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Microscopic behavior of the solutions of a transmission problem for the Helmholtz equation. A functional analytic approach

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Dedicated to the memory of Professor Gabriela Kohr

Abstract. Let Ω^i, Ω^o be bounded open connected subsets of $\mathbb{R}^n$ that contain the origin. Let $\Omega(\epsilon) \equiv \Omega^o \setminus \epsilon\overline{\Omega^i}$ for small $\epsilon > 0$. Then we consider a linear transmission problem for the Helmholtz equation in the pair of domains $\epsilon\Omega^i$ and $\Omega(\epsilon)$ with Neumann boundary conditions on $\partial\Omega^o$. Under appropriate conditions on the wave numbers in $\epsilon\Omega^i$ and $\Omega(\epsilon)$ and on the parameters involved in the transmission conditions on $\epsilon\partial\Omega^i$, the transmission problem has a unique solution $(u^i(\epsilon,\cdot), u^o(\epsilon,\cdot))$ for small values of $\epsilon > 0$. Here $u^i(\epsilon,\cdot)$ and $u^o(\epsilon,\cdot)$ solve the Helmholtz equation in $\epsilon\Omega^i$ and $\Omega(\epsilon)$, respectively. Then we prove that if $\xi \in \overline{\Omega^i}$ and $\xi \in \mathbb{R}^n \setminus \Omega^i$ then the rescaled solutions $u^i(\epsilon,\epsilon\xi)$ and $u^o(\epsilon,\epsilon\xi)$ can be expanded into a convergent power expansion of ϵ, $\kappa_n\epsilon\log\epsilon$, $\delta_{2,n}\log^{-1}\epsilon$, $\kappa_n\epsilon\log^2\epsilon$ for ϵ small enough. Here $\kappa_n = 1$ if n is even and $\kappa_n = 0$ if n is odd and $\delta_{2,2} \equiv 1$ and $\delta_{2,n} \equiv 0$ if $n \geq 3$.

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1. Introduction

In this paper we consider a linear transmission problem for the Helmholtz equation in a domain with a small inclusion. Problems of this type are motivated by the analysis of time-harmonic Maxwells Equations and we continue an analysis of [1] by analyzing the microscopic behavior of the solutions. For related problems for the Helmholtz equation, we refer to the papers [3] of Ammari, Vogelius and Volkov, [2] of

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Ammari, Iakovleva and Moskow, [4] of Ammari and Volkov, [12] of Hansen, Poignard and Vogelius, and [25] of Vogelius and Volkov.

First we introduce a problem with no hole (and no transmission), and then we consider the case with the hole. We consider $m \in \mathbb{N} \setminus \{0\}$, $n \in \mathbb{N} \setminus \{0, 1\}$, $\alpha \in]0, 1[$ and the following assumption.

$$\begin{aligned}&\textit{Let } \Omega \textit{ be a bounded open connected subset of } \mathbb{R}^n \textit{ of class } C^{m,\alpha}.\\ &\textit{Let } \mathbb{R}^n \setminus \overline{\Omega} \textit{ be connected. Let } 0 \in \Omega\,.\end{aligned} \tag{1.1}$$

Now let Ω^o be as in (1.1). Let

$$k_o \in \mathbb{C} \setminus]-\infty, 0]\,, \qquad \Im k_o \geq 0\,. \tag{1.2}$$

We also assume that k_o^2 is not a Neumann eigenvalue for $-\Delta$ in Ω^o. Then if

$$g^o \in C^{m-1,\alpha}(\partial\Omega^o)\,, \tag{1.3}$$

the Neumann problem

$$\begin{cases} \Delta u^o + k_o^2 u^o = 0 & \text{in } \Omega^o\,, \\ \frac{\partial}{\partial \nu_{\Omega^o}} u^o = g^o & \text{on } \partial\Omega^o \end{cases}$$

has a unique solution $\tilde{u}^o \in C^{m,\alpha}(\overline{\Omega}^o)$ (see for example Colton and Kress [9, Thm. 3.20] and classical Schauder regularity theory).

We now perturb singularly our problem. To do so, we consider another subset Ω^i of $\mathbb{R}^n$ as in (1.1). Then there exists

$$\epsilon_0 \in]0, 1[\text{ such that } \epsilon\overline{\Omega^i} \subseteq \Omega^o \qquad \forall \epsilon \in [-\epsilon_0, \epsilon_0]\,.$$

A known topological argument shows that $\Omega(\epsilon) \equiv \Omega^o \setminus \epsilon\overline{\Omega^i}$ is connected, and that $\mathbb{R}^n \setminus \overline{\Omega(\epsilon)}$ has exactly the two connected components $\epsilon\Omega^i$ and $\mathbb{R}^n \setminus \overline{\Omega^o}$, and that

$$\partial\Omega(\epsilon) = (\epsilon\partial\Omega^i) \cup \partial\Omega^o \qquad \forall \epsilon \in]-\epsilon_0, \epsilon_0[\setminus\{0\}\,.$$

Obviously, the outward unit normal ν_ϵ to $\partial\Omega(\epsilon)$ satisfies the equality

$$\begin{aligned}&\nu_\epsilon(x) = -\nu_{\Omega^i}(x/\epsilon)\,\mathrm{sgn}(\epsilon) \qquad \forall x \in \epsilon\partial\Omega^i\,,\\ &\nu_\epsilon(x) = \nu_{\Omega^o}(x) \qquad \forall x \in \partial\Omega^o\,,\end{aligned}$$

for all $\epsilon \in]-\epsilon_0, \epsilon_0[\setminus\{0\}$, where $\mathrm{sgn}(\epsilon) = 1$ if $\epsilon > 0$, $\mathrm{sgn}(\epsilon) = -1$ if $\epsilon < 0$. Then we introduce the constants

$$m^i, m^o \in]0, +\infty[\,, \qquad a \in]0, +\infty[\,,\ b \in \mathbb{R}\,,$$

and

$$k_i \in \mathbb{C} \setminus]-\infty, 0]\,, \qquad \Im k_i \geq 0\,, \tag{1.4}$$

and the datum

$$g^i \in C^{m-1,\alpha}(\partial\Omega^i)\,. \tag{1.5}$$

Then we consider the transmission problem

$$\begin{cases} \Delta u^i + k_i^2 u^i = 0 & \text{in } \epsilon\Omega^i\,, \\ \Delta u^o + k_o^2 u^o = 0 & \text{in } \Omega(\epsilon)\,, \\ u^o(x) - a u^i(x) = b & \forall x \in \epsilon\partial\Omega^i\,, \\ -\frac{1}{m^i}\frac{\partial}{\partial\nu_{\epsilon\Omega^i}} u^i(x) + \frac{1}{m^o}\frac{\partial}{\partial\nu_{\epsilon\Omega^i}} u^o(x) = g^i(x/\epsilon) & \forall x \in \epsilon\partial\Omega^i\,, \\ \frac{\partial}{\partial\nu_{\Omega^o}} u^o = g^o & \text{on } \partial\Omega^o\,, \end{cases} \tag{1.6}$$

in the unknown $(u^i, u^o) \in C^{m,\alpha}(\epsilon\overline{\Omega^i}) \times C^{m,\alpha}(\overline{\Omega(\epsilon)})$ for $\epsilon \in]0, \epsilon_0[$. By [1, Thm. 4.61], there exists $\epsilon' \in]0, \epsilon_0[$ such that problem (1.6) has a unique solution $(u^i(\epsilon, \cdot), u^o(\epsilon, \cdot)) \in C^{m,\alpha}(\epsilon\overline{\Omega^i}) \times C^{m,\alpha}(\overline{\Omega(\epsilon)})$. In [1, Thm. 5.1], we have analyzed the behavior of $u^o(\epsilon, \cdot)$ as ϵ approaches 0 and we have shown that if $x \in \Omega^o \setminus \{0\}$, then $u^o(\epsilon, x)$ can be expanded into a convergent power expansion of ϵ, $\kappa_n \epsilon \log \epsilon$, $\delta_{2,n} \log^{-1} \epsilon$ for ϵ small enough. Here $\kappa_n = 1$ if n is even and $\kappa_n = 0$ if n is odd and $\delta_{2,2} \equiv 1$ and $\delta_{2,n} \equiv 0$ if $n \geq 3$. In this paper we plan to consider the 'microscopic' behavior of our family of solutions, i.e., the behavior of the rescaled family

$$\{(u^i(\epsilon, \epsilon\cdot), u^o(\epsilon, \epsilon\cdot))\}_{\epsilon \in]0, \epsilon'[}$$

when ϵ is small enough. More precisely, we plan to answer the following two questions

(i) Let ξ be fixed in $\overline{\Omega^i}$. What can be said on the map $\epsilon \mapsto u^i(\epsilon, \epsilon\xi)$ when $\epsilon > 0$ is close to 0?

(ii) Let ξ be fixed in $\mathbb{R}^n \setminus \Omega^i$. What can be said on the map $\epsilon \mapsto u^o(\epsilon, \epsilon\xi)$ when $\epsilon > 0$ is close to 0?

Questions of this type have long been investigated for linear problems on domains with small holes with the methods of asymptotic analysis, which aim at proving complete asymptotic expansions in terms of the parameter ϵ. Although we cannot provide here a complete list of contributions, we mention the early works of of Cherepanov [6], [7] and the books of Nayfeh [22], Van Dyke [24], and Cole [8]. Then the description of the method of matching outer and inner asymptotic expansions of Il'in [13] and the Compound Expansion Method of Mazya, Nazarov and Plamenewskii [21] where the authors introduce a systematic approach for analyzing general Douglis and Nirenberg elliptic boundary value problems in domains with perforations and corners.

To analyze the problem and answer the above questions we resort to the Functional Analytic Approach (see reference [11] with Dalla Riva and Musolino) and we exploit the corresponding results of [1] and we prove that if $\xi \in \overline{\Omega^i}$ and $\xi \in \mathbb{R}^n \setminus \Omega^i$ then $u^i(\epsilon, \epsilon\xi)$ and $u^o(\epsilon, \epsilon\xi)$ can be expanded into a convergent power expansion of ϵ, $\kappa_n \epsilon \log \epsilon$, $\delta_{2,n} \log^{-1} \epsilon$, $\kappa_n \epsilon \log^2 \epsilon$ for ϵ small enough, respectively (see Theorem 5.1).

