+1\right)}{r}\right)^{\frac{1}{q}}\ \kappa_a^b(x).$$

6. *If $\alpha=\beta=s$ and $\psi(t)=\gamma=\delta=1$, where $s\in(0,1]$ in* (2.7), *then Ostrowski inequality for $s-$convex functions in $1^{st}$ kind:*

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)dt\right|\leq\frac{M}{(2)^{1-\frac{1}{q}}}\left(\frac{1}{s+2}+\frac{B\left(\frac{2}{s},2\right)}{s}\right)^{\frac{1}{q}}\ \kappa_a^b(x).$$

7. *If $\alpha=\delta\to 0$ and $\psi(t)=\beta=\gamma=1$ in* (2.7), *then Ostrowski inequality for $P-$convex functions:*

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)dt\right|\leq\frac{M}{(2)^{1-\frac{1}{q}}}\ \kappa_a^b(x).$$

8. *If $\psi(t)=\beta=\gamma=1$, $\alpha=\delta=s$ where $s\in[0,1]$, then* (2.7) *reduces to the inequality* (2.3) *of Theorem* 4 *in* [1].
9. *If $\psi(t)=\alpha=\beta=\gamma=\delta=1$, then* (2.7) *reduces to the inequality* (1.1).
10. *If $\alpha=\beta=\gamma=\delta=1$, in* (2.7), *then Ostrowski inequality for $\psi-$convex:*

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)dt\right|$$
$$\leq\frac{M}{2^{1-\frac{1}{q}}}\left(\int_0^1\left(t^2\psi(t)+t(1-t)\psi(1-t)\right)dt\right)^{\frac{1}{q}}\ \kappa_a^b(x).$$

11. *If $\alpha=\beta=\gamma=\delta=1,l(t)=t$, then if $h=l\psi$, in* (2.7), *then Ostrowski inequality for $h-$convex:*

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)dt\right|\leq\frac{M}{2^{1-\frac{1}{q}}}\left(\int_0^1\left(th(t)+th(1-t)\right)dt\right)^{\frac{1}{q}}\ \kappa_a^b(x).$$

12. *If* $\alpha=\beta=\gamma=\delta=1, \psi(t)=t^{-(s+1)}$ *in* (2.7), *then Ostrowski inequality for GL* $s-$*convex:*

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)dt\right| \leq \frac{M}{2^{1-\frac{1}{q}}}\left(\frac{1}{1-s}\right)^{\frac{1}{q}} \kappa_a^b(x).$$

13. *If* $\alpha=\beta=\gamma=\delta=1, \psi(t)=\frac{1}{2\sqrt{t(1-t)}}$ *in* (2.7), *then Ostrowski inequality for* $MT-$*convex:*

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)dt\right| \leq \frac{M\tau^{\frac{1}{q}}}{2^{1+\frac{1}{q}}} \kappa_a^b(x).$$

**Theorem 2.13.** *Let* $a<b, f\in AC[a,b], f'\in L_1[a,b]$, *and* $|f'|^q$ *is* $\psi-(\alpha,\beta,\gamma,\delta)-$*convex function for* $q>1$ *with* $|f'(x)|\leq M$, *then* $\forall x\in(a,b)$

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)dt\right|$$
$$\leq \frac{M}{(p+1)^{\frac{1}{p}}}\left(\int_0^1\left(t^{\alpha\gamma}\psi(t)+(1-t^{\beta})^{\delta}\psi(1-t)\right)dt\right)^{\frac{1}{q}} \kappa_a^b(x), \tag{2.11}$$

*where* $p^{-1}+q^{-1}=1$.

*Proof.* From the Lemma 2.8 and Hölder's inequality, we have

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)dt\right| \leq \frac{(x-a)^2}{b-a}\left(\int_0^1 t^p dt\right)^{\frac{1}{p}}\left(\int_0^1 |f'(tx+(1-t)a)|^q dt\right)^{\frac{1}{q}}$$
$$+\frac{(b-x)^2}{b-a}\left(\int_0^1 t^p dt\right)^{\frac{1}{p}}\left(\int_0^1 |f'(tx+(1-t)b)|^q dt\right)^{\frac{1}{q}}. \tag{2.12}$$

