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g-Loewner chains, Bloch functions and extension operators into the family of locally biholomorphic mappings in infinite dimensional spaces

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Abstract. In this paper, we survey recent results obtained by the authors on the preservations of the first elements of (g-) Loewner chains and the Bloch mappings by the Roper-Suffridge type extension operators, the Muir type extension operators and the Pfaltzgraff-Suffridge type extension operators into the mappings on the domains in the complex Banach spaces.

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1. Introduction

After Roper and Suffridge [46] introduced the following extension operator

$$\Phi(f)(z) = (f(z_1), \sqrt{f'(z_1)}\tilde{z}), \quad z = (z_1, \tilde{z}) \in \mathbb{B}^n,$$

which extends locally univalent functions on the unit disc $\mathbb{U}$ in $\mathbb{C}$ to locally biholomorphic mappings on the Euclidean unit ball $\mathbb{B}^n$ in $\mathbb{C}^n$, the preservation of starlike mappings, spirallike mappings, the first elements of Loewner chains and Bloch mappings by similar extension operators have been extensively studied (see e.g. [3], [10], [11], [14], [19], [20], [21], [22], [33], [35], [36], [40], [41], [46], [48], [49] and [50]).

The Roper-Suffridge extension operator Φ preserves the following geometric and analytic properties from the one dimensional case to higher dimensions:

(i) $\Phi(S^*(\mathbb{B}^1)) \subseteq S^*(\mathbb{B}^n)$, where $S^*(\mathbb{B}^n)$ denotes the family of normalized starlike (univalent) mappings on $\mathbb{B}^n$ ([20]).

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(ii) If $f \in S$, where S denotes the family of normalized univalent functions on $\mathbb{U}$, then $\Phi(f)$ can be embedded as the first element of a Loewner chain on $\mathbb{B}^n$ ([19], [22]).
(iii) Φ maps the family of normalized univalent Bloch functions on $\mathbb{U}$ with Bloch semi-norm 1 into the family of normalized univalent Bloch mappings on $\mathbb{B}^n$ ([20]).

For further properties of the Roper-Suffridge extension operator Φ, see e.g. [46].

Let $\alpha \geq 0$, $\beta \geq 0$ be given. Then the modification $\Phi_{n,\alpha,\beta}$ of the Roper-Suffridge extension operator ([19]) is given by:

$$\Phi_{n,\alpha,\beta}(f)(z) = \left(f(z_1), \left(\frac{f(z_1)}{z_1}\right)^{\alpha} (f'(z_1))^{\beta} \tilde{z}\right), \quad z = (z_1, \tilde{z}) \in \mathbb{B}^n,$$

for any $f \in \mathcal{L}S(\mathbb{U})$ such that $f(z_1) \neq 0$ for $z_1 \in \mathbb{U} \setminus \{0\}$, where $\mathcal{L}S(\mathbb{U})$ denotes the family of normalized locally univalent functions on $\mathbb{U}$. The branches of the power functions are chosen such that

$$\left(\frac{f(z_1)}{z_1}\right)^{\alpha}\bigg|_{z_1=0} = 1 \quad \text{and} \quad (f'(z_1))^{\beta}\big|_{z_1=0} = 1.$$

The extension operator $\Phi_{n,\alpha,\beta}$ has the following properties:

(i) $\Phi_{n,\alpha,\beta}(S^*(\mathbb{B}^1)) \subseteq S^*(\mathbb{B}^n)$, for $\alpha, \beta \geq 0$ with $\alpha \leq 1$, $\beta \leq 1/2$ and $\alpha + \beta \leq 1$ ([19]).
(ii) If $f \in S$, then $\Phi_{n,\alpha,\beta}(f)$ can be embedded as the first element of a Loewner chain on $\mathbb{B}^n$, for $\alpha, \beta \geq 0$ with $\alpha \leq 1$, $\beta \leq 1/2$ and $\alpha + \beta \leq 1$ ([19]).
(iii) $\Phi_{n,0,\beta}$ maps the family of normalized univalent Bloch functions on $\mathbb{U}$ with Bloch semi-norm 1 into the family of normalized univalent Bloch mappings on $\mathbb{B}^n$, for all $\beta \in [0, 1/2]$ ([22]).

The Muir extension operator $\Phi_{n,Q}$, which is another modification of the Roper-Suffridge extension operator, is given by ([40])

$$\Phi_{n,Q}(f)(z) = \left(f(z_1) + Q(\tilde{z})f'(z_1), \sqrt{f'(z_1)}\tilde{z}\right), \quad z = (z_1, \tilde{z}) \in \mathbb{B}^n,$$

where $f \in \mathcal{L}S(\mathbb{U})$ and $Q : \mathbb{C}^{n-1} \to \mathbb{C}$ is a homogeneous polynomial mapping of degree 2. The branch of the power function is chosen such that $\sqrt{f'(z_1)}\big|_{z_1=0} = 1$.

One of the properties of the Muir extension operator is as follows:

(i) $\Phi_{n,Q}(S^*(\mathbb{B}^1)) \subseteq S^*(\mathbb{B}^n)$ if and only if $\|Q\| \leq 1/4$ ([40]).

Muir [41] also studied the extension operator $\Phi_G : S \to S(\mathbb{B}^n)$ given by

$$\Phi_G(f)(z) = \left(f(z_1) + G\big(\sqrt{f'(z_1)}\tilde{z}\big), \sqrt{f'(z_1)}\tilde{z}\right), \quad z = (z_1, \tilde{z}) \in \mathbb{B}^n,$$

where $G : \mathbb{C}^{n-1} \to \mathbb{C}$ is a holomorphic function such that $G(0) = 0$ and $DG(0) = 0$, and the branch of the power function is chosen such that

$$\sqrt{f'(z_1)}\big|_{z_1=0} = 1.$$

Note that $DG(0)$ is the Frechét derivative of G at 0. One of the properties of the extension operator Φ_G is as follows:

(i) If $\alpha \in [0, 1)$ and $\Phi_G(S^*(\alpha)) \subseteq S^*(\mathbb{B}^n)$, where $S^*(\alpha)$ denotes the family of all normalized starlike functions of order α on $\mathbb{U}$, then G is a homogeneous polynomial of degree 2 from $\mathbb{C}^{n-1}$ into $\mathbb{C}$ and $\|G\| \leq 1/4$ ([41]).

Further study of the above operator has been given in [41], [50] (cf. [11]).