2. Preliminaries and notation

For standard definitions of Calculus in normed spaces, we refer to Cartan [5] and to Prodi and Ambrosetti [23]. The symbol $\mathbb{N}$ denotes the set of natural numbers including 0. Throughout the paper,

$$n \in \mathbb{N} \setminus \{0, 1\}.$$

Let $\mathbb{D} \subseteq \mathbb{R}^n$. Then $\overline{\mathbb{D}}$ denotes the closure of $\mathbb{D}$ and $\partial\mathbb{D}$ denotes the boundary of $\mathbb{D}$. For all $R > 0$, $x \in \mathbb{R}^n$, x_j denotes the j-th coordinate of x, $|x|$ denotes the Euclidean modulus of x in $\mathbb{R}^n$, and $\mathbb{B}_n(x, R)$ denotes the ball $\{y \in \mathbb{R}^n : |x - y| < R\}$. Let Ω be an open subset of $\mathbb{R}^n$. Then we find convenient to set

$$\Omega^+ \equiv \Omega, \qquad \Omega^- \equiv \mathbb{R}^n \setminus \overline{\Omega}.$$

The space of m times continuously differentiable complex-valued functions on Ω is denoted by $C^m(\Omega, \mathbb{C})$, or more simply by $C^m(\Omega)$. Let $r \in \mathbb{N} \setminus \{0\}$, $f \in (C^m(\Omega))^r$. The

s-th component of f is denoted f_s and the Jacobian matrix of f is denoted Df. Let $\eta \equiv (\eta_1, \ldots, \eta_n) \in \mathbb{N}^n$, $|\eta| \equiv \eta_1 + \cdots + \eta_n$. Then $D^\eta f$ denotes $\frac{\partial^{|\eta|} f}{\partial x_1^{\eta_1} \ldots \partial x_n^{\eta_n}}$. The subspace of $C^m(\Omega)$ of those functions f such that f and its derivatives $D^\eta f$ of order $|\eta| \leq m$ can be extended with continuity to $\overline{\Omega}$ is denoted $C^m(\overline{\Omega})$. The subspace of $C^m(\overline{\Omega})$ whose functions have m-th order derivatives that are Hölder continuous with exponent $\alpha \in]0,1]$ is denoted $C^{m,\alpha}(\overline{\Omega})$, (cf. *e.g.* [11, §2.11]). Let $\mathbb{D} \subseteq \mathbb{R}^n$. Then $C^{m,\alpha}(\overline{\Omega}, \mathbb{D})$ denotes the set $\left\{ f \in \left(C^{m,\alpha}(\overline{\Omega}) \right)^n : \ f(\overline{\Omega}) \subseteq \mathbb{D} \right\}$. We say that a bounded open subset of $\mathbb{R}^n$ is of class C^m or of class $C^{m,\alpha}$, if it is a manifold with boundary imbedded in $\mathbb{R}^n$ of class C^m or $C^{m,\alpha}$, respectively (cf. *e.g.*, [11, §2.13]). For standard properties of the functions of class $C^{m,\alpha}$ both on a domain of $\mathbb{R}^n$ or on a manifold imbedded in $\mathbb{R}^n$ we refer to [11, §2.11, 2.12, 2.14, 2.20] (see also [14, §2, Lem. 3.1, 4.26, Thm. 4.28], [18, §2].) We retain the standard notation of L^p spaces and of corresponding norms. We note that throughout the paper 'analytic' means 'real analytic'.

3. Some basic facts in potential theory

In the sequel, arg and log denote the principal branch of the argument and of the logarithm in $\mathbb{C} \setminus]-\infty, 0]$, respectively. Then we have

$$\arg(z) = \Im \log(z) \in]-\pi, \pi[\qquad \forall z \in \mathbb{C} \setminus]-\infty, 0] .$$

Then we set

$$J_\nu^\sharp(z) \equiv \sum_{j=0}^{\infty} \frac{(-1)^j z^j (1/2)^{2j} (1/2)^\nu}{\Gamma(j+1)\Gamma(j+\nu+1)} \qquad \forall z \in \mathbb{C} , \tag{3.1}$$

for all $\nu \in \mathbb{C} \setminus \{-j : j \in \mathbb{N} \setminus \{0\}\}$. Here $(1/2)^\nu = e^{\nu \log(1/2)}$. As is well known, if $\nu \in \mathbb{C} \setminus \{-j : j \in \mathbb{N} \setminus \{0\}\}$ then the function $J_\nu^\sharp(\cdot)$ is entire and

$$J_\nu^\sharp(z^2) = e^{-\nu \log z} J_\nu(z) \qquad \forall z \in \mathbb{C} \setminus]-\infty, 0] ,$$

where $J_\nu(\cdot)$ is the Bessel function of the first kind of index ν (cf. *e.g.*, Lebedev [20, Ch. 1, §5.3].) If $\nu \in \mathbb{N}$, we set

$$\begin{aligned} N_\nu^\sharp(z) \equiv & -\frac{2^\nu}{\pi} \sum_{0 \leq j \leq \nu - 1} \frac{(\nu - j - 1)!}{j!} z^j (1/2)^{2j} \\ & -\frac{z^\nu}{\pi} \sum_{j=0}^{\infty} \frac{(-1)^j z^j (1/2)^{2j} (1/2)^\nu}{j!(\nu+j)!} \left(2 \sum_{0 < l \leq j} \frac{1}{l} + \sum_{j < l \leq j + \nu} \frac{1}{l} \right) \qquad \forall z \in \mathbb{C} . \end{aligned}$$

As one can easily see, the $N_\nu^\sharp(\cdot)$ is an entire holomorphic function of the variable $z \in \mathbb{C}$ and

$$N_\nu(z) = \frac{2}{\pi}(\log(z) - \log 2 + \gamma) J_\nu(z) + z^{-\nu} N_\nu^\sharp(z^2) \qquad \forall z \in \mathbb{C} \setminus]-\infty, 0] ,$$

where γ is the Euler-Mascheroni constant, and where $N_\nu(\cdot)$ is the Neumann function of index ν, also known as Bessel function of second kind and index ν (cf. *e.g.*, Lebedev [20, Ch. 1, §5.5].) Let $k \in \mathbb{C}\setminus]-\infty,0]$, $n \in \mathbb{N}\setminus\{0,1\}$, $a_n \in \mathbb{C}$. Then we set

$$b_n \equiv \begin{cases} \pi^{1-(n/2)}2^{-1-(n/2)} & \text{if } n \text{ is even}\,, \\ (-1)^{\frac{n-1}{2}}\pi^{1-(n/2)}2^{-1-(n/2)} & \text{if } n \text{ is odd}\,, \end{cases}$$

and

$$\tilde{S}_{k,a_n}(x) = \begin{cases} k^{n-2}\Big\{a_n + \frac{2b_n}{\pi}(\log k - \log 2 + \gamma) + \frac{2b_n}{\pi}\log|x|\Big\} \\ \quad \times J^{\sharp}_{\frac{n-2}{2}}(k^2|x|^2) + b_n|x|^{2-n}N^{\sharp}_{\frac{n-2}{2}}(k^2|x|^2) \\ \hfill \text{if } n \text{ is even,} \\ a_n k^{n-2}J^{\sharp}_{\frac{n-2}{2}}(k^2|x|^2) + b_n|x|^{2-n}J^{\sharp}_{-\frac{n-2}{2}}(k^2|x|^2) \\ \hfill \text{if } n \text{ is odd,} \end{cases} \tag{3.2}$$

for all $x \in \mathbb{R}^n \setminus \{0\}$. As it is known and can be easily verified, the family $\{\tilde{S}_{k,a_n}\}_{a_n \in \mathbb{C}}$ coincides with the family of all radial fundamental solutions of $\Delta + k^2$.

Now we need to consider two specific fundamental solutions. For the first, which we denote by $S_{h,n}$, we need to choose a_n so that the resulting fundamental solution can be extended to an entire holomorphic function of the variable $k \in \mathbb{C}$. Then we introduce the following theorem. For a proof we refer to the paper [19, Prop. 3.3] with Rossi.

Theorem 3.1. *Let $n \in \mathbb{N}\setminus\{0,1\}$. Let $S_{h,n}(\cdot,\cdot)$ be the map from $(\mathbb{R}^n\setminus\{0\})\times\mathbb{C}$ to $\mathbb{C}$ defined by*

$$S_{h,n}(x,k) \equiv \begin{cases} b_n\Big\{\frac{2}{\pi}k^{n-2}J^{\sharp}_{\frac{n-2}{2}}(k^2|x|^2)\log|x| \\ \qquad\qquad + |x|^{-(n-2)}N^{\sharp}_{\frac{n-2}{2}}(k^2|x|^2)\Big\} & \textit{if } n \textit{ is even}\,, \\ b_n|x|^{-(n-2)}J^{\sharp}_{-\frac{n-2}{2}}(k^2|x|^2) & \textit{if } n \textit{ is odd}\,, \end{cases}$$

for all $(x,k) \in (\mathbb{R}^n\setminus\{0\})\times\mathbb{C}$. Then the following statements hold.

(i) *$S_{h,n}(\cdot,k)$ is a fundamental solution of $\Delta+k^2$ for all $k\in\mathbb{C}$ and $S_{h,n}(\cdot,0)$ coincides with the classical fundamental solution S_n of Δ, i.e.,*

$$S_{h,n}(x,0) = S_n(x) \equiv \begin{cases} \frac{1}{s_n}\log|x| & \forall x \in \mathbb{R}^n\setminus\{0\}, \quad \textit{if } n = 2\,, \\ \frac{1}{(2-n)s_n}|x|^{2-n} & \forall x \in \mathbb{R}^n\setminus\{0\}, \quad \textit{if } n > 2\,, \end{cases}$$

where s_n denotes the $(n-1)$ dimensional measure of $\partial\mathbb{B}_n(0,1)$.

(ii) *$S_{h,n}(\cdot,k)$ is real analytic in $\mathbb{R}^n\setminus\{0\}$. Moreover, if $x\in\mathbb{R}^n\setminus\{0\}$, then the map $S_{h,n}(x,\cdot)$ is holomorphic in $\mathbb{C}$.*

Next we introduce the second fundamental solution that we need. Let $k \in \mathbb{C}\setminus]-\infty,0]$, $\Im k \geq 0$. As well known in scattering theory, a function $u \in C^1(\mathbb{R}^n\setminus\{0\})$ satisfies the outgoing k-radiation condition provided that

$$\lim_{x\to\infty}|x|^{\frac{n-1}{2}}\Big(Du(x)\frac{x}{|x|} - iku(x)\Big) = 0\,.$$

Classically, one can prove that the fundamental solution of (3.2) satisfies the outgoing k-radiation condition if and only if

$$a_n \equiv \left\{ \begin{array}{ll} -ib_n & \text{if } n \text{ is even} \\ -e^{-i\frac{n-2}{2}\pi} b_n & \text{if } n \text{ is odd} \end{array} \right\} = -i\pi^{1-(n/2)} 2^{-1-(n/2)} \tag{3.3}$$

Then we introduce the following definition.

Definition 3.2. Let $n \in \mathbb{N} \setminus \{0,1\}$. Let $k \in \mathbb{C} \setminus]-\infty, 0]$. We denote by $S_{r,n}(\cdot, k)$ the function from $\mathbb{R}^n \setminus \{0\}$ to $\mathbb{C}$ defined by

$$S_{r,n}(x,k) \equiv \tilde{S}_{k,a_n}(x) \qquad \forall x \in \mathbb{R}^n \setminus \{0\} ,$$

with a_n as in (3.3) (cf. (3.2).)

As we have said above, if $k \in \mathbb{C} \setminus]-\infty, 0]$ and $\Im k \geq 0$, then $S_{r,n}(\cdot, k)$ satisfies the outgoing k-radiation condition. The subscript r stands for 'radiation'. Now we introduce the function γ_n from $\mathbb{C}$ to $\mathbb{C}$ defined by setting

$$\gamma_n(z) \equiv \left\{ \begin{array}{ll} [-i + \frac{2}{\pi}(z - \log 2 + \gamma)] b_n & \text{if } n \text{ is even} , \\ -e^{-i\frac{n-2}{2}\pi} b_n & \text{if } n \text{ is odd} , \end{array} \right. \tag{3.4}$$

for all $z \in \mathbb{C}$. Then we have

$$S_{r,n}(x,k) = S_{h,n}(x,k) + \gamma_n(\log k) k^{n-2} J^{\sharp}_{\frac{n-2}{2}}(k^2|x|^2) \qquad \forall x \in \mathbb{R}^n \setminus \{0\} ,$$

for all $k \in \mathbb{C} \setminus]-\infty, 0]$. Next we introduce the layer potential operators corresponding to a fundamental solution or to a smooth kernel.

