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Complex operators generated by q-Bernstein polynomials, $q \geq 1$

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Dedicated to the memory of Akif D. Gadjiev

Abstract. By using a univalent and analytic function τ in a suitable open disk centered in origin, we attach to analytic functions f, the complex Bernstein-type operators of the form $B_{n,q}^{\tau}(f) = B_{n,q}\left(f \circ \tau^{-1}\right) \circ \tau$, where $B_{n,q}$ denote the classical complex q-Bernstein polynomials, $q \geq 1$. The new complex operators satisfy the same quantitative estimates as $B_{n,q}$. As applications, for two concrete choices of τ, we construct complex rational functions and complex trigonometric polynomials which approximate f with a geometric rate.

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1. Introduction

Starting from the classical Bernstein polynomials defined for $f \in C[0,1]$ by

$$B_n(f)(x) = \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} f\left(\frac{k}{n}\right),$$

a new sequence of Bernstein-type operators of real variable is introduced in [1] by the formula

$$B_n^{\tau} f := B_n\left(f \circ \tau^{-1}\right) \circ \tau,$$

where τ is a real-valued function on $[0,1]$ which satisfies the following conditions:

(τ_1) τ is differentiable of any order on $[0,1]$,

(τ_2) $\tau(0) = 0$, $\tau(1) = 1$ and $\tau'(x) > 0$ on $[0,1]$.

Specifically, $B_n^{\tau}(f)$ in [1] is given by

$$B_n^{\tau}(f)(x) = \sum_{k=0}^{n} \binom{n}{k} \tau^k(x)(1-\tau(x))^{n-k} (f \circ \tau^{-1})(\tfrac{k}{n}), \; x \in [0,1].$$

According to [1], the sequence $B_n^{\tau}(f)$, $n \in \mathbb{N}$, converges uniformly to $f \in C[0,1]$.

In [6]-[7] and [2], the complex form of the q-Bernstein polynomials, $q \geq 1$, given by

$$B_{n,q}\left(f\right)\left(z\right)=\sum_{k=0}^{n}\binom{n}{k}_{q}z^{k}\cdot\Pi_{s=0}^{n-k-1}(1-q^{s}z)f\left(\frac{[k]_{q}}{[n]_{q}}\right),\ n\in\mathbb{N},$$

were intensively studied. Here f is a complex-valued analytic function in an open disk of radius ≥ 1 and centered in origin. Also, above we have

$$[n]_q=(q^n-1)/(q-1),$$

$$\binom{n}{k}_q=\frac{[n]_q!}{[k]_q!\cdot[n-k]_q!},$$

$$[n]_q!=[1]_q\cdot[2]_q\cdot\ldots\cdot[n]_q,\ [0]_q!=1.$$

Note that for $q=1$, $B_{n,q}(f)$ reduce to the classical Bernstein polynomials.

Inspired by the real case in [1], in this paper we consider the idea in the complex setting and introduce the complex operators defined by

$$B^{\tau}_{n,q}(f)(z)=B_{n,q}(f\circ\tau^{-1})(\tau(z)),n\in\mathbb{N},z\in\mathbb{C},q\geq 1,$$

where denoting $\mathbb{D}_R=\{z\in\mathbb{C};|z|<R\}$, now τ satisfies the following properties:

$$\tau:\mathbb{D}_R\to\mathbb{C},R>1,\text{ is analytic, univalent, }\tau(0)=0,\tau(1)=1,\\ \text{and there exists }R'>1\text{ such that }\mathbb{D}_{R'}\subset\tau(\mathbb{D}_R). \tag{1.1}$$

By using the approach in [2], for the complex operators $B^{\tau}_{n,q}$ we prove upper and lower estimates and a quantitative Voronovskaja-type result in some compact subsets generated by τ.

Also, two important examples for τ are considered, which generate sequences of complex rational operators and of trigonometric polynomials of complex variable, approximating for $q>1$ the function f with the geometric rate $\frac{1}{q^n}$ in some compact disks centered in origin.