On the other hand, g-Loewner chains have been extensively studied in [13], [15], [17], [31]. Chirilă ([3], [4]) studied the preservation of the first elements of g-Loewner chains by the extension operators $\Phi_{n,\alpha,\beta}$ and $\Phi_{n,Q}$ on $\mathbb{B}^n$, in the case that $g(\zeta) = \frac{1-\zeta}{1+(1-2\gamma)\zeta}$ for $\zeta \in \mathbb{U}$ and $\gamma \in (0,1)$.

Let $\Phi_{n,r} : \mathcal{L}S(\mathbb{B}^n) \to \mathcal{L}S(\mathbb{B}^{n+r})$ be the Pfaltzgraff-Suffridge type extension operator, where $\mathcal{L}S(\mathbb{B}^n)$ denotes the family of normalized locally univalent mappings from $\mathbb{B}^n$ to $\mathbb{C}^n$, given by (see [23] and [43], in the case $r = 1$)

$$\Phi_{n,r}(f)(z) = \left(f(x), [J_f(x)]^{\frac{1}{n+1}} y\right), \quad z = (x, y) \in \mathbb{B}^{n+r}, \tag{1.1}$$

where $J_f(x)$ is the Jacobian determinant of f at x, and $r \geq 1$ is an integer. The branch of the power function is chosen such that $[J_f(x)]^{1/(n+1)}\big|_{x=0} = 1$. We note that the operator $\Phi_{1,r}$ reduces to the Roper-Suffridge extension operator. The Pfaltzgraff-Suffridge type extension operator $\Phi_{n,r}$ has the following properties (see [23] in the case $r = 1$):

(i) $\Phi_{n,r}(S^*(\mathbb{B}^n)) \subseteq S^*(\mathbb{B}^{n+r})$.
(ii) If $f \in S(\mathbb{B}^n)$ can be embedded as the first element of a Loewner chain on $\mathbb{B}^n$, then $F = \Phi_{n,r}(f)$ can be embedded as the first element of a Loewner chain on $\mathbb{B}^{n+r}$.

Let Y be a complex Banach space and let $r \geq 1$. Recently, the authors [18] studied the Roper-Suffridge type extension operator $\Phi_{\alpha,\beta}$ that provides a way of extending a locally univalent function f on $\mathbb{U}$ to a locally biholomorphic mapping $F \in H(\Omega_r)$, where $\Omega_r = \{(z_1, w) \in \mathbb{C} \times Y : |z_1|^2 + \|w\|_Y^r < 1\}$ and proved the preservation result of the first element of a g-Loewner chain and the Bloch mappings by the Roper-Suffridge type extension operator $\Phi_{\alpha,\beta}$. They also studied the Muir type extension operator Φ_{P_k} that provides a way of extending a locally univalent function f on $\mathbb{U}$ to a locally biholomorphic mapping $F \in H(\Omega_k)$, where $k \geq 2$ is an integer and $P_k : Y \to \mathbb{C}$ is a homogeneous polynomial mapping of degree k, and proved the preservation result of the first element of a Loewner chain and the Bloch mappings by the Muir type extension operator Φ_{P_k}.

In [16], Graham, Hamada and Kohr have considered a generalization of the Pfaltzgraff-Suffridge extension operator on bounded symmetric domains in $\mathbb{C}^n$, and proved that if $\mathbb{B}_X$ is a bounded symmetric domain in $X = \mathbb{C}^n$, and $\mathfrak{F}_{n,\alpha}$ is an extension operator which maps normalized locally biholomorphic mappings on $\mathbb{B}_X$ to locally biholomorphic mappings on $\mathbb{D}_\alpha$, where $\mathbb{D}_\alpha \subseteq \mathbb{B}_X \times \mathbb{B}_Y$ is a certain domain with $\mathbb{B}_X \times \{0\} \subset \mathbb{D}_\alpha$, then $\mathfrak{F}_{n,\alpha}$ extends the first elements of Loewner chains from $\mathbb{B}_X$ to the first elements of Loewner chains on $\mathbb{D}_\alpha$, when $\alpha \geq n/(2c(\mathbb{B}_X))$, where $c(\mathbb{B}_X)$ is a constant defined by the Bergman metric on X (see (5.1)). Also, they proved that normalized locally univalent I-Bloch mappings, which have finite trace order on $\mathbb{B}_X$, are mapped into R-Bloch mappings on Ω_α by the operator $\mathfrak{F}_{n,\alpha}$ when $\alpha \geq 1/2$, where $\Omega_\alpha \subset X \times Y$ is a bounded balanced convex domain such that $\mathbb{B}_X \times \{0\} \subset \Omega_\alpha \subseteq \mathbb{D}_\alpha$.

In this paper, we survey the above results obtained in [16] and [18].

2. Preliminaries

Let X and Y be complex Banach spaces. Let $L(X,Y)$ denote the family of continuous linear operators from X to Y. The family $L(X,X)$ is denoted by $L(X)$, and the identity in $L(X)$ is denoted by I_X. Let $\Omega \subset X$ be a domain which contains the origin and let $H(\Omega)$ be the family of holomorphic mappings from Ω into X. If a mapping $f \in H(\Omega)$ satisfies $f(0)=0$, $Df(0)=I_X$, we say that f is normalized, where $Df(z)$ is the Fréchet derivative of f at z. Let $\mathcal{L}S(\Omega)$ denote the family of normalized locally biholomorphic mappings on Ω and let $S(\Omega)$ denote the family of normalized biholomorphic mappings on Ω. Also, let $S^*(\Omega)$ (respectively, $K(\Omega)$) be the subset of $S(\Omega)$ consisting of starlike (respectively, convex) mappings on Ω, where a mapping $f \in S(\Omega)$ is said to be starlike (respectively, convex) if $f(\Omega)$ is a starlike (respectively, convex) domain in X. The family $S(\mathbb{U})$ is denoted by S, where $\mathbb{U}$ is the unit disc in $\mathbb{C}$. The family $S^*(\mathbb{U})$ (respectively, $K(\mathbb{U})$) is denoted by S^* (respectively, K).

Definition 2.1 (cf. [29]). *Let X be a complex Banach space and let $\Omega \subseteq X$ be a bounded balanced domain. Also, let $\gamma \in (-\frac{\pi}{2}, \frac{\pi}{2})$ and let $f \in H(\Omega)$. We say that f is spirallike of type γ on Ω if $f \in S(\Omega)$ and $\exp(-e^{-i\gamma}t)f(\Omega) \subseteq f(\Omega)$, for all $t \geq 0$.*

In the case $\gamma = 0$, a spirallike mapping f of type 0 is starlike in the usual sense.

Let $\widehat{S}_\gamma(\Omega)$ denote the family of spirallike mappings of type γ on Ω.