Definition 3.3. Let $n \in \mathbb{N} \setminus \{0,1\}$, $k \in \mathbb{C}$. Let S be either a fundamental solution of $\Delta + k^2$ or a real analytic function from $\mathbb{R}^n$ to $\mathbb{C}$. Let Ω be a bounded open subset of $\mathbb{R}^n$ of class $C^{1,\alpha}$. Let $\mu \in C^0(\partial\Omega)$. Then we introduce the following notation.

(i) We denote by $v_\Omega[\mu, S]$ the function from $\mathbb{R}^n$ to $\mathbb{C}$ defined by

$$v_\Omega[\mu, S](x) \equiv \int_{\partial\Omega} S(x-y)\mu(y)\, d\sigma_y \qquad \forall x \in \mathbb{R}^n .$$

Then we denote by $v_\Omega^+[\mu, S]$, by $v_\Omega^-[\mu, S]$ and by $V_\Omega[\mu, S]$, the restriction of $v_\Omega[\mu, S]$ to $\overline{\Omega}$, to $\overline{\Omega^-}$ and to $\partial\Omega$, respectively.

(ii) We denote by $W_\Omega^t[\mu, S]$ the function from $\partial\Omega$ to $\mathbb{C}$ defined by

$$W_\Omega^t[\mu, S](x) \equiv \int_{\partial\Omega} \frac{\partial}{\partial\nu_{\Omega,x}} S(x-y)\mu(y)\, d\sigma_y \qquad \forall x \in \partial\Omega ,$$

where

$$\frac{\partial}{\partial\nu_{\Omega,x}} S(x-y) \equiv DS(x-y)\nu_\Omega(x) \qquad \forall (x,y) \in \partial\Omega \times \partial\Omega, x \neq y .$$

If $k \in \mathbb{C} \setminus]-\infty, 0]$, we set

$$v_\Omega[\mu, k] \equiv v_\Omega[\mu, S_{r,n}(\cdot, k)] ,$$

and we use corresponding abbreviations for V_Ω, $v_\Omega^\pm$, W_Ω^t. If $k \in \mathbb{C}$, we set

$$v_{\Omega,h}[\mu, k] = v_\Omega[\mu, S_{h,n}(\cdot, k)] ,$$

and we use corresponding abbreviations for $V_{\Omega,h}$, $v^{\pm}_{\Omega,h}$, $W^{t}_{\Omega,h}$. If $\lambda \in \mathbb{C}$, we set

$$v_{\Omega,J}[\mu,\lambda] = v_{\Omega}[\mu, J^{\sharp}_{\frac{n-2}{2}}(\lambda|\cdot|^2)]\,,$$

and we use corresponding abbreviations for $V_{\Omega,J}$, $v^{\pm}_{\Omega,J}$, $W^{t}_{\Omega,J}$. For the regularity results on acoustic layer potentials that we need, we refer the reader to [10] (which is a generalization of [19]), to [16, Thm. A.3] and to [1, §3]. If $m \in \mathbb{N}\setminus\{0\}$, $\alpha \in]0,1[$, we set

$$\begin{aligned}\tilde{W}^{t}_{\Omega,J}&[\mu,\lambda](x)\\ &\equiv 2\int_{\partial\Omega}(J^{\sharp}_{\frac{n-2}{2}})'(\lambda(x-y)(x-y))(x-y)\nu_{\Omega}(x)\mu(y)\,d\sigma_y\,,\end{aligned}$$

for all $x \in \partial\Omega$ and for all $(\mu,\lambda) \in C^{m-1,\alpha}(\partial\Omega)\times\mathbb{C}$. Then we have

$$W^{t}_{\Omega,J}[\mu,\lambda](x) = \lambda\tilde{W}^{t}_{\Omega,J}[\mu,\lambda](x) \qquad \forall x \in \partial\Omega\,,$$

for all $(\mu,\lambda) \in C^{m-1,\alpha}(\partial\Omega)\times\mathbb{C}$. By our abbreviations, we have

$$v^{\pm}_{\Omega}[\mu,k] = v^{\pm}_{\Omega,h}[\mu,k] + \gamma_n(\log k)k^{n-2}v^{\pm}_{\Omega,J}[\mu,k^2] \tag{3.5}$$

on $\overline{\Omega^{\pm}}$ for all μ in $C^{m-1,\alpha}(\partial\Omega)$ and $k \in \mathbb{C}\setminus]-\infty,0]$ (cf. [1, Cor. 3.25]).

Next we observe that the fundamental solution $S_{r,n}$ satisfies the following scaling property, which can be verified by exploiting the definition of $S_{r,n}$ and elementary computations.

Lemma 3.4. *Let $n \in \mathbb{N}\setminus\{0,1\}$, $k \in \mathbb{C}\setminus]-\infty,0]$. Then the following equalities hold*

$$\begin{aligned}\epsilon^{n-2}S_{r,n}(\epsilon x,k) &= S_{r,n}(x,\epsilon k)\,,\\ \epsilon^{n-1}DS_{r,n}(\epsilon x,k) &= DS_{r,n}(x,\epsilon k)\end{aligned}$$

for all $x \in \mathbb{R}^n\setminus\{0\}$, $\epsilon \in]0,+\infty[$.

Then we note that the following elementary equality holds

$$\gamma_n(\log(\epsilon k)) = \frac{2b_n}{\pi}\kappa_n\log\epsilon + \gamma_n(\log k)\,, \tag{3.6}$$

for all $k \in \mathbb{C}\setminus]-\infty,0]$ and $\epsilon \in]0,+\infty[$ (cf. (3.4).)

4. Existence of a family of solutions $\{(u^i(\epsilon,\cdot),u^o(\epsilon,\cdot))\}_{\epsilon\in]0,\epsilon'[}$

We first transform problem (1.6) into a problem for integral equations on the boundaries $\partial\Omega^i$ and $\partial\Omega^o$. To do so, we first set

$$Y_{m-1,\alpha} \equiv C^{m-1,\alpha}(\partial\Omega^i)_0 \times \mathbb{C} \times C^{m-1,\alpha}(\partial\Omega^i)_0 \times \mathbb{C} \times C^{m-1,\alpha}(\partial\Omega^o)\,,$$

where

$$C^{m-1,\alpha}(\partial\Omega^i)_0 \equiv \left\{\theta \in C^{m-1,\alpha}(\partial\Omega^i) : \int_{\partial\Omega^i}\theta\,d\sigma = 0\right\}$$

and we mention that we can choose $\theta^\sharp \in C^{m-1,\alpha}(\partial\Omega^i)$ such that

$$\theta^\sharp \text{ is real valued}\,, \int_{\partial\Omega^i} \theta^\sharp \, d\sigma = 1\,, \qquad -\frac{1}{2}\theta^\sharp + W^t_{\Omega^i,h}[\theta^\sharp, 0] = 0 \quad \text{on } \partial\Omega^i \tag{4.1}$$

and accordingly that

$$v^\sharp \equiv V_{\Omega^i,h}[\theta^\sharp, 0] \text{ is constant on } \partial\Omega^i \tag{4.2}$$

(cf. *e.g.*, [11, Prop. 6.18, Thms. 6.24, 6.25], [15, Thm. 5.1]). Then we also have

$$V_{\Omega^i,J}[\theta^\sharp, 0] = J^\sharp_{\frac{n-2}{2}}(0) \qquad \text{on } \partial\Omega^i \tag{4.3}$$

and

$$\int_{\partial\Omega^i} \frac{1}{2}\phi + W^t_{\Omega^i,h}[\phi, 0] d\sigma = \int_{\partial\Omega^i} \phi \, d\sigma \qquad \forall \phi \in C^{m-1,\alpha}(\partial\Omega^i)$$

(cf. *e.g.*, [11, Lem. 6.11]). To shorten our notation, we find convenient to introduce the polynomial function ϱ_n from $\mathbb{R}^2$ to $\mathbb{R}$ defined by

$$\varrho_n(\epsilon, \epsilon_1) \equiv [(1-\delta_{2,n})\epsilon^{n-3} + \delta_{2,n}][(1-\delta_{2,n})\epsilon_1 + \kappa_n \delta_{2,n}] \qquad \forall (\epsilon, \epsilon_1) \in \mathbb{R}^2\,, \tag{4.4}$$

and we observe that

$$\varrho_n(\epsilon, \kappa_n \epsilon \log \epsilon) = \epsilon^{n-2} \kappa_n \frac{\log \epsilon}{\log^{\delta_{2,n}} \epsilon} \qquad \forall \epsilon \in]0,1[\,. \tag{4.5}$$

Then we set

$$Z_{m-1,\alpha} \equiv C^{m,\alpha}(\partial\Omega^i) \times C^{m-1,\alpha}(\partial\Omega^i) \times C^{m-1,\alpha}(\partial\Omega^o)$$

and we introduce the map $\mathcal{M} \equiv (\mathcal{M}_l)_{l=1,2,3}$ from $]-\epsilon_0, \epsilon_0[\times \mathbb{R}^2 \times Y_{m-1,\alpha}$ to $Z_{m-1,\alpha}$ defined by

$$\begin{aligned}
\mathcal{M}_1[\epsilon, \epsilon_1, \epsilon_2, \zeta, c^i, \varsigma^i, c, \theta^o](\xi) \equiv \int_{\partial\Omega^i} S_{h,n}(\xi - \eta, \epsilon k_o) \varsigma^i(\eta) \, d\sigma_\eta \\
+\epsilon^{n-1} k_o^n \left[\frac{2b_n}{\pi}\epsilon_1 + \epsilon \gamma_n (\log k_o)\right] \int_0^1 \frac{\partial}{\partial\lambda} V_{\Omega^i,J}[\varsigma^i, t\epsilon^2 k_o^2](\xi) \, dt \\
+ \int_{\partial\Omega^i} S_{h,n}(\xi - \eta, \epsilon k_o) c^i \theta^\sharp(\eta) \, d\sigma_\eta \\
+\epsilon^{n-1} k_o^n \left[\frac{2b_n}{\pi}\epsilon_1 + \epsilon \gamma_n (\log k_o)\right] c^i \int_0^1 \frac{\partial}{\partial\lambda} V_{\Omega^i,J}[\theta^\sharp, t\epsilon^2 k_o^2](\xi) \, dt \\
+\epsilon^{n-2} k_o^{n-2} \gamma_n (\log k_o) c^i V_{\Omega^i,J}[\theta^\sharp, 0](\xi) + \int_{\partial\Omega^o} S_{r,n}(\epsilon\xi - y, k_o) \theta^o(y) \, d\sigma_y \\
-a \int_{\partial\Omega^i} S_{h,n}(\xi - \eta, \epsilon k_i) \zeta(\eta) \, d\sigma_\eta \\
-a\epsilon^{n-1} k_i^n \left[\frac{2b_n}{\pi}\epsilon_1 + \epsilon \gamma_n (\log k_i)\right] \int_0^1 \frac{\partial}{\partial\lambda} V_{\Omega^i,J}[\zeta, t\epsilon^2 k_i^2](\xi) \, dt \\
-(k_o^{n-2}/k_i^{n-2}) c^i \int_{\partial\Omega^i} S_{h,n}(\xi - \eta, \epsilon k_i) \theta^\sharp(\eta) \, d\sigma_\eta \\
-a \int_{\partial\Omega^i} S_{h,n}(\xi - \eta, \epsilon k_i) c[(1-\delta_{2,n}) + \epsilon_2] \theta^\sharp(\eta) \, d\sigma_\eta
\end{aligned} \tag{4.6}$$