2. Approximation results

In this section, we present the main approximation properties of the operators $B^{\tau}_{n,q}$. Firstly, we consider the case when $q=1$. We have:

Theorem 2.1. *Let τ be satisfying the conditions in (1.1) and $f:\mathbb{D}_R\to\mathbb{C}$ be analytic in $\mathbb{D}_R$, $R>1$. Since $g:\mathbb{D}_{R'}\to\mathbb{C}$ defined by $g(w)=(f\circ\tau^{-1})(w)$ is analytic on the disk $\mathbb{D}_{R'}$, $R'>1$, let us write $g(w)=\sum_{k=0}^{\infty}c_k w^k$, for all $w\in\mathbb{D}_{R'}$.*

Let $1\leq r'<R'$ be arbitrary fixed. Then, for all $z\in\mathbb{D}_R$ with $|\tau(z)|\leq r'$ and for all $n\in\mathbb{N}$, we have:

(i) (Upper estimate)

$$\left|B^{\tau}_{n,1}\left(f\right)\left(z\right)-f\left(z\right)\right|\leq\frac{C^{\tau}_{r'}}{n}, \tag{2.1}$$

where $C^{\tau}_{r'}=\frac{3r'(r'+1)}{2}\sum_{k=2}^{\infty}|c_k|\,k\left(k-1\right)\left(r'\right)^{k-2}<\infty.$

(ii) (Voronovskaja-type result)

$$\left| B_{n,1}^{\tau}(f)(z) - f(z) - \frac{\tau(z)(1-\tau(z))}{2n} D_{\tau}^{2}(f)(z) \right| \leq \frac{5(1+r')^2 M_{r'}^{\tau}}{2n^2} \tag{2.2}$$

where $D_{\tau}^{2} f(z) := \left(f \circ \tau^{-1}\right)''(\tau(z)) = g''(\tau(z))$ *is detailed by*

$$D_{\tau}^{2}(f)(z) = \frac{f''(z)}{(\tau'(z))^2} - \frac{\tau''(z) f'(z)}{(\tau'(z))^3} = \frac{1}{\tau'(z)} \left(\frac{f'(z)}{\tau'(z)} \right)'$$

and

$$M_{r'}^{\tau} = \sum_{k=3}^{\infty} |c_k| \, k(k-1)(k-2)^2 \cdot (r')^{k-2} < \infty.$$

(iii) If f is not a polynomial in τ of degree ≤ 1, then

$$\left\| B_{n,1}^{\tau}(f) - f \right\|_{r',\tau} \sim \frac{1}{n},$$

where $\|F\|_{r',\tau} = \sup\{|F(z)|; |z| < R, |\tau(z)| \leq r'\}$ *and the constants in the equivalence depend only on f, τ and r'.*

Proof. Let $g(w) = \sum_{k=0}^{\infty} c_k w^k$ be an analytic function in a disk $\mathbb{D}_{R'}$ with $R' > 1$. Also, for simplicity, denote the classical Bernstein polynomials $B_{n,1}(g)(w)$ by $B_n(g)(w)$.

(*i*) According to Theorem 1.1.2, (i), page 6 in [2], for all $1 \leq r' < R'$, $n \in \mathbb{N}$ and $|w| \leq r'$, we have

$$|B_n(g)(w) - g(w)| \leq \frac{C_{r'}}{n},$$

where $C_{r'} = \frac{3r'(1+r')}{2} \sum_{k=2}^{\infty} k(k-1)|c_k|(r')^{k-2}$.

Now, if above we replace g by $f \circ \tau^{-1}$ and w by $\tau(z)$, then we easily arrive at the required estimate (2.1).