Assumption 2.1. Let $g : \mathbb{U} \to \mathbb{C}$ be a univalent holomorphic function such that $g(0) = 1$ and $\Re g(\zeta) > 0$ on $\mathbb{U}$.

Next we recall the notions of subordination and Loewner chain on a complex Banach space X (see e.g. [16], [18], [21] and [45]).

Definition 2.2. *Let X be a complex Banach space and let $\Omega \subseteq X$ be a domain which contains the origin.*

(i) *If $f, g \in H(\Omega)$, we say that f is subordinate to g (denoted by $f \prec g$) if there exists a Schwarz mapping v (i.e. $v \in H(\Omega)$, $v(0) = 0$ and $v(\Omega) \subseteq \Omega$) such that $f = g \circ v$.*

(ii) *A mapping $f : \Omega \times [0,\infty) \to X$ is called a univalent subordination chain if $f(\cdot,t)$ is univalent on Ω, $f(0,t) = 0$ for $t \geq 0$, and $f(\cdot,s) \prec f(\cdot,t)$, $0 \leq s \leq t < \infty$. A univalent subordination chain $f : \Omega \times [0,\infty) \to X$ is called a Loewner chain if $f(\cdot,t)$ is biholomorphic on Ω and $Df(0,t) = e^t I_X$, for all $t \geq 0$.*

Remark 2.3. Note that if $f : \Omega \times [0,\infty) \to X$ is a Loewner chain, then the subordination condition is equivalent to the existence of a unique biholomorphic Schwarz mapping $v = v(\cdot,s,t)$, called the transition mapping associated with $f(x,t)$, such that $f(x,s) = f(v(x,s,t),t)$ for $x \in \Omega$ and $t \geq s \geq 0$. Also, $Dv(0,s,t) = e^{s-t}I_X$ for $t \geq s \geq 0$ (see e.g. [21]).

For various applications of the Loewner theory in the study of univalent mappings in higher dimensions, see e.g. [21, Chapter 8].

For $x \in X \setminus \{0\}$, we define

$$T(x) = \big\{ l_x \in L(X,\mathbb{C}) : \; l_x(x) = \|x\|_X, \; \|l_x\| = 1 \big\}.$$

Then $T(x) \neq \emptyset$ in view of the Hahn-Banach theorem.

Let $\mathbb{B}_X$ be the unit ball of a complex Banach space X. Next, we recall the definition of the Carathéodory family $\mathcal{M} = \mathcal{M}(\mathbb{B}_X)$ in $H(\mathbb{B}_X)$ (see [47]):

$$\mathcal{M}(\mathbb{B}_X) = \big\{h \in H(\mathbb{B}_X) : h(0) = 0, Dh(0) = I_X,$$

$$\Re l_x(h(x)) > 0, \forall x \in \mathbb{B}_X \setminus \{0\}, \forall l_x \in T(x)\big\}.$$

If $X = \mathbb{C}$, then $f \in \mathcal{M}(\mathbb{U})$ if and only if $f(x)/x \in \mathcal{P}$, where

$$\mathcal{P} = \big\{p \in H(\mathbb{U}) : p(0) = 1, \Re p(z_1) > 0, \forall z_1 \in \mathbb{U}\big\}$$

is the Carathéodory family on $\mathbb{U}$.

Definition 2.4 (cf. [1]**,** [9]**).** *Let X be a complex Banach space. A mapping $h = h(x,t) : \mathbb{B}_X \times [0,\infty) \to X$ is called a generating vector field (Herglotz vector field) if the following conditions hold:*

(i) *$h(\cdot,t) \in \mathcal{M}(\mathbb{B}_X)$, for a.e. $t \geq 0$;*
(ii) *$h(x,\cdot)$ is strongly measurable on $[0,\infty)$, for all $x \in \mathbb{B}_X$.*

Definition 2.5 (see e.g. [13] **and** [15]**).** *Let $g : \mathbb{U} \to \mathbb{C}$ satisfy Assumption* 2.1. *Also, let $h \in H(\mathbb{B}_X)$ be normalized. We say that h belongs to the family $\mathcal{M}_g = \mathcal{M}_g(\mathbb{B}_X)$ if*

$$\frac{1}{\|x\|_X} l_x(h(x)) \in g(\mathbb{U}), \quad \forall x \in \mathbb{B}_X \setminus \{0\}, \quad \forall l_x \in T(x).$$

Further, we define the notion of a g-Loewner chain in the case of complex Banach spaces (not necessarily reflexive), where $g : \mathbb{U} \to \mathbb{C}$ satisfies Assumption 2.1. In the case $X = \mathbb{C}^n$, see [13], [15].

Definition 2.6. *Let $g : \mathbb{U} \to \mathbb{C}$ satisfy Assumption* 2.1. *We say that a mapping $f = f(x,t) : \mathbb{B}_X \times [0,\infty) \to X$ is a g-Loewner chain if the following conditions hold:*

(i) *$f(x,t)$ is a Loewner chain such that $\{e^{-t}f(\cdot,t)\}_{t\geq 0}$ is uniformly bounded on each ball $\rho\mathbb{B}_X$ $(0 < \rho < 1)$;*
(ii) *there exists a null set $E \subset [0,\infty)$ such that $\frac{\partial f}{\partial t}(x,t)$ exists for $t \in [0,\infty) \setminus E$ and for all $x \in \mathbb{B}_X$, and there exists a generating vector field $h = h(x,t) : \mathbb{B}_X \times [0,\infty) \to X$ with $h(\cdot,t) \in \mathcal{M}_g(\mathbb{B}_X)$ for $t \in [0,\infty) \setminus E$, such that*

$$\frac{\partial f}{\partial t}(x,t) = Df(x,t)h(x,t), \quad t \in [0,\infty) \setminus E, \forall x \in \mathbb{B}_X. \tag{2.1}$$

Remark 2.7. In general, if X is a complex Banach space and if $f(x,t)$ satisfies condition (i) of Definition 2.6, it is not known whether $\frac{\partial f}{\partial t}(x,t)$ exists for $x \in \mathbb{B}_X$ and $t \in [0,\infty) \setminus E$, where $E \subset [0,\infty)$ is a null set. Also, if $\frac{\partial f}{\partial t}(x,t)$ exists for $x \in \mathbb{B}_X$ and $t \in [0,\infty) \setminus E$, it is not known whether there exists a generating vector field $h(x,t)$ such that the Loewner differential equation (2.1) holds. However, positive answers to these questions may be obtained in the case of separable reflexive complex Banach spaces. A discussion of Loewner chains and the associated Loewner differential equation in the case of separable reflexive complex Banach spaces may be found in [32]. In the finite dimensional case $X = \mathbb{C}^n$, see [44, Chapter 6] for $n = 1$; see [1], [9], and [13], in the case $n \geq 2$.