$$-\epsilon^{n-1}k_o^{n-2}c^i k_i^2\left[\frac{2b_n}{\pi}\epsilon_1+\epsilon\gamma_n(\log k_i)\right]\int_0^1\frac{\partial}{\partial\lambda}V_{\Omega^i,J}[\theta^\sharp,t\epsilon^2k_i^2](\xi)\,dt$$

$$-a\epsilon^{n-1}k_i^n\left[\frac{2b_n}{\pi}\epsilon_1+\epsilon\gamma_n(\log k_i)\right]c[(1-\delta_{2,n})+\epsilon_2]$$

$$\times\int_0^1\frac{\partial}{\partial\lambda}V_{\Omega^i,J}[\theta^\sharp,t\epsilon^2k_i^2](\xi)\,dt-\epsilon^{n-2}k_o^{n-2}c^i\gamma_n(\log k_i)V_{\Omega^i,J}[\theta^\sharp,0](\xi)$$

$$-ak_i^{n-2}\left[\frac{2b_n}{\pi}\varrho_n(\epsilon,\epsilon_1)+\epsilon^{n-2}[(1-\delta_{2,n})+\epsilon_2]\gamma_n(\log k_i)\right]cV_{\Omega^i,J}[\theta^\sharp,0](\xi)-b,\ \forall\xi\in\partial\Omega^i,$$

$$\mathcal{M}_2[\epsilon,\epsilon_1,\epsilon_2,\zeta,c^i,\varsigma^i,c,\theta^o](\xi)\tag{4.7}$$

$$\equiv-\frac{1}{m^i}\Big\{-\frac{1}{2}\left(\zeta(\xi)+a^{-1}(k_o^{n-2}/k_i^{n-2})c^i\theta^\sharp(\xi)+c[(1-\delta_{2,n})+\epsilon_2]\theta^\sharp(\xi)\right)$$

$$+\int_{\partial\Omega^i}DS_{h,n}(\xi-\eta,\epsilon k_i)\nu_{\Omega^i}(\xi)$$

$$\times\left(\zeta(\eta)+a^{-1}(k_o^{n-2}/k_i^{n-2})c^i\theta^\sharp(\eta)+c[(1-\delta_{2,n})+\epsilon_2]\theta^\sharp(\eta)\right)\,d\sigma_\eta$$

$$+\epsilon^{n-1}k_i^n\left[\frac{2b_n}{\pi}\epsilon_1+\epsilon\gamma_n(\log k_i)\right]$$

$$\times\tilde{W}^t_{\Omega^i,J}[\zeta+a^{-1}(k_o^{n-2}/k_i^{n-2})c^i\theta^\sharp+c[(1-\delta_{2,n})+\epsilon_2]\theta^\sharp,\epsilon^2k_i^2](\xi)\Big\}$$

$$-\frac{1}{m^o}\Big\{-\frac{1}{2}\left(\varsigma^i(\xi)+c^i\theta^\sharp(\xi)\right)$$

$$-\int_{\partial\Omega^i}DS_{h,n}(\xi-\eta,\epsilon k_o)\nu_{\Omega^i}(\xi)\left(\varsigma^i(\eta)+c^i\theta^\sharp(\eta)\right)\,d\sigma_\eta$$

$$-\epsilon^{n-1}k_o^n\left[\frac{2b_n}{\pi}\epsilon_1+\epsilon\gamma_n(\log k_o)\right]\tilde{W}^t_{\Omega^i,J}[\varsigma^i+c^i\theta^\sharp,\epsilon^2k_o^2](\xi)$$

$$-\epsilon\int_{\partial\Omega^o}DS_{r,n}(\epsilon\xi-y,k_o)\nu_{\Omega^i}(\xi)\theta^o(y)\,d\sigma_y\Big\}-\epsilon g^i(\xi),\ \forall\xi\in\partial\Omega^i,$$

$$\mathcal{M}_3[\epsilon,\epsilon_1,\epsilon_2,\zeta,c^i,\varsigma^i,c,\theta^o](x)\tag{4.8}$$

$$\equiv-\frac{1}{2}\theta^o(x)+\int_{\partial\Omega^i}DS_{r,n}(x-\epsilon\eta,k_o)\nu_{\Omega^o}(x)\left(\varsigma^i(\eta)+c^i\theta^\sharp(\eta)\right)\,d\sigma_\eta\epsilon^{n-2}$$

$$+\int_{\partial\Omega^o}DS_{r,n}(x-y,k_o)\nu_{\Omega^o}(x)\theta^o(y)\,d\sigma_y-g^o(x),\ \forall x\in\partial\Omega^o$$

for all $(\epsilon,\epsilon_1,\epsilon_2,\zeta,c^i,\varsigma^i,c,\theta^o)\in]-\epsilon_0,\epsilon_0[\times\mathbb{R}^2\times Y_{m-1,\alpha}$. Here $\frac{\partial}{\partial\lambda}V_{\Omega^i,J}$ denotes the partial differential of the analytic map $V_{\Omega^i,J}[\cdot,\cdot]$ with respect to its second argument (cf. [1, Thm. 3.22]). Then we have the following statement of [1, Thms. 4.18, 4.47] that shows that for $\epsilon\in]0,\epsilon_0[$ small problem (1.6) is equivalent to equation $\mathcal{M}[\epsilon,\epsilon_1,\epsilon_2,\zeta,c^i,\varsigma^i,c,\theta^o]=0$ provided that we choose $\epsilon_1=\kappa_n\epsilon\log\epsilon$, $\epsilon_2=\frac{\delta_{2,n}}{\log\epsilon}$.

Theorem 4.1. *Let $m \in \mathbb{N} \setminus \{0\}$, $n \in \mathbb{N} \setminus \{0,1\}$, $\alpha \in]0,1[$. Let Ω^i, Ω^o be as in (1.1). Let m^i, m^o, $a \in]0,+\infty[$, $b \in \mathbb{R}$. Let g^i, g^o be as in (1.3), (1.5). Let k_i, k_o be as in (1.2), (1.4). Assume that k_o^2 is not a Neumann eigenvalue for $-\Delta$ in Ω^o. Let $\theta^\sharp \in C^{m-1,\alpha}(\partial\Omega^i)$ be as in (4.1). Let $\mathcal{M} \equiv (\mathcal{M}_l)_{l=1,2,3}$ be the map from $]-\epsilon_0, \epsilon_0[\times\mathbb{R}^2\times Y_{m-1,\alpha}$ to $Z_{m-1,\alpha}$ defined by (4.6)–(4.8). Then the following statements hold.*

(i) *If $\epsilon = \epsilon_1 = \epsilon_2 = 0$, then equation*

$$\mathcal{M}[0,0,0,\zeta,c^i,\varsigma^i,c,\theta^o] = 0 \tag{4.9}$$

has one and only one solution $(\tilde{\zeta}, \tilde{c}^i, \tilde{\varsigma}^i, \tilde{c}, \tilde{\theta}^o)$ in $Y_{m-1,\alpha}$. Moreover, $\tilde{c}^i = 0$.

(ii) *There exists $\epsilon^* \in]0, \epsilon_0[$ such that the map from the subset of $Y_{m-1,\alpha}$ consisting of the 5-tuples $(\zeta, c^i, \varsigma^i, c, \theta^o)$ that solve the equation*

$$\mathcal{M}[\epsilon, \kappa_n \epsilon \log\epsilon, \frac{\delta_{2,n}}{\log\epsilon}, \zeta, c^i, \varsigma^i, c, \theta^o] = 0$$

onto the set of solutions (u^i, u^o) in $C^{m,\alpha}(\epsilon\overline{\Omega^i}) \times C^{m,\alpha}(\overline{\Omega(\epsilon)})$, which satisfy problem (1.6), which takes $(\zeta, c^i, \varsigma^i, c, \theta^o)$ to the pair of functions

$$(u^i[\epsilon,\zeta,c^i,\varsigma^i,c,\theta^o], u^o[\epsilon,\zeta,c^i,\varsigma^i,c,\theta^o])$$

defined by

$$\begin{aligned} u^i[\epsilon,\zeta,c^i,\varsigma^i,c,\theta^o](x) &= \frac{1}{\epsilon} v^+_{\epsilon\Omega^i}[\zeta(\cdot/\epsilon), k_i](x) \\ &\quad + \frac{1}{\epsilon}\Big(a^{-1}(k_o^{n-2}/k_i^{n-2})c^i + \frac{c}{(\log\epsilon)^{\delta_{2,n}}}\Big) v^+_{\epsilon\Omega^i}[\theta^\sharp(\cdot/\epsilon), k_i](x) \qquad \forall x \in \epsilon\overline{\Omega^i}\,, \\ u^o[\epsilon,\zeta,c^i,\varsigma^i,c,\theta^o](x) &= v^+_{\Omega^o}[\theta^o, k_o](x) + \frac{1}{\epsilon} v^-_{\epsilon\Omega^i}[\varsigma^i(\cdot/\epsilon), k_o](x) \\ &\quad + \frac{c^i}{\epsilon} v^-_{\epsilon\Omega^i}[\theta^\sharp(\cdot/\epsilon), k_o](x) \qquad \forall x \in \overline{\Omega(\epsilon)}\,, \end{aligned} \tag{4.10}$$

is a bijection.

The equation (4.9) can be shown to be equivalent to a boundary value problem in the sense of the following statement of [1, Thm. 4.32].