Since $|f'|^q$ is $\psi-(\alpha,\beta,\gamma,\delta)-$convex and $|f'(x)|\leq M$, we get

$$\int_0^1 |f'(tx+(1-t)a)|^q dt \leq M^q\int_0^1\left(t^{\alpha\gamma}\psi(t)+(1-t^{\beta})^{\delta}\psi(1-t)\right)dt, \tag{2.13}$$

and

$$\int_0^1 |f'(tx+(1-t)b)|^q dt \leq M^q\int_0^1\left(t^{\alpha\gamma}\psi(t)+(1-t^{\beta})^{\delta}\psi(1-t)\right)dt. \tag{2.14}$$

Using inequalities (2.12) – (2.14), we get (2.11). □

**Corollary 2.14.** *In Theorem* 2.13, *one can see the following.*

1. *If* $\psi(t)=1$, *in* (2.11), *then Ostrowski inequality for* $(\alpha,\beta,\gamma,\delta)-$*convex in mixed kind:*

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)dt\right| \leq \frac{M}{(p+1)^{\frac{1}{p}}}\left(\frac{1}{\alpha\gamma+1}+\frac{B\left(\frac{1}{\beta},\delta+1\right)}{\beta}\right)^{\frac{1}{q}} \kappa_a^b(x).$$

2. *If* $\psi(t) = \gamma = \delta = 1$, $\alpha \in [0,1]$ *and* $\beta \in (0,1]$, *in* (2.11), *then Ostrowski inequality for* $(\alpha,\beta)-$*convex in* $1^{st}$ *kind:*

$$\left|f(x) - \frac{1}{b-a}\int_a^b f(t)dt\right| \leq \frac{M}{(p+1)^{\frac{1}{p}}}\left(\frac{1}{\alpha+1} + \frac{B\left(\frac{1}{\beta},2\right)}{\beta}\right)^{\frac{1}{q}} \kappa_a^b(x).$$

3. *If* $\psi(t) = \beta = \gamma = 1$, $\alpha \in [0,1]$ *and* $\delta \in [0,1]$, *in* (2.11), *then Ostrowski inequality for* $(\alpha,\delta)-$*convex in* $2^{nd}$ *kind:*

$$\left|f(x) - \frac{1}{b-a}\int_a^b f(t)dt\right| \leq \frac{M}{(p+1)^{\frac{1}{p}}}\left(\frac{1}{\alpha+1} + \frac{1}{\delta+1}\right)^{\frac{1}{q}} \kappa_a^b(x).$$

4. *If* $\psi(t) = 1, \alpha = \delta = s$, $\beta = \gamma = r$, *where* $s \in [0,1]$ *and* $r \in (0,1]$ *in* (2.11), *then Ostrowski inequality for* $(s,r)-$*convex in mixed kind:*

$$\left|f(x) - \frac{1}{b-a}\int_a^b f(t)dt\right| \leq \frac{M}{(p+1)^{\frac{1}{p}}}\left(\frac{1}{rs+1} + \frac{B\left(\frac{1}{r},s+1\right)}{r}\right)^{\frac{1}{q}} \kappa_a^b(x).$$

5. *If* $\alpha = \beta = s$ *and* $\psi(t) = \gamma = \delta = 1$, *where* $s \in (0,1]$ *in* (2.11), *then Ostrowski inequality for* $s-$*convex in* $1^{st}$ *kind:*

$$\left|f(x) - \frac{1}{b-a}\int_a^b f(t)dt\right| \leq \frac{M}{(p+1)^{\frac{1}{p}}}\left(\frac{1}{s+1} + \frac{B\left(\frac{1}{s},2\right)}{s}\right)^{\frac{1}{q}} \kappa_a^b(x).$$

6. *If* $\psi(t) = \beta = \gamma = 1$, $\alpha = \delta = s$ *where* $s \in [0,1]$, *then* (2.11) *reduces to the inequality* (2.2) *of Theorem* 3 *in* [1].
7. *If* $\alpha = \delta \to 0$ *and* $\psi(t) = \beta = \gamma = 1$ *in* (2.11), *then Ostrowski inequality for* $P-$*convex:*

$$\left|f(x) - \frac{1}{b-a}\int_a^b f(t)dt\right| \leq \frac{(2)^{\frac{1}{q}}M}{(p+1)^{\frac{1}{p}}} \kappa_a^b(x).$$

8. *If* $\psi(t) = \alpha = \beta = \gamma = \delta = 1$ *in* (2.11), *then Ostrowski inequality for convex:*

$$\left|f(x) - \frac{1}{b-a}\int_a^b f(t)dt\right| \leq \frac{M}{(p+1)^{\frac{1}{p}}} \kappa_a^b(x).$$