Definition 2.8 (see [26]). *Let $g : \mathbb{U} \to \mathbb{C}$ satisfy the conditions of Assumption 2.1. A mapping $f \in \mathcal{L}S(\mathbb{B}_X)$ is said to be g-starlike if $h \in \mathcal{M}_g(\mathbb{B}_X)$, where*

$$h(x) = [Df(x)]^{-1} f(x), \quad x \in \mathbb{B}_X.$$

Let $S_g^*(\mathbb{B}_X)$ denote the class of g-starlike mappings on $\mathbb{B}_X$.

Definition 2.9 (see e.g. [5], and [37]). *A complex Banach space X is called a JB*-triple if X is a complex Banach space equipped with a continuous Jordan triple product*

$$X \times X \times X \to X \qquad (x, y, z) \mapsto \{x, y, z\}$$

satisfying

(J_1) *$\{x, y, z\}$ is symmetric bilinear in the outer variables, but conjugate linear in the middle variable,*
(J_2) $\{a, b, \{x, y, z\}\} = \{\{a, b, x\}, y, z\} - \{x, \{b, a, y\}, z\} + \{x, y, \{a, b, z\}\}$,
(J_3) *$x\Box x \in L(X, X)$ is a hermitian operator with spectrum $\geqq 0$,*
(J_4) $\|\{x, x, x\}\| = \|x\|^3$

for $a, b, x, y, z \in X$, where the box operator $x\Box y : X \to X$ is defined by

$$x\Box y(\cdot) = \{x, y, \cdot\},$$

and $\|\cdot\|$ is the norm on X.

A complex Banach space X is a JB*-triple if and only if the open unit ball of X is homogeneous (see e.g. [5, Section 3.3]).

Next we recall the notion of R-Bloch mappings on the unit ball of a complex Banach space X and also that of I-Bloch mappings on the unit ball of a JB*-triple.

Definition 2.10. (i) (*cf.* [25]) *Let $\mathbb{B}_X$ be the unit ball of a complex Banach space X and let $f : \mathbb{B}_X \to Y$ be a holomorphic mapping. We say that f is an R-Bloch mapping on $\mathbb{B}_X$ if*

$$\sup_{x \in \mathbb{B}_X} (1 - \|x\|^2)\|Df(x)x\| < \infty. \tag{2.2}$$

(ii) (*cf.* [6], [7], [24]) *Let $\mathbb{B}_X$ be the unit ball of a JB*-triple X and let $f : \mathbb{B}_X \to Y$ be a holomorphic mapping. We say that f is an I-Bloch mapping on $\mathbb{B}_X$ if*

$$\sup_{g \in \mathrm{Aut}(\mathbb{B}_X)} \|D(f \circ g)(0)\| < \infty, \tag{2.3}$$

where $\mathrm{Aut}(\mathbb{B}_X)$ *denotes the family of biholomorphic automorphisms of $\mathbb{B}_X$.*

Remark 2.11. (i) When $\mathbb{B}_X$ is the unit ball of a JB*-triple X, I-Bloch mappings are R-Bloch mappings by [34, Corollary 3.6] (cf. [7, Corollary 3.5], [24]). Chu, Hamada, Honda and Kohr [8, Example 2.9] and Miralles [39, Proposition 2.5] independently gave an example such that the converse is not true for $\mathbb{B}_X = \mathbb{U}^2$.

(ii) When $\mathbb{B}_X$ is a Hilbert ball and $Y = \mathbb{C}$, then conditions (2.2) and (2.3) are equivalent to the following relation:

$$\sup_{x \in \mathbb{B}_X} (1 - \|x\|^2)\|Df(x)\| < \infty, \tag{2.4}$$

by [2, Proposition 2.4, Theorems 2.6 and 3.8] (cf, [25, Theorem 2.8]). Moreover, (2.2), (2.3) and (2.4) give equivalent semi-norms for a holomorphic function $f : \mathbb{B}_X \to \mathbb{C}$ which satisfies one of the relations (2.2), (2.3) and (2.4). Then for $f \in H(\mathbb{B}_X)$, by considering the function $f_a = \langle f, a\rangle$ with $\|a\| = 1$, we obtain that conditions (2.2), (2.3) and (2.4) are equivalent. Namely, the notions of R-Bloch mappings and I-Bloch mappings are equivalent to the usual notion of Bloch mappings on the Hilbert ball. In particular, $f \in H(\mathbb{U})$ is a Bloch function if and only if

$$\sup_{\zeta\in\mathbb{U}}(1 - |\zeta|^2)|f'(\zeta)| < \infty.$$

Next, we recall the notion of a linearly invariant family (L.I.F.) and the trace-order of a L.I.F. on the unit ball $\mathbb{B}_X$ of a finite-dimensional complex Banach space X ([28]; cf. [42], [21, Chapter 10]).

Definition 2.12. *Let X be a complex Banach space and let $\mathbb{B}_X$ be the open unit ball of X. A family $\mathcal{F} \subseteq H(\mathbb{B}_X)$ is called a linearly invariant family (L.I.F.) if the following conditions hold:*

(i) $\mathcal{F} \subseteq \mathcal{L}S(\mathbb{B}_X)$;
(ii) $\Lambda_\phi(f) \in \mathcal{F}$, *for all* $f \in \mathcal{F}$ *and* $\phi \in \mathrm{Aut}(\mathbb{B}_X)$,

where $\Lambda_\phi(f)$ is the Koebe transform given by

$$\Lambda_\phi(f)(x) = [D\phi(0)]^{-1}[Df(\phi(0))]^{-1}(f(\phi(x)) - f(\phi(0))), \quad \forall\, x \in \mathbb{B}_X.$$

Definition 2.13 ([28]; cf. [42]). *If $\mathcal{F}$ is a linearly invariant family on the unit ball of a finite dimensional complex Banach space X, we define the trace order of $\mathcal{F}$, by*

$$\mathrm{ord}\,\mathcal{F} = \sup_{f\in\mathcal{F}}\sup_{\|y\|=1}\left\{\frac{1}{2}\left|\mathrm{trace}\left[D^2 f(0)(y,\cdot)\right]\right|\right\}.$$

Since the trace is a similarity invariant, the above definition is well-defined. When $X = \mathbb{C}$ and $\mathbb{B}_X = \mathbb{U}$, the trace order is the usual order of a linearly invariant family on $\mathbb{U}$.