(*ii*) According to Theorem 1.1.3, (ii), page 9 in [2], for all $1 \leq r' < R'$, $n \in \mathbb{N}$ and $|w| \leq r'$, we have

$$\left| B_n(g)(w) - g(w) - \frac{w(1-w)}{2n} g''(w) \right| \leq \frac{5(1+r')^2 M_{r'}}{2n^2},$$

where $M_{r'} = \sum\limits_{k=3}^{\infty} |c_k| \, k(k-1)(k-2)^2 \cdot (r')^{k-2}$. Take $g(w) = (f \circ \tau^{-1})(w) = f[\tau^{-1}(w)]$. Since

$$g'(w) = f'[\tau^{-1}(w)] \cdot (\tau^{-1}(w))' = f'[\tau^{-1}(w)] \cdot \frac{1}{\tau'(\tau^{-1}(w))},$$

differentiating once again, we easily get

$$g''(w) = \frac{f''(\tau^{-1}(w))}{[\tau'(\tau^{-1}(w))]^2} - \frac{f'(\tau^{-1}(w)) \cdot \tau''(\tau^{-1}(w))}{[\tau'(\tau^{-1}(w))]^3}.$$

Now, replacing in the above estimate g by $f \circ \tau^{-1}$ and w by $\tau(z)$, we immediately get (2.2).

(*iii*) According to Corollary 1.1.5, page 14 in [2], it follows that for all $1 \leq r' < R'$ we have

$$\|B_n(g) - g\|_{r'} = \sup\{|B_n(g)(w) - g(w)|; |w| \leq r'\} \sim \frac{1}{n}.$$

But

$$\|B_n(g)-g\|_{r'} \geq \sup\{|B_n(g)(\tau(z))-g(\tau(z))|; |z|<R, |\tau(z)|\leq r'\}$$
$$= \|B^{\tau}_{n,1}(f)-f\|_{r',\tau},$$

which does not imply the required equivalence in the statement.

For this reason, we have to use here the standard method in [2] and the estimates (2.1) and (2.2). Thus, for all $z\in\mathbb{D}_R$ with $|\tau(z)|\leq r'$ and $n\in\mathbb{N}$ we can write

$$B^{\tau}_{n,1}(f)(z)-f(z)=\frac{1}{n}\left\{\frac{\tau(z)(1-\tau(z))}{2}D^2_{\tau}(f)(z)\right.$$
$$\left.+\frac{1}{n}\left[n^2\left(B^{\tau}_{n,1}(f)(z)-f(z)-\frac{\tau(z)(1-\tau(z))}{2n}D^2_{\tau}(f)(z)\right)\right]\right\}.$$

Then, the obvious inequality $\|F+G\|_{r',\tau}\geq\|F\|_{r',\tau}-\|G\|_{r',\tau}$ implies

$$\|B^{\tau}_{n,1}(f)-f\|_{r',\tau}\geq\frac{1}{n}\left\{\left\|\frac{\tau(1-\tau)}{2}D^2_{\tau}(f)\right\|_{r',\tau}\right.$$
$$\left.-\frac{1}{n}\left[n^2\left(\left\|B^{\tau}_{n,1}(f)-f-\frac{\tau(1-\tau)}{2n}D^2_{\tau}(f)\right\|_{r',\tau}\right)\right]\right\}.$$

By the hypothesis on f we immediately get that $g(\tau(z))$ is not a polynomial in $\tau(z)$ of degree ≤ 1. Then, by the formula $D^2_{\tau}(f)(z)=g''(\tau(z))$ we easily get

$$\left\|\frac{\tau(1-\tau)}{2}D^2_{\tau}(f)\right\|_{r',\tau}>0.$$

Indeed, supposing the contrary, it follows the obvious contradiction $g''(\tau(z))=0$, for all $z\in\mathbb{D}_R$.

Since by (2.2) there exists a constant $C>0$ with

$$n^2\left(\left\|B^{\tau}_{n,1}(f)-f-\frac{\tau(1-\tau)}{2n}D^2_{\tau}(f)\right\|_{r',\tau}\right)\leq C,$$

it is clear that there exists $n_0\in\mathbb{N}$ such that

$$\|B^{\tau}_{n,1}(f)-f\|_{r',\tau} \geq \frac{1}{2n}\left\|\frac{\tau(1-\tau)}{2}D^2_{\tau}(f)\right\|_{r',\tau}, \text{ for all } n\geq n_0.$$