9. *If* $\alpha = \beta = \gamma = \delta = 1$, *in* (2.11), *then Ostrowski inequality for* $\psi-$*convex:*

$$\left|f(x) - \frac{1}{b-a}\int_a^b f(t)dt\right|$$
$$\leq \frac{M}{(p+1)^{\frac{1}{p}}}\left(\int_0^1 (t\psi(t) + (1-t)\psi(1-t))\,dt\right)^{\frac{1}{q}} \kappa_a^b(x).$$

*10. If $\alpha=\beta=\gamma=\delta=1, l(t)=t$, then if $h=l\psi$, in (2.11), then Ostrowski inequality for $h-$convex:*

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)dt\right| \le \frac{M}{(p+1)^{\frac{1}{p}}}\left(\int_0^1 (h(t)+h(1-t))\,dt\right)^{\frac{1}{q}} \kappa_a^b(x).$$

*11. If $\alpha=\beta=\gamma=\delta=1, \psi(t)=t^{-(s+1)}$ where $s\in[0,1)$ in (2.11), then Ostrowski inequality for GL $s-$convex:*

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)dt\right| \le \frac{M}{(p+1)^{\frac{1}{p}}}\left(\frac{2}{1-s}\right)^{\frac{1}{q}} \kappa_a^b(x).$$

*12. If $\alpha=\beta=\gamma=\delta=1, \psi(t)=\frac{1}{2\sqrt{t(1-t)}}$ in (2.11), then Ostrowski inequality for MT$-$convex:*

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)dt\right| \le \frac{M\left(\frac{\pi}{2}\right)^{\frac{1}{q}}}{(1+p)^{\frac{1}{p}}} \kappa_a^b(x).$$

## 3. Applications of midpoint Ostrowski type inequalities via $\psi-(\alpha,\beta,\gamma,\delta)-$convex

If we replace $f$ by $-f$ and $x=\frac{a+b}{2}$ in Theorem 2.6, then the functional generalization of Ostrowski midpoint inequality for $\psi-(\alpha,\beta,\gamma,\delta)-$ convex functions:

$$\tau\left(\frac{1}{b-a}\int_a^b f(t)dt - f\left(\frac{a+b}{2}\right)\right)$$
$$\le \frac{\psi\left(\frac{1}{2}\right)}{b-a}\left[\frac{1}{2^{\alpha\gamma-1}}\int_a^{\frac{a+b}{2}} \tau[(a-t)f'(t)]dt + \frac{\left(2^\beta-1\right)^\delta}{2^{\beta\delta-1}}\int_{\frac{a+b}{2}}^b \tau[(b-t)f'(t)]dt\right]. \tag{3.1}$$

**Remark 3.1.** Assume that $\tau:[a,b]\subseteq\mathbb{R}\to\mathbb{R}$ be an $\psi-(\alpha,\beta,\gamma,\delta)-$convex function in mixed kind:

1. If $f(t)=\frac{1}{t}$ in inequality (3.1) where $t\in[a,b]\subset(0,\infty)$, then

$$\tau\left[\frac{A(a,b)-L(a,b)}{A(a,b)L(a,b)}\right]$$
$$\le \frac{\psi\left(\frac{1}{2}\right)}{b-a}\left[\frac{1}{2^{\alpha\gamma-1}}\int_a^{\frac{a+b}{2}} \tau\left[\frac{t-a}{t^2}\right]dt + \frac{\left(2^\beta-1\right)^\delta}{2^{\beta\delta-1}}\int_{\frac{a+b}{2}}^b \tau\left[\frac{t-b}{t^2}\right]dt\right].$$

2. If $f(t)=-\ln t$ in inequality (3.1), where $t\in[a,b]\subset(0,\infty)$, then

$$\tau\left[\ln\left(\frac{A(a,b)}{I(a,b)}\right)\right]$$
$$\le \frac{\psi\left(\frac{1}{2}\right)}{b-a}\left[\frac{1}{2^{\alpha\gamma-1}}\int_a^{\frac{a+b}{2}} \tau\left[\frac{t-a}{t}\right]dt + \frac{\left(2^\beta-1\right)^\delta}{2^{\beta\delta-1}}\int_{\frac{a+b}{2}}^b \tau\left[\frac{t-b}{t}\right]dt\right].$$