Let $\Lambda[\{f\}]$ be the linearly invariant family generated by $f \in \mathcal{L}S(\mathbb{B}_X)$ (see [28]; cf. [42]). In this case, $\mathrm{ord}\,\Lambda[\{f\}]$ is called the trace order of f.

3. Roper-Suffridge type extension operators

Let Y be a complex Banach space and let $r \geq 1$. Also, let

$$\Omega_r = \left\{(z_1, w) \in Z = \mathbb{C} \times Y : |z_1|^2 + \|w\|_Y^r < 1\right\}. \tag{3.1}$$

Then, the Minkowski function of Ω_r is a complete norm $\|\cdot\|_Z$ on Z and Ω_r is the unit ball of Z with respect to this norm. Let $\alpha, \beta \geq 0$ and let $\Phi_{\alpha,\beta} : S \to S(\Omega_r)$ be the Roper-Suffridge type extension operator given by

$$\Phi_{\alpha,\beta}(f)(z_1, w) = \left(f(z_1), \left(\frac{f(z_1)}{z_1}\right)^\alpha (f'(z_1))^\beta w\right), \quad (z_1, w) \in \Omega_r. \tag{3.2}$$

The branches of the power functions are chosen such that

$$\left(\frac{f(z_1)}{z_1}\right)^\alpha\Bigg|_{z_1=0} = 1 \quad \text{and} \quad (f'(z_1))^\beta|_{z_1=0} = 1.$$

3.1. g-Loewner chains and Roper-Suffridge type extension operators

Let $g : \mathbb{U} \to \mathbb{C}$ be a convex (univalent) function which satisfies Assumption 2.1. In the first part of this section, we are concerned with preservation of the first elements of g-Loewner chains from $\mathbb{U}$ into Ω_r under the Roper-Suffridge type extension operators $\Phi_{\alpha,\beta}$, where $r \geq 1$ (cf. [1, Theorem 7.1], [3, Theorem 2.1], [14, Corollary 2.9], [19, Theorem 2.1], [22, Theorem 2.1]).

Theorem 3.1. *Let $g : \mathbb{U} \to \mathbb{C}$ be a convex (univalent) function which satisfies Assumption* 2.1. *Let Y be a complex Banach space and let Ω_r be the unit ball of $Z = \mathbb{C} \times Y$ given by* (3.1)*, where $r \geq 1$. Let $\Phi_{\alpha,\beta}$ be the Roper-Suffridge type extension operator given by* (3.2)*. Assume that $f \in S$ can be embedded as the first element of a g-Loewner chain on $\mathbb{U}$. Then $F = \Phi_{\alpha,\beta}(f) \in S(\Omega_r)$ can be embedded as the first element of a g-Loewner chain on Ω_r for $\alpha \in [0,1]$, $\beta \in [0,1/r]$, $\alpha + \beta \leq 1$.*

As a corollary of Theorem 3.1, we obtain the following preservation of the first elements of Loewner chains from $\mathbb{U}$ into the unit ball Ω_r under the Roper-Suffridge type extension operators $\Phi_{\alpha,\beta}$ (cf. [14, Corollary 2.9], [19, Theorem 2.1], [22, Theorem 2.1], [36]).

Corollary 3.2. *Let Ω_r and $\Phi_{\alpha,\beta}$ be as in Theorem* 3.1*. If $f \in S$, then $F = \Phi_{\alpha,\beta}(f) \in S(\Omega_r)$ can be embedded as the first element of a Loewner chain on Ω_r for $\alpha \in [0,1]$, $\beta \in [0,1/r]$, $\alpha + \beta \leq 1$.*

As another consequence of Theorem 3.1, we obtain that the Roper-Suffridge type extension operators $\Phi_{\alpha,\beta}$ preserve g-starlike mappings. This result is a generalization of [20, Theorem 2.2], in the case $Y = \mathbb{C}^{n-1}$, $r = 2$ and $g(\zeta) = \frac{1-\zeta}{1+\zeta}$, $\zeta \in \mathbb{U}$ (cf. [3, Corollary 2.2], [4, Corollary 2.3]).

Corollary 3.3. *Let Ω_r, $\Phi_{\alpha,\beta}$ and g be as in Theorem* 3.1*. If f is a g-starlike mapping on $\mathbb{U}$, then $F = \Phi_{\alpha,\beta}(f) \in S(\Omega_r)$ is also a g-starlike mapping on Ω_r for $\alpha \in [0,1]$, $\beta \in [0,1/r]$, $\alpha + \beta \leq 1$.*

As particular cases of Corollary 3.3, we obtain that strongly starlike mappings of order $d \in (0,1]$ and almost starlike mappings of order $d \in [0,1)$ (see e.g. [21]) are preserved by the Roper-Suffridge type extension operators $\Phi_{\alpha,\beta}$ for $\alpha \in [0,1]$, $\beta \in [0,1/r]$, $\alpha + \beta \leq 1$.

In the case $\beta = 0$, [26, Theorem 5.1] can be generalized as follows.

Theorem 3.4. *Let Ω_r and $\Phi_{\alpha,\beta}$ be as in Theorem* 3.1*. Let g be a univalent holomorphic function on $\mathbb{U}$ which satisfies Assumption* 2.1 *such that $g(\mathbb{U})$ is a starlike domain with respect to* 1*. Assume that $f \in S$ can be embedded as the first element of a g-Loewner chain on $\mathbb{U}$. Then $F = \Phi_{\alpha,0}(f) \in S(\Omega_r)$ can be embedded as the first element of a g-Loewner chain on Ω_r for $\alpha \in [0,1]$.*

As a corollary of Theorem 3.4, we obtain the following generalization of [27, Theorem 5.3] to certain complex Banach spaces.

Corollary 3.5. *Let Ω_r and $\Phi_{\alpha,\beta}$ be as in Theorem* 3.1*. If f is a parabolic starlike mapping of order $d \in [0,1)$ on $\mathbb{U}$, then $F = \Phi_{\alpha,0}(f) \in S(\Omega_r)$ is also a parabolic starlike mapping of order d on Ω_r for $\alpha \in [0,1]$.*

3.2. Bloch mappings and Roper-Suffridge type extension operators

In the second part of this section, we show that normalized univalent Bloch functions on $\mathbb{U}$ (respectively normalized uniformly locally univalent Bloch functions on $\mathbb{U}$) are extended to R-Bloch mappings on Ω_r by the Roper-Suffridge type extension operators $\Phi_{\alpha,\beta}$, for $\alpha > 0$ and $\beta \in [0, 1/r)$ (respectively for $\alpha = 0$ and $\beta \in [0, 1/r]$).

The following theorem is a generalization of [20, Theorem 2.6] and [22, Theorem 4.1] to certain complex Banach spaces (cf. [10, Proposition 6.1]).