Then, for $1\leq n\leq n_0-1$ we obviously have

$$\|B^{\tau}_{n,1}(f)-f\|_{r',\tau}\geq\frac{M_{r',n,\tau}(f)}{n},$$

with $M_{r',n,\tau}(f)=n\cdot\|B^{\tau}_{n,1}(f)-f\|_{r',\tau}>0$, which finally leads to

$$\|B^{\tau}_{n,1}(f)-f\|_{r',\tau}\geq\frac{C_{r',\tau}(f)}{n}, \text{ for all } n\in\mathbb{N},$$

where

$$C_{r',\tau}(f)=\min\left\{M_{r',1,\tau}, M_{r',2,\tau}(f), ..., M_{r',n_0-1,\tau}(f), \left\|\frac{\tau(1-\tau)}{4}D^2_{\tau}(f)\right\|_{r',\tau}\right\}.$$

Combining now with the estimate (2.1) from the point (i), we get the required equivalence. □

In the case $q > 1$ we have the following upper estimate of the geometric order $\frac{1}{q^n}$.

Theorem 2.2. *Let $f : \mathbb{D}_R \to \mathbb{C}$ be analytic in $\mathbb{D}_R$, $R > q$ and τ satisfying the conditions in (1.1). Denote*

$$g(w) = (f \circ \tau^{-1})(w) = \sum_{k=0}^{\infty} c_k w^k, \;\; w \in \mathbb{D}_{R'}.$$

For all $q \in (1, R')$, $1 \leq r' < \frac{R'}{q}$, $n \in \mathbb{N}$ and $z \in \mathbb{D}_R$ with $|\tau(z)| \leq r'$, we have

$$\left|B^{\tau}_{n,q}(f)(z) - f(z)\right| \leq \frac{M^{\tau}_{r',q}}{[n]_q} \leq \frac{q \cdot M^{\tau}_{r',q}}{q^n},$$

where $M^{\tau}_{r',q} = 2 \sum\limits_{k=2}^{\infty} |c_k| \, (k-1) \, [k-1]_q (r')^k < \infty$.

Proof. According to Theorem 1.5.1, page 51 in [2] we have

$$|B_{n,q}(g)(w) - g(w)| \leq \frac{M_{r',q}}{[n]_q} \leq \frac{q \cdot M^{\tau}_{r',q}}{q^n}, \text{ for all } 1 \leq r' < R', n \in \mathbb{N}, |w| \leq r',$$

where $M_{r',n} = \frac{3r'(1+r')}{2} \sum_{k=2}^{\infty} k(k-1)|c_k|(r')^{k-2}$.

Now, if above we replace g by $f \circ \tau^{-1}$ and w by $\tau(z)$, then we easily arrive at the required estimate. □

Remark 2.3. In a similar manner with Theorem 2.1, (ii), applying the results in, e.g., [10], for $B^{\tau}_{n,q}(f)$ we may deduce a quantitative Voronovskaja-type result of order $\frac{1}{q^{2n}}$.

3. Applications

In this section we apply the previous results to the cases of two concrete examples for τ. As consequences, we construct sequences of complex rational functions and complex trigonometric polynomials, convergent to f with a geometric rate. The first result is the following.

Theorem 3.1. *Let $f : \mathbb{D}_R \to \mathbb{C}$ be analytic in $\mathbb{D}_R$ with $R > 1 + \sqrt{2}$ and denote*

$$\tau(z) = \frac{Rz}{R + 1 - z}, \;\; |z| < R.$$

Then, with the notations in Theorems 2.1 and 2.2 we have:

(i) $B^{\tau}_{n,1}(f)(z)$ and $B^{\tau}_{n,q}(f)(z)$, $q > 1$, are complex rational functions on $\mathbb{D}_R$;

(ii) τ satisfies the conditions in (1.1) with $R' = \frac{R^2}{2R+1} > 1$;

(iii) if $1 \leq r' < R'$ then $1 \leq \frac{r'(R+1)}{R+r'} < R$ and for all $|z| \leq r = \frac{r'(R+1)}{R+r'}$, the upper estimates (2.1), (2.2) in Theorem 2.1, (i)-(ii) and the equivalence $\|B^{\tau}_{n,1}(f) - f\|_r \sim \frac{1}{n}$ hold.