Theorem 4.2. *Let $m \in \mathbb{N} \setminus \{0\}$, $n \in \mathbb{N} \setminus \{0,1\}$, $\alpha \in]0,1[$. Let Ω^i, Ω^o be as in (1.1). Let m^i, m^o, $a \in]0,+\infty[$, $b \in \mathbb{R}$. Let g^i, g^o be as in (1.3), (1.5). Let k_i, k_o be as in (1.2), (1.4). Assume that k_o^2 is not a Neumann eigenvalue for $-\Delta$ in Ω^o. Then the limiting boundary value problem*

$$\begin{cases} \Delta u_1^{i,r} = 0 & \text{in } \Omega^i\,, \\ \Delta u_1^{o,r} = 0 & \text{in } \Omega^{i-}\,, \\ \Delta u^o + k_o^2 u^o = 0 & \text{in } \Omega^o\,, \\ u_1^{o,r}(x) + u^o(0) - a u_1^{i,r}(x) = b & \forall x \in \partial\Omega^i\,, \\ -\frac{1}{m^i}\frac{\partial}{\partial\nu_{\Omega^i}} u_1^{i,r}(x) + \frac{1}{m^o}\frac{\partial}{\partial\nu_{\Omega^i}} u_1^{o,r}(x) = 0 & \forall x \in \partial\Omega^i\,, \\ \frac{\partial}{\partial\nu_{\Omega^o}} u^o = g^o & \text{on } \partial\Omega^o\,, \\ \lim_{\xi\to\infty} u_1^{o,r}(\xi) = 0\,, \end{cases}$$

has one and only one solution $(\tilde{u}_1^{i,r}, \tilde{u}_1^{o,r}, \tilde{u}^o)$ *in*

$$C^{m,\alpha}(\overline{\Omega^i}) \times C^{m,\alpha}_{\mathrm{loc}}(\overline{\Omega^{i-}}) \times C^{m,\alpha}(\overline{\Omega^o}),$$

which is delivered by the following formulas

$$\begin{aligned} &\tilde{u}_1^{i,r} = v^+_{\Omega^i,h}[\tilde{\zeta}, 0] + \tilde{C} \quad \text{in } \overline{\Omega^i}, \qquad \tilde{u}_1^{o,r} = v^-_{\Omega^i,h}[\tilde{\varsigma}^i, 0] \quad \text{in } \overline{\Omega^{i-}}, \\ &\tilde{u}^o = v^+_{\Omega^o}[\tilde{\theta}^o, k_o] \qquad\quad \text{in } \overline{\Omega^o} \end{aligned} \tag{4.11}$$

where $(\tilde{\zeta}, \tilde{c}^i, \tilde{\varsigma}^i, \tilde{c}, \tilde{\theta}^o)$ *is the only solution in* $Y_{m-1,\alpha}$ *of equation* (4.9) *and*

$$\tilde{C} = \left(\frac{\delta_{2,n}}{2\pi} + (1-\delta_{2,n})\, v^\sharp\right)\tilde{c}$$

(see (4.2) *for the constant* $v^\sharp \equiv V_{\Omega^i,h}[\theta^\sharp, 0]$*).*

Next we turn to equation $\mathcal{M} = 0$. One can show that one can solve equation $\mathcal{M}[\epsilon, \epsilon_1, \epsilon_2, \zeta, c^i, \varsigma^i, c, \theta^o] = 0$ in the unknown $(\zeta, c^i, \varsigma^i, c, \theta^o)$ in terms of $(\epsilon, \epsilon_1, \epsilon_2)$ by mean of the following statement of [1, Thm. 4.53, Rmk. 4.58].

Theorem 4.3. *Let* $m \in \mathbb{N}\setminus\{0\}$, $n \in \mathbb{N}\setminus\{0,1\}$, $\alpha \in]0,1[$. *Let* Ω^i, Ω^o *be as in* (1.1)*. Let* m^i, m^o, $a \in]0,+\infty[$, $b \in \mathbb{R}$. *Let* g^i, g^o *be as in* (1.3)*,* (1.5)*. Let* k_i, k_o *be as in* (1.2)*,* (1.4)*. Assume that* k_o^2 *is not a Neumann eigenvalue for* $-\Delta$ *in* Ω^o*. Let* $\epsilon_* \in]0, \epsilon_0[$ *be as in Theorem* 4.1*. Let* $\mathcal{M} \equiv (\mathcal{M}_l)_{l=1,2,3}$ *be the map from* $]-\epsilon_0, \epsilon_0[\times\mathbb{R}^2 \times Y_{m-1,\alpha}$ *to* $Z_{m-1,\alpha}$ *defined by* (4.6)*–*(4.8)*. Then there exists* $\epsilon' \in]0, \epsilon_*[$*, an open neighbourhood* $\tilde{U}$ *of* $(0,0)$ *in* $\mathbb{R}^2$ *and an open neighbourhood* $\tilde{V}$ *of* $(\tilde{\zeta}, \tilde{c}^i, \tilde{\varsigma}^i, \tilde{c}, \tilde{\theta}^o)$ *in* $Y_{m-1,\alpha}$ *and a real analytic map*

$$\left(Z, C^i, S^i, C, \Theta^o\right)$$

from $]-\epsilon', \epsilon'[\times\tilde{U}$ *to* $\tilde{V}$ *such that*

$$\left(\kappa_n \epsilon \log\epsilon, \frac{\delta_{2,n}}{\log\epsilon}\right) \in \tilde{U}, \ \ \forall \epsilon \in]0, \epsilon'[,$$

and such that the set of zeros of $\mathcal{M}$ *in* $]-\epsilon', \epsilon'[\times\tilde{U}\times\tilde{V}$ *coincides with the graph of the map* $(Z, C^i, S^i, C, \Theta^o)$*. In particular,*

$$\left(Z[0,0,0], C^i[0,0,0], S^i[0,0,0], C[0,0,0], \Theta^o[0,0,0]\right) = (\tilde{\zeta}, \tilde{c}^i, \tilde{\varsigma}^i, \tilde{c}, \tilde{\theta}^o),$$

where $(\tilde{\zeta}, \tilde{c}^i, \tilde{\varsigma}^i, \tilde{c}, \tilde{\theta}^o)$ *is the only solution in* $Y_{m-1,\alpha}$ *of equation* (4.9)*. Moreover,*

$$\frac{\partial C^i}{\partial \epsilon_1}[0,0,0] = 0, \ \ \frac{\partial C^i}{\partial \epsilon_2}[0,0,0] = 0. \tag{4.12}$$

For the sake of brevity, we set

$$\Xi_n[\epsilon] \equiv \left(\kappa_n \epsilon \log\epsilon, \frac{\delta_{2,n}}{\log\epsilon}\right), \qquad \forall \epsilon \in]0,1[. \tag{4.13}$$

Then we have the following existence and uniqueness theorem for problem (1.6) for $\epsilon \in]0, \epsilon'[$ (cf. [1, Thm. 4.61].)

Theorem 4.4. *Let $m \in \mathbb{N} \setminus \{0\}$, $n \in \mathbb{N} \setminus \{0, 1\}$, $\alpha \in]0, 1[$. Let Ω^i, Ω^o be as in (1.1). Let m^i, m^o, $a \in]0, +\infty[$, $b \in \mathbb{R}$. Let g^i, g^o be as in (1.3), (1.5). Let k_i, k_o be as in (1.2), (1.4). Assume that k_o^2 is not a Neumann eigenvalue for $-\Delta$ in Ω^o.*

Let $\epsilon' \in]0, \epsilon_0[$ be as in Theorem 4.3. If $\epsilon \in]0, \epsilon'[$, then the transmission problem (1.6) has one and only one solution $(u^i(\epsilon, \cdot), u^o(\epsilon, \cdot)) \in C^{m,\alpha}(\epsilon\overline{\Omega^i}) \times C^{m,\alpha}(\overline{\Omega(\epsilon)})$ and the following formula holds

$$
\begin{aligned}
&u^i(\epsilon, \cdot) \qquad (4.14)\\
&\quad = u^i[\epsilon, Z[\epsilon, \Xi_n[\epsilon]], C^i[\epsilon, \Xi_n[\epsilon]], S^i[\epsilon, \Xi_n[\epsilon]], C[\epsilon, \Xi_n[\epsilon]], \Theta^o[\epsilon, \Xi_n[\epsilon]]](\cdot)\\
&u^o(\epsilon, \cdot)\\
&\quad = u^o[\epsilon, Z[\epsilon, \Xi_n[\epsilon]], C^i[\epsilon, \Xi_n[\epsilon]], S^i[\epsilon, \Xi_n[\epsilon]], C[\epsilon, \Xi_n[\epsilon]], \Theta^o[\epsilon, \Xi_n[\epsilon]]](\cdot)
\end{aligned}
$$

for all $\epsilon \in]0, \epsilon'[$ (cf. (4.10)).

5. Microscopic representation for $\{(u^i(\epsilon, \cdot), u^o(\epsilon, \cdot))\}_{\epsilon \in]0, \epsilon'[}$

We now analyze the microscopic behavior of our family of solutions, i.e., the behavior of the rescaled family $\{(u^i(\epsilon, \epsilon \cdot), u^o(\epsilon, \epsilon \cdot))\}_{\epsilon \in]0, \epsilon'[}$.

Theorem 5.1. *With the assumptions of Theorem 4.3, the following statements hold.*

(i) *There exist real analytic maps $\mathcal{U}_1^i$, $\mathcal{U}_2^i$ from $]-\epsilon', \epsilon'[\times \tilde{U}$ to $C^{m,\alpha}(\overline{\Omega^i})$ such that*

$$
u^i(\epsilon, \epsilon\xi) = \mathcal{U}_1^i[\epsilon, \Xi_n[\epsilon]] + (\kappa_n \epsilon \log^2 \epsilon)\mathcal{U}_2^i[\epsilon, \Xi_n[\epsilon]] \quad \forall \xi \in \overline{\Omega^i}
$$

for all $\epsilon \in]0, \epsilon'[$ (cf. (4.13) for the definition of Ξ_n). Moreover,

$$
\mathcal{U}_1^i[0, 0, 0] = \tilde{u}_1^{i,r}\,, \qquad \mathcal{U}_2^i[0, 0, 0] = 0\,,
$$

where $\tilde{u}_1^{i,r}$ has been defined in Theorem 4.2.

(ii) *Let Ω_m be a bounded open subset of $\mathbb{R}^n \setminus \overline{\Omega^i}$. Then there exist $\epsilon_m \in]0, \epsilon'[$, and two real analytic maps $\mathcal{U}_{m,1}^o$, $\mathcal{U}_{m,2}^o$ from $]-\epsilon_m, \epsilon_m[\times \tilde{U}$ to $C^{m,\alpha}(\overline{\Omega_m})$ such that*

$$
\epsilon\overline{\Omega_m} \subseteq \Omega^o \ \forall \epsilon \in]-\epsilon_m, \epsilon_m[\,,
$$

$$
\begin{aligned}
u^o(\epsilon, \epsilon\xi) = &\,\mathcal{U}_{m,1}^o[\epsilon, \Xi[\epsilon]](\xi)\\
&+ (\kappa_n \epsilon \log^2 \epsilon)\mathcal{U}_{m,2}^o[\epsilon, \Xi[\epsilon]](\xi) \qquad \forall \xi \in \overline{\Omega_m}\,,\ \epsilon \in]0, \epsilon_m[.
\end{aligned}
$$

Moreover,

$$
\mathcal{U}_{m,1}^o[0, 0, 0](\xi) = \tilde{u}^o(0) + \tilde{u}_1^{o,r}(\xi)\,, \quad \mathcal{U}_{m,2}^o[0, 0, 0](\xi) = 0 \quad \forall \xi \in \overline{\Omega_m}\,,
$$

where $\tilde{u}^o$ and $\tilde{u}_1^{o,r}$ are as in Theorem 4.2.