Theorem 3.6. *Let Ω_r and $\Phi_{\alpha,\beta}$ be as in Theorem 3.1. If $f \in S$ is a Bloch function on $\mathbb{U}$, then $F = \Phi_{\alpha,\beta}(f) \in S(\Omega_r)$ is an R-Bloch mapping on Ω_r for $\alpha > 0$ and $\beta \in [0, 1/r)$.*

In the case $\alpha = 0$ and $\beta \in [0, 1/r]$, we obtain that uniformly locally univalent Bloch functions on $\mathbb{U}$ are extended to R-Bloch mappings on Ω_r by the extension operator $\Phi_{0,\beta}$. This result is a generalization of [20, Theorem 2.6] and [22, Theorem 4.1] to certain complex Banach spaces and also is an improvement of Theorem 3.6.

Theorem 3.7. *Let Ω_r and $\Phi_{\alpha,\beta}$ be as in Theorem 3.1. If $f \in \mathcal{L}S(\mathbb{U})$ is a uniformly locally univalent Bloch function on $\mathbb{U}$, then $F = \Phi_{0,\beta}(f) \in \mathcal{L}S(\Omega_r)$ is an R-Bloch mapping on Ω_r for $\beta \in [0, 1/r]$.*

4. Muir type extension operators

Let $k \geq 2$ be an integer and let Y be a complex Banach space and let Ω_k be the unit ball of $Z = \mathbb{C} \times Y$ given by (3.1). Let $P_k : Y \to \mathbb{C}$ be a homogeneous polynomial mapping of degree k. The Muir type extension operator Φ_{P_k} is defined by (cf. [40])

$$\Phi_{P_k}(f)(z) = \Big(f(z_1) + P_k(w)f'(z_1), (f'(z_1))^{\frac{1}{k}} w\Big), \quad z = (z_1, w) \in \Omega_k, \tag{4.1}$$

where f is a locally univalent function on $\mathbb{U}$, normalized by $f(0) = f'(0) - 1 = 0$. The branch of the power function is chosen such that $(f'(z_1))^{\frac{1}{k}}|_{z_1=0} = 1$.

4.1. g-Loewner chains and Muir type extension operators

We begin this section with the following preservation result of the first elements of g-Loewner chains by the Muir type extension operators Φ_{P_k}, where g is a convex function on $\mathbb{U}$ which satisfies Assumption 2.1. In the case $Y = \mathbb{C}^{n-1}$, $k = 2$ and $g(\zeta) = \frac{1-\zeta}{1+(1-2\gamma)\zeta}$, $\zeta \in \mathbb{U}$, where $\gamma \in (0, 1)$, see [4, Theorem 3.1] (cf. [33, Theorem 5.6], [35, Theorem 2.1 and Corollary 2.2]).

Theorem 4.1. *Let $k \geq 2$ be an integer. Let Y be a complex Banach space and let Ω_k be the unit ball of $Z = \mathbb{C} \times Y$ given by (3.1). Let $P_k : Y \to \mathbb{C}$ be a homogeneous polynomial mapping of degree k and let Φ_{P_k} be the Muir type extension operator given by (4.1). Let g be a convex function on $\mathbb{U}$ which satisfies Assumption 2.1. Assume that $f \in S$ can be embedded as the first element of a g-Loewner chain on $\mathbb{U}$ and that $\|P_k\| \leq d(1, \partial g(\mathbb{U}))/4$, where*

$$d(1, \partial g(\mathbb{U})) = \inf_{\zeta \in \partial g(\mathbb{U})} |\zeta - 1|.$$

Then $F = \Phi_{P_k}(f) \in S(\Omega_k)$ can be embedded as the first element of a g-Loewner chain on Ω_k.

As a corollary of Theorem 4.1, we obtain the following result. This result is a generalization of [35, Theorem 2.1 and Corollary 2.2], in the case $Y = \mathbb{C}^{n-1}$, $k = 2$ and $g(\zeta) = \frac{1-\zeta}{1+\zeta}$, $\zeta \in \mathbb{U}$ to certain complex Banach spaces (cf. [33, Theorem 5.6]).

Corollary 4.2. *Let Ω_k and Φ_{P_k} be as in Theorem 4.1, where $\|P_k\| \le 1/4$. If $f \in S$, then $F = \Phi_{P_k}(f)$ can be embedded as the first element of a Loewner chain on Ω_k.*

In view of Theorem 4.1, it would be interesting to give an answer to the following questions:

Question 4.3. *Under the assumptions of Theorem 4.1, is the coefficient bound $\|P_k\| \le d(1, \partial g(\mathbb{U}))/4$ also necessary for the preservation of the first elements of g-Loewner chains under the Muir type extension operator Φ_{P_k}?*

Question 4.4. *Under the assumptions of Theorem 4.1, is the coefficient bound $\|P_k\| \le d(1, \partial g(\mathbb{U}))/4$ sharp for the preservation of the first elements of g-Loewner chains under the Muir type extension operator Φ_{P_k}?*

In the case that $f \in K$ can be embedded as the first element of a g-Loewner chain $f(z_1, t)$ on $\mathbb{U}$ such that $f(\cdot, t)$ is convex on $\mathbb{U}$ for $t \ge 0$, then Theorem 4.1 may be refined as follows (cf. [4], [35], [40]).

Proposition 4.5. *Let Ω_k and Φ_{P_k} be as in Theorem 4.1. Let g be a convex function on $\mathbb{U}$ which satisfies Assumption 2.1. Assume that $f \in K$ can be embedded as the first element of a g-Loewner chain $f(z_1, t)$ on $\mathbb{U}$, such that $e^{-t} f(\cdot, t) \in K$, for all $t \ge 0$. If $\|P_k\| \le d(1, \partial g(\mathbb{U}))/2$, then $F = \Phi_{P_k}(f) \in S(\Omega_k)$ can be embedded as the first element of a g-Loewner chain on Ω_k.*

Let g be a linear fractional transformation with real coefficients, which satisfies Assumption 2.1. Then the image $g(\mathbb{U})$ is one of the following sets:

$$g(\mathbb{U}) = \left\{\zeta \in \mathbb{C} : \left|\zeta - \frac{1}{2\gamma}\right| < \frac{\delta}{2\gamma}\right\}, \ \gamma > 0,\ \delta \in (0, 1],\ |2\gamma - 1| < \delta,$$
$$g(\mathbb{U}) = \{\zeta \in \mathbb{C} : \Re\zeta > \delta\},\ \delta \in [0, 1).$$

As a corollary of Theorem 4.1, we obtain the following results.