(iv) If $1 < q < R'$ and $1 \leq r' < \frac{R'}{q}$, then the estimate in Theorem 2.2 holds for all $|z| \leq r = \frac{r'(R+1)}{R+r'}$.

Proof. (i) It is clear that both kinds of operators $B^{\tau}_{n,1}(f)(z)$ and $B^{\tau}_{n,q}(f)$, $q > 1$, are complex rational functions on $\mathbb{D}_R$.

(ii) We are interested on the image of $\mathbb{D}_R$ through the analytic and univalent mapping τ. Writing $w = \frac{Rz}{R+1-z}$, we get $z = \frac{(R+1)w}{w+R}$, so that $|z| < R$ is equivalent to

$$\left|\frac{(R+1)w}{w+R}\right| < R.$$

Denoting now $w = u + iv$, the previous inequality is equivalent to

$$\frac{(R+1)\sqrt{u^2+v^2}}{\sqrt{(u+R)^2+v^2}} < R,$$

which is equivalent to the inequality $(R+1)^2(u^2+v^2) < R^2[(u+R)^2+v^2]$. Simple calculations lead this last inequality to the following list of equivalent inequalities:

$$u^2[(R+1)^2 - R^2] + v^2[(R+1)^2 - R^2] < 2R^3u + R^4,$$

$$u^2 - 2u\frac{R^3}{2R+1} + v^2 < \frac{R^4}{2R+1},$$

$$\left(u - \frac{R^3}{2R+1}\right)^2 + v^2 < \left[\frac{R^2(R+1)}{2R+1}\right]^2.$$

This last inequality represents a disk of center $(R^3/(2R+1), 0)$ and of radius

$$R^2(R+1)/(2R+1).$$

Now, simple geometric reasonings lead to the fact that the above disk includes the disk of center in origin and of radius

$$\left|\frac{R^3}{2R+1} - \frac{R^2(R+1)}{2R+1}\right| = \frac{R^2}{2R+1},$$

where by the hypothesis $R > 1+\sqrt{2}$ we immediately get $R^2/(2R+1) > 1$. Concluding, since also we have $\tau(0) = 0$ and $\tau(1) = 1$, it follows that τ satisfies (1.1) with $R' = R^2/(2R+1)$.

(iii) Let $1 \le r' < R'$. Evidently that $\frac{r'(R+1)}{R+r'} \ge 1$ and since the function

$$F(x) = \frac{(R+1)x}{R+x}$$

is strictly increasing as function of $x \ge 0$, it follows

$$\frac{r'(R+1)}{R+r'} < \frac{R'(R+1)}{R+R'} = \frac{R^3+R^2}{3R^2+R} < R.$$

Then, since $\frac{R|z|}{R+1-|z|} \le r'$ is equivalent with the inequality $|z| \le r = \frac{r'(R+1)}{R+r'}$, by the obvious inequality $|\tau(z)| = \frac{R|z|}{|R+1-z|} \le \frac{R|z|}{R+1-|z|}$, $|z| < R$, it follows that the inequality $|z| \le \frac{r'(R+1)}{R+r'}$ implies $|\tau(z)| \le r'$ and therefore Theorem 2.1, (i), (ii) holds for these z.

In order to prove the equivalence, we use exactly the same reasonings as in the proof of Theorem 2.1, (iii), taking into account that (2.1) and (2.2) hold for all $|z| \le r = \frac{r'(R+1)}{R+r'}$.

(iv) If $1 < q < R'$ and $1 \leq r' < \frac{R'}{q}$, then reasoning as in the previous case $q = 1$, we immediately get the desired conclusion. □

Theorem 3.2. *Let $f : \mathbb{D}_{\pi/2} \to \mathbb{C}$ be analytic in $\mathbb{D}_{\pi/2}$ and $\tau(z) = \frac{\sin(z)}{\sin(1)}$, $|z| < \frac{\pi}{2}$. Then, with the notations in Theorems 2.1 and 2.2 we have:*

(i) $B^{\tau}_{n,1}(f)(z)$ and $B^{\tau}_{n,q}(f)(z)$, $q > 1$, are trigonometric polynomials of complex variable on $\mathbb{D}_{\pi/2}$;

(ii) τ satisfies the conditions in (1.1) with $R = \frac{\pi}{2}$ and $R' = \frac{1}{\sin(1)} > 1$;

(iii) for any $1 \leq r' < \frac{1}{\sin(1)}$ and for all $|z| \leq r := \frac{\pi r' \sin(1)}{2\cosh(\pi/2)} < \frac{\pi}{2}$, the upper estimates (2.1), (2.2) in Theorem 2.1, (i)-(ii) and the equivalence $\|B^{\tau}_{n,1}(f) - f\|_r \sim \frac{1}{n}$ hold.