Proof. By the first formulas of (4.10) and (4.14), we have

$$
\begin{aligned}
u^i(\epsilon, \epsilon\xi) = &\,\epsilon^{n-2} \int_{\partial\Omega^i} S_{r,n}(\epsilon\xi - \epsilon\eta, k_i) Z[\epsilon, \Xi_n[\epsilon]](\eta)\, d\sigma_\eta\\
&+ \epsilon^{n-2} a^{-1} \left(k_o^{n-2}/k_i^{n-2}\right) C^i[\epsilon, \Xi_n[\epsilon]] \int_{\partial\Omega^i} S_{r,n}(\epsilon\xi - \epsilon\eta, k_i)\theta^\sharp(\eta)\, d\sigma_\eta\\
&+ \epsilon^{n-2} \log^{-\delta_{2,n}} \epsilon\ C[\epsilon, \Xi_n[\epsilon]] \int_{\partial\Omega^i} S_{r,n}(\epsilon\xi - \epsilon\eta, k_i)\theta^\sharp(\eta)\, d\sigma_\eta \quad \forall \xi \in \overline{\Omega^i}
\end{aligned}
$$

for all $\epsilon \in]0, \epsilon'[$. Then Lemma 3.4 implies that

$$\begin{aligned} u^i(\epsilon, \epsilon\xi) = & v^+_{\Omega^i}[Z[\epsilon, \Xi_n[\epsilon]], \epsilon k_i](\xi) \\ & + a^{-1} \left(k_o^{n-2}/k_i^{n-2}\right) C^i[\epsilon, \Xi_n[\epsilon]] v^+_{\Omega^i}[\theta^\sharp, \epsilon k_i](\xi) \\ & + \log^{-\delta_{2,n}} \epsilon \; C[\epsilon, \Xi_n[\epsilon]] v^+_{\Omega^i}[\theta^\sharp, \epsilon k_i](\xi) \quad \forall \xi \in \overline{\Omega^i} \end{aligned}$$

for all $\epsilon \in]0, \epsilon'[$. Then equality (3.5) implies that

$$\begin{aligned} u^i(\epsilon, \epsilon\xi) = & v^+_{\Omega^i,h}[Z[\epsilon, \Xi_n[\epsilon]], \epsilon k_i](\xi) \\ & + \gamma_n(\log(\epsilon k_i)) \epsilon^{n-2} k_i^{n-2} v^+_{\Omega^i,J}[Z[\epsilon, \Xi_n[\epsilon]], \epsilon^2 k_i^2](\xi) \\ & + a^{-1} \left(k_o^{n-2}/k_i^{n-2}\right) C^i[\epsilon, \Xi_n[\epsilon]] \\ & \times \left(v^+_{\Omega^i,h}[\theta^\sharp, \epsilon k_i](\xi) + \gamma_n(\log(\epsilon k_i)) \epsilon^{n-2} k_i^{n-2} v^+_{\Omega^i,J}[\theta^\sharp, \epsilon^2 k_i^2](\xi)\right) \\ & + \log^{-\delta_{2,n}} \epsilon \; C[\epsilon, \Xi_n[\epsilon]] \\ & \times \left(v^+_{\Omega^i,h}[\theta^\sharp, \epsilon k_i](\xi) + \gamma_n(\log(\epsilon k_i)) \epsilon^{n-2} k_i^{n-2} v^+_{\Omega^i,J}[\theta^\sharp, \epsilon^2 k_i^2](\xi)\right) \end{aligned}$$

for all $\xi \in \overline{\Omega^i}$ and $\epsilon \in]0, \epsilon'[$. By equality (3.6), we have

$$\begin{aligned} u^i(\epsilon, \epsilon\xi) = & v^+_{\Omega^i,h}[Z[\epsilon, \Xi_n[\epsilon]], \epsilon k_i](\xi) \\ & + \left[\frac{2b_n}{\pi} \kappa_n \log \epsilon + \gamma_n(\log k_i)\right] \epsilon^{n-2} k_i^{n-2} v^+_{\Omega^i,J}[Z[\epsilon, \Xi_n[\epsilon]], \epsilon^2 k_i^2](\xi) \\ & + a^{-1} \left(k_o^{n-2}/k_i^{n-2}\right) C^i[\epsilon, \Xi_n[\epsilon]] \Bigg(v^+_{\Omega^i,h}[\theta^\sharp, \epsilon k_i](\xi) \\ & + \left[\frac{2b_n}{\pi} \kappa_n \log \epsilon + \gamma_n(\log k_i)\right] \epsilon^{n-2} k_i^{n-2} v^+_{\Omega^i,J}[\theta^\sharp, \epsilon^2 k_i^2](\xi) \Bigg) \\ & + \log^{-\delta_{2,n}} \epsilon \; C[\epsilon, \Xi_n[\epsilon]] \Bigg(v^+_{\Omega^i,h}[\theta^\sharp, \epsilon k_i](\xi) \\ & + \left[\frac{2b_n}{\pi} \kappa_n \log \epsilon + \gamma_n(\log k_i)\right] \epsilon^{n-2} k_i^{n-2} v^+_{\Omega^i,J}[\theta^\sharp, \epsilon^2 k_i^2](\xi) \Bigg) \qquad \forall \xi \in \overline{\Omega^i} \end{aligned}$$

for all $\epsilon \in]0, \epsilon'[$. Since there exists an entire function $J^{\sharp,1}_{\frac{n-2}{2}}$ such that

$$J^\sharp_{\frac{n-2}{2}}(z) = J^\sharp_{\frac{n-2}{2}}(0) + z J^{\sharp,1}_{\frac{n-2}{2}}(z) \qquad \forall z \in \mathbb{C} \tag{5.1}$$

(see (3.1)), then we have

$$v^+_{\Omega^i,J}[Z[\epsilon, \Xi_n[\epsilon]], \epsilon^2 k_i^2](\xi) = \int_{\partial\Omega^i} J^\sharp_{\frac{n-2}{2}}(\epsilon^2 k_i^2 |\xi - \eta|^2) Z[\epsilon, \Xi_n[\epsilon]](\eta) d\sigma_\eta \tag{5.2}$$

$$\begin{aligned} & = J^\sharp_{\frac{n-2}{2}}(0) \int_{\partial\Omega^i} Z[\epsilon, \Xi_n[\epsilon]](\eta) d\sigma_\eta \\ & \quad + \epsilon^2 k_i^2 \int_{\partial\Omega^i} |\xi - \eta|^2 J^{\sharp,1}_{\frac{n-2}{2}}(\epsilon^2 k_i^2 |\xi - \eta|^2) Z[\epsilon, \Xi_n[\epsilon]](\eta) d\sigma_\eta \\ & = \epsilon^2 k_i^2 \int_{\partial\Omega^i} |\xi - \eta|^2 J^{\sharp,1}_{\frac{n-2}{2}}(\epsilon^2 k_i^2 |\xi - \eta|^2) Z[\epsilon, \Xi_n[\epsilon]](\eta) d\sigma_\eta \qquad \forall \epsilon \in]0, \epsilon'[. \end{aligned}$$

Indeed, $\int_{\partial\Omega^i} Z[\epsilon, \Xi_n[\epsilon]](\eta)d\sigma_\eta = 0$ for all $\epsilon \in]0, \epsilon'[$. Also, the Fundamental Theorem of Calculus and equality $C^i[0,0,0] = \tilde{c}^i = 0$ imply that

$$
\begin{aligned}
&(\kappa_n \log\epsilon)\ C^i[\epsilon, \kappa_n\epsilon\log\epsilon, \delta_{2,n}/\log\epsilon] \\
&\quad = (\kappa_n\epsilon\log\epsilon)\int_0^1 \frac{\partial C^i}{\partial t_1}[s\epsilon, s\kappa_n\epsilon\log\epsilon, s\delta_{2,n}/\log\epsilon]\,ds \\
&\qquad + \kappa_n\epsilon\log^2\epsilon\int_0^1 \frac{\partial C^i}{\partial t_2}[s\epsilon, s\kappa_n\epsilon\log\epsilon, s\delta_{2,n}/\log\epsilon]\,ds \\
&\qquad + \delta_{2,n}\int_0^1 \frac{\partial C^i}{\partial t_3}[s\epsilon, s\kappa_n\epsilon\log\epsilon, s\delta_{2,n}/\log\epsilon]\,ds \qquad \forall\epsilon\in]0,\epsilon'[.
\end{aligned} \tag{5.3}
$$

Next introduce the following analytic functions

$$
C_1[\epsilon, \epsilon_1, \epsilon_2] \equiv \epsilon_1 \int_0^1 \frac{\partial C^i}{\partial t_1}[s\epsilon, s\epsilon_1, s\epsilon_2]\,ds + \delta_{2,n}\int_0^1 \frac{\partial C^i}{\partial t_3}[s\epsilon, s\epsilon_1, s\epsilon_2]\,ds
$$

and

$$
C_2[\epsilon, \epsilon_1, \epsilon_2] \equiv \int_0^1 \frac{\partial C^i}{\partial t_2}[s\epsilon, s\epsilon_1, s\epsilon_2]\,ds \qquad \forall(\epsilon, \epsilon_1, \epsilon_2) \in]-\epsilon', \epsilon'[\times\tilde{U}.
$$

By equality (5.3), we have

$$
\begin{aligned}
&(\kappa_n\log\epsilon)\ C^i[\epsilon, \kappa_n\epsilon\log\epsilon, \delta_{2,n}/\log\epsilon] \\
&\quad = C_1[\epsilon, \kappa_n\epsilon\log\epsilon, \delta_{2,n}/\log\epsilon] + \kappa_n\epsilon\log^2\epsilon\, C_2[\epsilon, \kappa_n\epsilon\log\epsilon, \delta_{2,n}/\log\epsilon].
\end{aligned} \tag{5.4}
$$

By (5.2), (5.4), and by the elementary equality

$$
\log^{-\delta_{2,n}}\epsilon = (1-\delta_{2,n}) + \frac{\delta_{2,n}}{\log\epsilon} \qquad \forall\epsilon\in]0,1[\,,
$$

we have

$$
\begin{aligned}
u^i(\epsilon, \epsilon\xi) &= v^+_{\Omega^i,h}[Z[\epsilon, \Xi_n[\epsilon]], \epsilon k_i](\xi) \\
&+ \frac{2b_n}{\pi}k_i^n\kappa_n\epsilon^n\log\epsilon\int_{\partial\Omega^i}|\xi-\eta|^2 J^{\sharp,1}_{\frac{n-2}{2}}(\epsilon^2k_i^2|\xi-\eta|^2)Z[\epsilon, \Xi_n[\epsilon]](\eta)d\sigma_\eta \\
&+ \gamma_n(\log k_i)k_i^{n-2}\epsilon^{n-2}v^+_{\Omega^i,J}[Z[\epsilon, \Xi_n[\epsilon]], \epsilon^2k_i^2](\xi) \\
&+ a^{-1}\left(k_o^{n-2}/k_i^{n-2}\right)C^i[\epsilon, \Xi_n[\epsilon]]v^+_{\Omega^i,h}[\theta^\sharp, \epsilon k_i](\xi) \\
&+ a^{-1}k_o^{n-2}\frac{2b_n}{\pi}C_1[\epsilon, \Xi_n[\epsilon]]\,\epsilon^{n-2}v^+_{\Omega^i,J}[\theta^\sharp, \epsilon^2k_i^2](\xi) \\
&+ a^{-1}k_o^{n-2}\frac{2b_n}{\pi}\kappa_n\epsilon\log^2\epsilon\, C_2[\epsilon, \Xi_n[\epsilon]]\epsilon^{n-2}v^+_{\Omega^i,J}[\theta^\sharp, \epsilon^2k_i^2](\xi) \\
&+ a^{-1}k_o^{n-2}C^i[\epsilon, \Xi_n[\epsilon]]\gamma_n(\log k_i)\epsilon^{n-2}v^+_{\Omega^i,J}[\theta^\sharp, \epsilon^2k_i^2](\xi) \\
&+ \left[(1-\delta_{2,n}) + \frac{\delta_{2,n}}{\log\epsilon}\right]C[\epsilon, \Xi_n[\epsilon]]v^+_{\Omega^i,h}[\theta^\sharp, \epsilon k_i](\xi) \\
&+ \frac{2b_n}{\pi}\varrho_n(\epsilon, \kappa_n\epsilon\log\epsilon)k_i^{n-2}C[\epsilon, \Xi_n[\epsilon]]v^+_{\Omega^i,J}[\theta^\sharp, \epsilon^2k_i^2](\xi) \\
&+ \left[(1-\delta_{2,n}) + \frac{\delta_{2,n}}{\log\epsilon}\right]\gamma_n(\log k_i)C[\epsilon, \Xi_n[\epsilon]]\epsilon^{n-2}k_i^{n-2}v^+_{\Omega^i,J}[\theta^\sharp, \epsilon^2k_i^2](\xi),\ \forall\xi\in\overline{\Omega^i}
\end{aligned}
$$