Corollary 4.6. *Let Ω_k and Φ_{P_k} be as in Theorem 4.1. Let g be a linear fractional transformation with real coefficients which satisfies Assumption 2.1. Assume that $f \in S$ can be embedded as the first element of a g-Loewner chain on $\mathbb{U}$.*

(i) If $g(\mathbb{U}) = \left\{\zeta \in \mathbb{C} : \left|\zeta - \frac{1}{2\gamma}\right| < \frac{\delta}{2\gamma}\right\}$, where $\gamma > 0$, $\delta \in (0, 1]$, and $|2\gamma - 1| < \delta$, and if $\|P_k\| \le (\delta - |2\gamma - 1|)/(8\gamma)$, then $F = \Phi_{P_k}(f) \in S(\Omega_k)$ can be embedded as the first element of a g-Loewner chain on Ω_k.

(ii) If $g(\mathbb{U}) = \{\zeta \in \mathbb{C} : \Re\zeta > \delta\}$, where $\delta \in [0, 1)$, and if $\|P_k\| \le (1 - \delta)/4$, then $F = \Phi_{P_k}(f) \in S(\Omega_k)$ can be embedded as the first element of a g-Loewner chain on Ω_k.

As in Corollary 3.3, we obtain the following result (cf. [4, Corollary 3.3], [35, Corollary 2.3] [40, Theorem 4.1]).

Corollary 4.7. *Let Ω_k, Φ_{P_k} and g be as in Theorem 4.1. If f is a g-starlike mapping on $\mathbb{U}$ and if $\|P_k\| \le d(1, \partial g(\mathbb{U}))/4$, then $F = \Phi_{P_k}(f) \in S(\Omega_k)$ is also a g-starlike mapping on Ω_k.*

In particular, we have the following corollary.

Corollary 4.8. *Let Ω_k and Φ_{P_k} be as in Theorem 4.1.*

(i) If $f:\mathbb{U}\to\mathbb{C}$ is a strongly starlike mapping of order $d\in(0,1]$ on $\mathbb{U}$ and if $\|P_k\|\leq\sin\left(\frac{\pi}{2}d\right)/4$, then $F=\Phi_{P_k}(f)\in S(\Omega_k)$ is also a strongly starlike mapping of order d on Ω_k.

(ii) If $f:\mathbb{U}\to\mathbb{C}$ is an almost starlike mapping of order $d\in[0,1)$ on $\mathbb{U}$ and if $\|P_k\|\leq(1-d)/4$, then $F=\Phi_{P_k}(f)\in S(\Omega_k)$ is also an almost starlike mapping of order d on Ω_k.

Taking into account Corollary 4.7, it would be interesting to give an answer to the following question.

Question 4.9. *Under the same assumptions of Corollary 4.7, is the condition $\|P_k\|\leq d(1,\partial g(\mathbb{U}))/4$ necessary for the preservation of g-starlikeness under the Muir type extension operator Φ_{P_k}?*

Note that if $g(\zeta)=\frac{1-\zeta}{1+\zeta}$, $\zeta\in\mathbb{U}$, $k=2$ and $Y=\mathbb{C}^{n-1}$, the answer is positive, in view of [40, Theorem 4.1].

Next, let $G:Y\to\mathbb{C}$ be a holomorphic function such that $G(0)=0$ and $DG(0)=0$. Also, let $\Phi_{G,k}:\mathcal{LS}(\mathbb{U})\to\mathcal{LS}(\Omega_k)$ be the following modification of the Muir extension operator (cf. [41])

$$\Phi_{G,k}(f)(z)=\Big(f(z_1)+G\big((f'(z_1))^{\frac{1}{k}}w\big),(f'(z_1))^{\frac{1}{k}}w\Big),\quad z=(z_1,w)\in\Omega_k, \tag{4.2}$$

where Ω_k is the unit ball of $Z=\mathbb{C}\times Y$ given by (3.1). The branch of the power function is chosen such that $(f'(z_1))^{\frac{1}{k}}\big|_{z_1=0}=1$.

It is natural to ask the following question, in connection with Corollary 4.7 (cf. [41], [50]):

Question 4.10. *Let $k\geq 2$ be an integer and let Ω_k be the unit ball of $Z=\mathbb{C}\times Y$ given by (3.1). Assume that $g:\mathbb{U}\to\mathbb{C}$ is a univalent function, which satisfies Assumption 2.1. Let $G:Y\to\mathbb{C}$ be a holomorphic function such that $G(0)=0$ and $DG(0)=0$. If $\Phi_{G,k}(S_g^*(\mathbb{U}))\subseteq S^*(\Omega_k)$, what conditions for G must be satisfied ?*

The following result provides an answer to the above question (cf. [41, Theorem 5.1], [50, Theorem 3.1]).

Theorem 4.11. *Let Ω_k be as in Theorem 4.1. Let g be a univalent function with real coefficients on $\mathbb{U}$, which satisfies Assumption 2.1. Assume that there exists the limit*

$$a:=\liminf_{r\to1^-}\frac{g(r)}{1-r}<+\infty. \tag{4.3}$$

Let $G:Y\to\mathbb{C}$ be a holomorphic function such that $G(0)=0$ and $DG(0)=0$ and $\Phi_{G,k}$ be the extension operator given in (4.2). Let f be a g-starlike function on $\mathbb{U}$ such that $\frac{f(\zeta)}{\zeta f'(\zeta)}=g(\zeta)$ for $\zeta\in\mathbb{U}$. If $\Phi_{G,k}(f)$ is a starlike mapping on Ω_k, then G is a polynomial of degree at most k.

As a corollary of Theorem 4.11, we obtain the following result (cf. [41, Corollary 5.2], [50, Corollary 3.2]).

Corollary 4.12. *Let Ω_k, $\Phi_{G,k}$ and g be as in Theorem 4.11. If $\Phi_{G,k}(S_g^*(\mathbb{U}))\subseteq S^*(\Omega_k)$, then G is a polynomial of degree at most k.*

4.2. Bloch mappings and Muir type extension operators

The next result shows that normalized uniformly locally univalent Bloch functions on $\mathbb{U}$ are extended to normalized locally univalent R-Bloch mappings on Ω_k by the Muir type extension operators Φ_{P_k} given by (4.1).

Theorem 4.13. *Let Ω_k and Φ_{P_k} be as in Theorem 4.1. If $f \in \mathcal{LS}(\mathbb{U})$ is a uniformly locally univalent Bloch function on $\mathbb{U}$, then $F = \Phi_{P_k}(f) \in \mathcal{LS}(\Omega_k)$ is an R-Bloch mapping on Ω_k.*

As a corollary of Theorem 4.13, we obtain the following result.