(iv) If $1 < q < R'$ and $1 \leq r' < \frac{R'}{q}$, then the estimate in Theorem 2.2 holds for all $|z| \leq r = \frac{\pi r' \sin(1)}{2\cosh(\pi/2)}$.

Proof. (i) It is clear that both kinds of operators $B^{\tau}_{n,1}(f)(z)$ and $B^{\tau}_{n,q}(f)$, $q > 1$, are trigonometric polynomials of complex variable on $\mathbb{D}_{\pi/2}$.

(ii) From the well-known facts that $\sin(z)$ is univalent in $\mathbb{D}_{\pi/2}$ and that its inverse $\arcsin(z)$ exists in $\mathbb{C} \setminus ((-\infty, 1) \cup (1, +\infty))$ (see, e.g., [3], p. 164 and [8], pp. 90-91), it is immediate that $\tau(z)$ satisfies (1.1) with $R = \pi/2$ and $R' = \frac{1}{\sin(1)} > 1$.

(iii) For any $r' \in [1, R')$, we are interested to find a disk centered in origin and contained in the set $\{z \in \mathbb{D}_{\pi/2}; |\tau(z)| \leq r'\}$.

Firstly, we observe that for all $|z| < \pi/2$ we have

$$|\tau(z)| = \frac{|\sin z|}{\sin(1)} = \left|\frac{e^{iz} - e^{-iz}}{2i\sin(1)}\right| \leq \frac{1}{\sin(1)} \frac{e^{-y} + e^{y}}{2} = \frac{1}{\sin(1)} \cosh y < \frac{\cosh\frac{\pi}{2}}{\sin(1)}.$$

Now, we will use the following version of the Schwarz's lemma (see, e.g., [9], p. 218): if f is analytic in $\mathbb{D}_R$, $f(0) = 0$ and $|f(z)| < M$ for all $|z| < R$, then $|f(z)| \leq \frac{M}{R}|z|$, for all $|z| < R$.

Taking above $R = \frac{\pi}{2}$ and $M = \frac{\cosh\frac{\pi}{2}}{\sin(1)}$, we immediately get that for all $|z| < \frac{\pi}{2}$ we have $|\tau(z)| \leq \frac{2}{\pi} \frac{\cosh\frac{\pi}{2}}{\sin(1)} |z|$.

Now, if we put the condition $\frac{2}{\pi} \frac{\cosh\frac{\pi}{2}}{\sin(1)} |z| \leq r'$, then we easily obtain that for all $|z| \leq r = \frac{\pi r' \sin(1)}{2\cosh(\pi/2)}$ it follows $|\tau(z)| \leq r'$ and therefore Theorem 2.1, (i) and (ii) hold for these values of z.

Note here that for any $1 \leq r' < \frac{1}{\sin(1)}$, we still have $\frac{\pi r' \sin(1)}{2\cosh(\pi/2)} < \frac{\pi}{2}$.

The equivalence is immediate from Theorem 2.1, (iii).

(iv) If $1 < q < R'$ and $1 \leq r' < \frac{R'}{q}$, then reasoning as in the previous case $q = 1$, we easily get the desired conclusion. □

Remark 3.3. The hypothesis $\tau(0) = 0$ and $\tau(1) = 1$ in (1.1) imply that the new defined τ-operators coincide with the function f at the points 0 and 1.

Remark 3.4. Evidently that the considerations in this paper can be applied to other choices of the mapping τ and to other complex q-Benstein-type operators like, for example, those studied in [4]-[5].

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