for all $\epsilon \in]0, \epsilon'[$ (see (4.5)). Thus, we find natural to set

$$\begin{aligned}
\mathcal{U}_1^i[\epsilon,\epsilon_1,\epsilon_2](\xi) &\equiv v^+_{\Omega^i,h}[Z[\epsilon,\epsilon_1,\epsilon_2],\epsilon k_i](\xi) \\
&+\frac{2b_n}{\pi}k_i^n\epsilon_1\epsilon^{n-1}\int_{\partial\Omega^i}|\xi-\eta|^2 J^{\sharp,1}_{\frac{n-2}{2}}(\epsilon^2k_i^2|\xi-\eta|^2)Z[\epsilon,\epsilon_1,\epsilon_2](\eta)d\sigma_\eta \\
&+\gamma_n(\log k_i)\epsilon^{n-2}k_i^{n-2}v^+_{\Omega^i,J}[Z[\epsilon,\epsilon_1,\epsilon_2],\epsilon^2k_i^2](\xi) \\
&+a^{-1}\left(k_o^{n-2}/k_i^{n-2}\right)C^i[\epsilon,\epsilon_1,\epsilon_2]v^+_{\Omega^i,h}[\theta^\sharp,\epsilon k_i](\xi) \\
&+a^{-1}k_o^{n-2}\frac{2b_n}{\pi}C_1[\epsilon,\epsilon_1,\epsilon_2]\,\epsilon^{n-2}v^+_{\Omega^i,J}[\theta^\sharp,\epsilon^2k_i^2](\xi) \\
&+a^{-1}k_o^{n-2}C^i[\epsilon,\epsilon_1,\epsilon_2]\gamma_n(\log k_i)\epsilon^{n-2}v^+_{\Omega^i,J}[\theta^\sharp,\epsilon^2k_i^2](\xi) \\
&+\left[(1-\delta_{2,n})+\epsilon_2\right]C[\epsilon,\epsilon_1,\epsilon_2]v^+_{\Omega^i,h}[\theta^\sharp,\epsilon k_i](\xi) \\
&+\frac{2b_n}{\pi}\varrho_n(\epsilon,\epsilon_1)k_i^{n-2}C[\epsilon,\epsilon_1,\epsilon_2]v^+_{\Omega^i,J}[\theta^\sharp,\epsilon^2k_i^2](\xi)+\left[(1-\delta_{2,n})+\epsilon_2\right] \\
&\times\gamma_n(\log k_i)C[\epsilon,\epsilon_1,\epsilon_2]\epsilon^{n-2}k_i^{n-2}v^+_{\Omega^i,J}[\theta^\sharp,\epsilon^2k_i^2](\xi)\quad\forall\xi\in\overline{\Omega^i}, \\
\mathcal{U}_2^i[\epsilon,\epsilon_1,\epsilon_2](\xi) &\equiv a^{-1}k_o^{n-2}\frac{2b_n}{\pi}C_2[\epsilon,\epsilon_1,\epsilon_2]\epsilon^{n-2}v^+_{\Omega^i,J}[\theta^\sharp,\epsilon^2k_i^2](\xi)\quad\forall\xi\in\overline{\Omega^i}
\end{aligned}\tag{5.5}$$

for all $(\epsilon,\epsilon_1,\epsilon_2)\in]\epsilon',\epsilon'[\times\tilde{U}$ (see Theorem 4.3). By [1, Thm. 3.21 (i)], $v^+_{\Omega^i,h}[\cdot,\cdot]$ defines a real analytic map from $C^{m-1,\alpha}(\partial\Omega^i)\times\mathbb{C}$ to $C^{m,\alpha}(\overline{\Omega^i})$. Then Theorem 4.3 implies that the map from $]-\epsilon',\epsilon'[\times\tilde{U}$ to $C^{m,\alpha}(\overline{\Omega^i})$ which takes $(\epsilon,\epsilon_1,\epsilon_2)$ to $v^+_{\Omega^i,h}\left[Z[\epsilon,\epsilon_1,\epsilon_2],\epsilon k_i\right]$ is real analytic. Similarly, the map from $]-\epsilon',\epsilon'[\times\tilde{U}$ to $C^{m,\alpha}(\overline{\Omega^i})$ which takes $(\epsilon,\epsilon_1,\epsilon_2)$ to $v^+_{\Omega^i,h}\left[\theta^\sharp,\epsilon k_i\right]$ is real analytic. Since $|\xi-\eta|^2J^{\sharp,1}_{\frac{n-2}{2}}(\epsilon^2k_i^2|\xi-\eta|^2)$ is analytic in the variable (ξ,η,ϵ) in an open neighbourhood of $\overline{\Omega^i}\times\partial\Omega^i\times]-\epsilon',\epsilon'[$ then Proposition 4.1 (i) of paper [17] with Musolino on integral operators with real analytic kernel implies that the map from $]-\epsilon',\epsilon'[\times L^1(\partial\Omega^i)$ to $C^{m,\alpha}(\overline{\Omega^i})$ that takes (ϵ,f) to the function $\int_{\partial\Omega^i}|\xi-\eta|^2J^{\sharp,1}_{\frac{n-2}{2}}(\epsilon^2k_i^2|\xi-\eta|^2)f(\eta)d\sigma_\eta$ of the variable $\xi\in\partial\Omega^i$ is real analytic. Since Z is real analytic and $C^{m-1,\alpha}(\partial\Omega^i)$ is continuously imbedded into $L^1(\partial\Omega^i)$, we conclude that the map from $]-\epsilon',\epsilon'[\times\tilde{U}$ to $C^{m,\alpha}(\overline{\Omega^i})$ which takes $(\epsilon,\epsilon_1,\epsilon_2)$ to the second summand of the right-hand side of (5.5) is analytic. By [1, Thm. 3.22 (ii)], $v^+_{\Omega^i,J}[\cdot,\cdot]$ defines a real analytic map from $C^{m-1,\alpha}(\partial\Omega^i)\times\mathbb{C}$ to $C^{m,\alpha}(\overline{\Omega^i})$. Then Theorem 4.3 implies that the map from $]-\epsilon',\epsilon'[\times\tilde{U}$ to $C^{m,\alpha}(\overline{\Omega^i})$ which takes $(\epsilon,\epsilon_1,\epsilon_2)$ to $v^+_{\Omega^i,J}\left[Z[\epsilon,\epsilon_1,\epsilon_2],\epsilon^2k_i^2\right]$ is real analytic. Similarly, the map from $]-\epsilon',\epsilon'[\times\tilde{U}$ to $C^{m,\alpha}(\overline{\Omega^i})$ which takes $(\epsilon,\epsilon_1,\epsilon_2)$ to $v^+_{\Omega^i,J}[\theta^\sharp,\epsilon^2k_i^2]$ is real analytic. Finally C^i is real analytic by Theorem 4.3 and thus, C_1 and C_2 are real analytic as well. Hence $\mathcal{U}_1^i$ and $\mathcal{U}_2^i$ are real analytic. Moreover, (4.3), (4.4) and Theorems 4.2, 4.3 imply that

$$\mathcal{U}_1^i[0,0,0](\xi)=v^+_{\Omega^i,h}[Z[0,0,0],0](\xi)+\gamma_n(\log k_i)\delta_{2,n}k_i^{n-2}v^+_{\Omega_i,J}[Z[0,0,0],0](\xi)$$

$$+a^{-1}\left(k_o^{n-2}/k_i^{n-2}\right)C^i[0,0,0]v^+_{\Omega^i,h}[\theta^\sharp,0](\xi)+a^{-1}k_o^{n-2}\frac{2b_n}{\pi}C_1[0,0,0]\,\delta_{2,n}v^+_{\Omega^i,J}[\theta^\sharp,0](\xi)$$

$$+a^{-1}k_o^{n-2}C^i[0,0,0]\gamma_n(\log k_i)\delta_{2,n}v^+_{\Omega^i,J}[\theta^\sharp,0](\xi)+(1-\delta_{2,n})C[0,0,0]v^+_{\Omega^i,h}[\theta^\sharp,0](\xi)$$

$$
\begin{aligned}
&+\frac{2b_n}{\pi}\delta_{2,n}k_i^{n-2}C[0,0,0]v^+_{\Omega^i,J}[\theta^\sharp,0](\xi)\\
&+(1-\delta_{2,n})\gamma_n(\log k_i)C[0,0,0]\delta_{2,n}k_i^{n-2}v^+_{\Omega^i,J}[\theta^\sharp,0](\xi)=v^+_{\Omega^i,h}[Z[0,0,0],0](\xi)\\
&+\left((1-\delta_{2,n})v^+_{\Omega^i,h}[\theta^\sharp,0](\xi)+\delta_{2,n}k_i^{n-2}\frac{2b_n}{\pi}J^\sharp_{\frac{n-2}{2}}(0)\right)C[0,0,0]\\
&+a^{-1}k_o^{n-2}\frac{2b_n}{\pi}C_1[0,0,0]\,\delta_{2,n}J^\sharp_{\frac{n-2}{2}}(0)\\
&=v^+_{\Omega^i,h}[\tilde{\zeta},0](\xi)+\tilde{C}+a^{-1}k_o^{n-2}\frac{2b_n}{\pi}C_1[0,0,0]\,\delta_{2,n}J^\sharp_{\frac{n-2}{2}}(0)\\
&=\tilde{u}_1^{i,r}(\xi)+a^{-1}k_o^{n-2}\frac{2b_n}{\pi}C_1[0,0,0]\,\delta_{2,n}J^\sharp_{\frac{n-2}{2}}(0)\\
&\mathcal{U}_2^i[0,0,0](\xi)=a^{-1}k_o^{n-2}\frac{2b_n}{\pi}C_2[0,0,0]\delta_{2,n}J^\sharp_{\frac{n-2}{2}}(0),\ \forall\xi\in\overline{\Omega^i}.
\end{aligned}
$$

By the definition of C_1, C_2 and by equality (4.12), we have

$$C_1[0,0,0]=C_2[0,0,0]=0\,, \tag{5.6}$$

and thus the proof of (i) is complete. We now consider statement (ii). Let ϵ_m be such that

$$\epsilon\overline{\Omega_m}\subseteq\Omega^o \qquad \forall\epsilon\in[-\epsilon_m,\epsilon_m].$$

Then we have

$$\overline{\Omega_m}\subseteq\frac{1}{\epsilon}\overline{\Omega(\epsilon)} \qquad \forall\epsilon\in[-\epsilon_m,\epsilon_m]\setminus\{0\}.$$