3. If $f(t) = t^p, p \in \mathbb{R} \setminus \{0, -1\}$ in inequality (3.1), where $t \in [a, b] \subset (0, \infty)$, then

$$\tau \left[L_p^p(a,b) - A^p(a,b)\right] \leq \frac{\psi\left(\frac{1}{2}\right)}{b-a}\left[\frac{1}{2^{\alpha\gamma-1}}\int_a^{\frac{a+b}{2}} \tau\left[\frac{p(a-t)}{t^{1-p}}\right]dt + \frac{\left(2^\beta-1\right)^\delta}{2^{\beta\delta-1}}\int_{\frac{a+b}{2}}^b \tau\left[\frac{p(b-t)}{t^{1-p}}\right]dt\right].$$

**Remark 3.2.** In Theorem 2.11, one can see the following.

1. Let $x = \frac{a+b}{2}$, $0 < a < b$, $q \geq 1$ and $f : \mathbb{R} \to \mathbb{R}^+$, $f(t) = t^n$ in (2.7). Then

$$|A^n(a,b) - L_n^n(a,b)| \leq \frac{M(b-a)}{(2)^{2-\frac{1}{q}}}\left(\int_0^1 \left(t^{\alpha\gamma+1}\psi(t) + t(1-t^\beta)^\delta\psi(1-t)\right)dt\right)^{\frac{1}{q}}.$$

2. Let $x = \frac{a+b}{2}$, $0 < a < b$, $q \geq 1$ and $f : (0,1] \to \mathbb{R}$, $f(t) = -\ln t$ in (2.7). Then

$$\left|\ln\left(\frac{A(a,b)}{I(a,b)}\right)\right| \leq \frac{M(b-a)}{(2)^{2-\frac{1}{q}}}\left(\int_0^1 \left(t^{\alpha\gamma+1}\psi(t) + t(1-t^\beta)^\delta\psi(1-t)\right)dt\right)^{\frac{1}{q}}.$$

**Remark 3.3.** In Theorem 2.13, one can see the following.

1. Let $x = \frac{a+b}{2}$, $0 < a < b$, $p^{-1} + q^{-1} = 1$ and $f : \mathbb{R} \to \mathbb{R}^+$, $f(t) = t^n$ in (2.11). Then

$$|A^n(a,b) - L_n^n(a,b)| \leq \frac{M(b-a)}{2(p+1)^{\frac{1}{p}}}\left(\int_0^1 \left(t^{\alpha\gamma}\psi(t) + (1-t^\beta)^\delta\psi(1-t)\right)dt\right)^{\frac{1}{q}}.$$

2. Let $x = \frac{a+b}{2}$, $0 < a < b$, $p^{-1} + q^{-1} = 1$ and $f : (0,1] \to \mathbb{R}$, $f(t) = -\ln t$ in (2.11). Then

$$\left|\ln\left(\frac{A(a,b)}{I(a,b)}\right)\right| \leq \frac{M(b-a)}{2(p+1)^{\frac{1}{p}}}\left(\int_0^1 \left(t^{\alpha\gamma}\psi(t) + (1-t^\beta)^\delta\psi(1-t)\right)dt\right)^{\frac{1}{q}}.$$

## 4. Conclusion and remarks

### 4.1. Conclusion

Ostrowski inequality is one of the most celebrated inequalities, we can find its various generalizations and variants in literature. In this paper, we presented the generalized notion of $\psi - (\alpha, \beta, \gamma, \delta)-$convex functions in mixed kind. This class of functions contains many important classes. We have started our first main result in section 2, the generalization of Ostrowski inequality via Montgomery identity with $\psi - (\alpha, \beta, \gamma, \delta)-$convex functions in mixed kind. Further, we used different techniques including Hölder's inequality and power mean inequality for generalization of Ostrowski inequality[15]. Finally, we have given some applications in terms of special means including arithmetic, geometric, harmonic, logarithmic, identric, and $p-$logarithmic means by using the midpoint inequalities.

### 4.2. Remarks and future ideas

1. One may also do similar work by using various different classes of convex functions.
2. One may do similar work to generalize all results stated in this research work by applying weights.
3. One may also state all results stated in this research work by higher-order derivatives.
4. One may also state all results stated in this research work by multivariable functions.
5. One may try to state all results stated in this research work for generalized fractional integral operators.
6. One may try to state all results stated in this research work for Jensen-Steffensen inequality and their different types of variants.
7. One may also do the similar work by using various different generalized forms for the Korkine's and Montgomery identities, improved power means inequality, Hölder's Iscan inequality, Jensen's integral inequality with weights, generalized fuzzy metric spaces on the set of all fuzzy numbers.

## References


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