By the second formulas of (4.10) and (4.14), we have

$$
\begin{aligned}
u^o(\epsilon,\epsilon\xi)=&\int_{\partial\Omega^o}S_{r,n}(\epsilon\xi-y,k_o)\Theta^o[\epsilon,\Xi_n[\epsilon]](y)d\sigma_y\\
&+\epsilon^{n-2}\int_{\partial\Omega^i}S_{r,n}(\epsilon\xi-\epsilon\eta,k_o)S^i[\epsilon,\Xi_n[\epsilon]](\eta)\,d\sigma_\eta\\
&+\epsilon^{n-2}\int_{\partial\Omega^i}S_{r,n}(\epsilon\xi-\epsilon\eta,k_o)C^i[\epsilon,\Xi_n[\epsilon]]\theta^\sharp(\eta)\,d\sigma_\eta \qquad \xi\in\overline{\Omega_m}
\end{aligned}
$$

for all $\epsilon\in]0,\epsilon_m[$. By Lemma 3.4 and equalities (3.5), (3.6), we have

$$
\begin{aligned}
u^o(\epsilon,\epsilon\xi)=&\int_{\partial\Omega^o}S_{r,n}(\epsilon\xi-y,k_o)\Theta^o[\epsilon,\Xi_n[\epsilon]](y)d\sigma_y\\
&+v^-_{\Omega^i,h}[S^i[\epsilon,\Xi_n[\epsilon]],\epsilon k_o](\xi)+\left[\frac{2b_n}{\pi}\kappa_n\log\epsilon+\gamma_n(\log k_o)\right]\epsilon^{n-2}k_o^{n-2}\\
&\times v^-_{\Omega^i,J}[S^i[\epsilon,\Xi_n[\epsilon]],\epsilon^2k_o^2](\xi)\\
&+C^i[\epsilon,\Xi_n[\epsilon]]\Big(v^-_{\Omega^i,h}[\theta^\sharp,\epsilon k_o](\xi)+\left[\frac{2b_n}{\pi}\kappa_n\log\epsilon+\gamma_n(\log k_o)\right]\epsilon^{n-2}k_o^{n-2}\\
&\times v^-_{\Omega^i,J}[\theta^\sharp,\epsilon^2k_o^2](\xi)\Big) \qquad \forall\xi\in\overline{\Omega_m}
\end{aligned}
$$

for all $\epsilon \in]0, \epsilon_m[$. Thus by exploiting (5.1) and (5.4) we find natural to set

$$
\begin{aligned}
&\mathcal{U}^o_{m,1}[\epsilon, \epsilon_1, \epsilon_2](\xi) \qquad (5.7)\\
&\equiv \int_{\partial\Omega^o} S_{r,n}(\epsilon\xi - y, k_o)\Theta^o[\epsilon, \epsilon_1, \epsilon_2](y)d\sigma_y + v^-_{\Omega^i,h}[S^i[\epsilon, \epsilon_1, \epsilon_2], \epsilon k_o](\xi)\\
&+\frac{2b_n}{\pi}\epsilon_1\epsilon^{n-1}k_o^n \int_{\partial\Omega^i} |\xi - \eta|^2 J^{\sharp,1}_{\frac{n-2}{2}}(\epsilon^2 k_o^2|\xi - \eta|^2)S^i[\epsilon, \epsilon_1, \epsilon_2](\eta)d\sigma_\eta\\
&+\gamma_n(\log k_o)\epsilon^{n-2}k_o^{n-2}v^-_{\Omega^i,J}[S^i[\epsilon, \epsilon_1, \epsilon_2], \epsilon^2 k_o^2](\xi)\\
&+C^i[\epsilon, \epsilon_1, \epsilon_2]v^-_{\Omega^i,h}[\theta^\sharp, \epsilon k_o](\xi)\\
&+\gamma_n(\log k_o)\epsilon^{n-2}k_o^{n-2}C^i[\epsilon, \epsilon_1, \epsilon_2]v^-_{\Omega^i,J}[\theta^\sharp, \epsilon^2 k_o^2](\xi)\\
&+\frac{2b_n}{\pi}\epsilon^{n-2}k_o^{n-2}C_1[\epsilon, \epsilon_1, \epsilon_2]v^-_{\Omega^i,J}[\theta^\sharp, \epsilon^2 k_o^2](\xi) \qquad \forall\xi \in \overline{\Omega_m}
\end{aligned}
$$

$$
\mathcal{U}^o_{m,2}[\epsilon, \epsilon_1, \epsilon_2](\xi) \equiv \epsilon^{n-2}k_o^{n-2}\frac{2b_n}{\pi}C_2[\epsilon, \epsilon_1, \epsilon_2]v^-_{\Omega^i,J}[\theta^\sharp, \epsilon^2 k_o^2](\xi) \qquad \forall\xi \in \overline{\Omega_m}
$$

for all $(\epsilon, \epsilon_1, \epsilon_2) \in]-\epsilon', \epsilon[\times\tilde{U}$. Since $S_{r,n}(\epsilon\xi - y, k_o)$ is real analytic in the variable (ξ, y, ϵ) in an open neighbourhood of $\overline{\Omega_m} \times \partial\Omega^o\times]-\epsilon_m, \epsilon_m[$ then by Proposition 4.1 (i) of [17] on the integral operators with real analytic kernel, the map from $]-\epsilon_m, \epsilon_m[\times L^1(\partial\Omega^o)$ to $C^{m,\alpha}(\overline{\Omega_m})$ which takes a pair (ϵ, h) to $\int_{\partial\Omega^o} S_{r,n}(\epsilon\cdot -y, k_o)h(y)d\sigma_y$ is analytic. Since Θ^o is real analytic and $C^{m-1,\alpha}(\partial\Omega^o)$ is continuously imbedded into $L^1(\partial\Omega^o)$, we conclude that the map from $]-\epsilon_m, \epsilon_m[\times\tilde{U}$ to $C^{m,\alpha}(\overline{\Omega_m})$ which takes $(\epsilon, \epsilon_1, \epsilon_2)$ to the first summand of the right-hand side of (5.7) is real analytic. By [1, Thm. 3.22 (ii)], $v^-_{\Omega^i,h}[\cdot,\cdot]$ defines a real analytic map from $C^{m-1,\alpha}(\partial\Omega^i) \times \mathbb{C}$ to $C^{m,\alpha}(\overline{\Omega_m})$. Then Theorem 4.3 implies that the map from $]-\epsilon', \epsilon'[\times\tilde{U}$ to $C^{m,\alpha}(\overline{\Omega_m})$ which takes $(\epsilon, \epsilon_1, \epsilon_2)$ to $v^-_{\Omega^i,h}\left[S^i[\epsilon, \epsilon_1, \epsilon_2], \epsilon k_o\right]$ is real analytic. Similarly, the map $]-\epsilon_m, \epsilon_m[\times\tilde{U}$ to $C^{m,\alpha}(\overline{\Omega_m})$ which takes $(\epsilon, \epsilon_1, \epsilon_2)$ to $v^-_{\Omega^i,h}[\theta^\sharp, \epsilon k_o]$ is real analytic. Since $|\xi - \eta|^2 J^{\sharp,1}_{\frac{n-2}{2}}(\epsilon^2 k_o^2|\xi - \eta|)$ is analytic in the variable (ξ, η, ϵ) in an open neighbourhood of $\overline{\Omega_m} \times \partial\Omega^i\times]-\epsilon_m, \epsilon_m[$ then by Proposition 4.1 (i) of [17] on integral operators with real analytic kernel the map from $]-\epsilon_m, \epsilon_m[\times L^1(\partial\Omega^i)$ to $C^{m,\alpha}(\overline{\Omega_m})$ that takes (ϵ, h) to the function

$$
\int_{\partial\Omega^i} |\xi - \eta|^2 J^{\sharp,1}_{\frac{n-2}{2}}(\epsilon^2 k_o^2|\xi - \eta|^2)h(\eta)d\sigma_\eta \qquad \forall\xi \in \partial\Omega^i
$$

is real analytic. Since S^i is real analytic and $C^{m-1,\alpha}(\partial\Omega^i)$ is continuously imbedded into $L^1(\partial\Omega^i)$, we conclude that the map from $]-\epsilon_m, \epsilon_m[\times\tilde{U}$ to $C^{m,\alpha}(\overline{\Omega_m})$ which takes $(\epsilon, \epsilon_1, \epsilon_2)$ to the third summand of the right-hand side of (5.7) is real analytic. By [1, Thm. 3.22 (iii)], $v^-_{\Omega^i,J}[\cdot,\cdot]$ defines a real analytic map from $C^{m-1,\alpha}(\partial\Omega^i) \times \mathbb{C}$ to $C^{m,\alpha}(\overline{\Omega_m})$. Then Theorem 4.3 (ii) implies that the map from $]-\epsilon_m, \epsilon_m[\times\tilde{U}$ to $C^{m,\alpha}(\overline{\Omega_m})$ which takes $(\epsilon, \epsilon_1, \epsilon_2)$ to $v^-_{\Omega^i,J}\left[S^i[\epsilon, \epsilon_1, \epsilon_2], \epsilon^2 k_o^2\right]$ is real analytic. Similarly, the map from $]-\epsilon_m, \epsilon_m[\times\tilde{U}$ to $C^{m,\alpha}(\overline{\Omega_m})$ which takes $(\epsilon, \epsilon_1, \epsilon_2)$ to $v^-_{\Omega^i,J}[\theta^\sharp, \epsilon^2 k_o^2]$ is real analytic. Finally, C^i is real analytic by Theorem 4.3 and thus C_1 and C_2 are analytic. Hence $\mathcal{U}^o_{m,1}$ and $\mathcal{U}^o_{m,2}$ are analytic. Then by (4.11), Theorem 4.3 and equality (5.6), we have

$$\begin{aligned}
\mathcal{U}^o_{m,1}[0,0,0](\xi) &= \int_{\partial\Omega^o} S_{r,n}(-y,k_o)\Theta^o[0,0,0](y)d\sigma_y + v^-_{\Omega^i,h}[S^i[0,0,0],0](\xi) \\
&\quad +\gamma_n(\log k_o)\delta_{2,n}k_o^{n-2}v^-_{\Omega^i,J}[S^i[0,0,0],0](\xi) + C^i[0,0,0]v^-_{\Omega^i,h}[\theta^\sharp,0](\xi) \\
&\quad +\gamma_n(\log k_o)\delta_{2,n}k_o^{n-2}C^i[0,0,0]v^-_{\Omega^i,J}[\theta^\sharp,0](\xi) \\
&\quad +\frac{2b_n}{\pi}\delta_{2,n}k_o^{n-2}C_1[0,0,0]v^-_{\Omega^i,J}[\theta^\sharp,0](\xi) \\
&= v^+_{\Omega^o}[\tilde{\theta}^o,k_o](0) + v^-_{\Omega^i,h}[\tilde{\varsigma}^i,0](\xi) + \frac{2b_n}{\pi}\delta_{2,n}k_o^{n-2}C_1[0,0,0]J^\sharp_{\frac{n-2}{2}}(0) \\
&= \tilde{u}^o(0) + \tilde{u}^{o,r}_1(\xi)\,, \\
\mathcal{U}^o_{m,2}[0,0,0](\xi) &= 0 \qquad \forall \xi \in \overline{\Omega_m}\,.
\end{aligned}$$

□

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