Corollary 4.14. *Let Ω_k and Φ_{P_k} be as in Theorem 4.1. If $f \in S$ is a Bloch function on $\mathbb{U}$, then $F = \Phi_{P_k}(f) \in S(\Omega_k)$ is an R-Bloch mapping on Ω_k.*

5. Pfaltzgraff-Suffridge type extension operators

In this section, let X be an n-dimensional JB*-triple. Also, let $\mathbb{B}_X$ be the open unit ball of X with respect to the norm $\|\cdot\|_X$ and for every $x, y \in X$, let $B(x,y) \in L(X)$ be the Bergman operator defined by

$$B(x,y)(z) = z - 2(x \square y)(z) + \{x, \{y,z,y\}, x\}, \quad z \in X.$$

If $f \in H(\mathbb{B}_X)$, let $J_f(x) = \det Df(x)$, $x \in \mathbb{B}_X$. Also, let h_0 be the Bergman metric on X at 0, and let $c(\mathbb{B}_X)$ be the constant given by (see [28])

$$c(\mathbb{B}_X) = \frac{1}{2} \sup_{x,y \in \mathbb{B}_X} |h_0(x,y)|. \tag{5.1}$$

In view of [24, Lemma 2.4] (cf. [30, Lemma 2.2]), the following distortion result holds:

$$\det B(x,x) \geq (1 - \|x\|_X^2)^{2c(\mathbb{B}_X)}, \quad x \in \mathbb{B}_X. \tag{5.2}$$

Equality holds for every $x \in X$ such that $x/\|x\|_X$ is a maximal tripotent in X.

Next, let Y be a complex Banach space with the norm $\|\cdot\|_Y$, and let $\mathbb{B}_Y$ be the unit ball of Y. For $\alpha > 0$, let

$$\mathbb{D}_\alpha = \Big\{(x,y) \in \mathbb{B}_X \times Y : \|y\|_Y < [\det B(x,x)]^{1/(4\alpha c(\mathbb{B}_X))}\Big\} \tag{5.3}$$

and

$$\Omega_\alpha = \Big\{(x,y) \in X \times Y : \|x\|_X^2 + \|y\|_Y^{2\alpha} < 1\Big\}. \tag{5.4}$$

Also, for $\alpha > 0$, let $\mathfrak{F}_{n,\alpha} : \mathcal{LS}(\mathbb{B}_X) \to \mathcal{LS}(\mathbb{D}_\alpha)$ be the Pfaltzgraff-Suffridge type extension operator given by

$$\mathfrak{F}_{n,\alpha}(f)(z) = \Big(f(x), [J_f(x)]^{1/(2\alpha c(\mathbb{B}_X))} y\Big), \quad z = (x,y) \in \mathbb{D}_\alpha. \tag{5.5}$$

The branch of the power function is chosen such that $[J_f(x)]^{1/(2\alpha c(\mathbb{B}_X))}|_{x=0} = 1$. Note that this branch is well defined on $\mathbb{B}_X$, since $\mathbb{B}_X$ is a starlike domain with respect to the origin in $X = \mathbb{C}^n$. It is not difficult to deduce that if $f \in \mathcal{LS}(\mathbb{B}_X)$ and $F = \mathfrak{F}_{n,\alpha}(f)$, then $F \in H(\mathbb{D}_\alpha)$ and the Frechét derivative $DF(z)$ has a bounded inverse at each point $z \in \mathbb{D}_\alpha$, i.e. F is locally biholomorphic on $\mathbb{D}_\alpha$. Hence the Pfaltzgraff-Suffridge type extension operator $\mathfrak{F}_{n,\alpha}$ is well defined and extends normalized locally biholomorphic mappings on $\mathbb{B}_X$ into normalized locally biholomorphic mappings on the domain $\mathbb{D}_\alpha$.

Example 5.1. (i) If X is the space $\mathbb{C}^n$ with the Euclidean norm $\|\cdot\|_e$, then $\mathbb{B}_X = \mathbb{B}^n$, $\det B(x,x) = (1-\|x\|_e^2)^{n+1}$, and $c(\mathbb{B}^n) = \frac{n+1}{2}$ (see e.g. [28]). Therefore, we have $\mathbb{D}_\alpha = \Omega_\alpha$ for $\alpha > 0$, that is

$$\mathbb{D}_\alpha = \Big\{(x,y) \in \mathbb{C}^n \times Y : \|x\|_e^2 + \|y\|_Y^{2\alpha} < 1\Big\}.$$

In this case, the operator $\mathfrak{F}_{n,\alpha}$ will be denoted by $\Gamma_{n,\alpha}$. Thus, we obtain the extension operator $\Gamma_{n,\alpha} : \mathcal{L}S(\mathbb{B}^n) \to \mathcal{L}S(\Omega_\alpha)$ given by (see [14, Definition 2.7]):

$$\Gamma_{n,\alpha}(f)(z) = \Big(f(x), [J_f(x)]^{1/(\alpha(n+1))}y\Big),\ \forall\, f \in \mathcal{L}S(\mathbb{B}^n),\ z = (x,y) \in \Omega_\alpha. \tag{5.6}$$

If $\alpha = 1$, $\mathbb{B}_X = \mathbb{B}^n$ and $\mathbb{B}_Y = \mathbb{B}^r$, then $\Omega_1 = \mathbb{B}^{n+r}$ and the operator $\Gamma_{n,1}$ reduces to the Pfaltzgraff-Suffridge type extension operator $\Phi_{n,r}$. On the other hand, if $n = 1$ and $\alpha = 1$, then the operator $\Gamma_{1,1}$ reduces to the Roper-Suffridge extension operator $\Psi : \mathcal{L}S(\mathbb{B}^1) \to \mathcal{L}S(\mathbb{B})$ given by (cf. [46]; see also [14])

$$\Psi(f)(z) = \Big(f(x), \sqrt{f'(x)}y\Big),\ z = (x,y) \in \mathbb{B},$$

where $\mathbb{B} = \big\{(x,y) \in \mathbb{C} \times Y : |x|^2 + \|y\|_Y^2 < 1\big\}$.

(ii) If $X = \mathbb{C}^n$ with respect to the maximum norm $\|\cdot\|_\infty$, then $c(\mathbb{U}^n) = n$ (see [28]), and $\det B(x,x) = \prod_{j=1}^n (1-|x_j|^2)^2$, $x = (x_1,\ldots,x_n) \in \mathbb{U}^n$. Denoting the domain $\mathbb{D}_\alpha$ by Δ_α for $\alpha > 0$, we obtain that

$$\Delta_\alpha = \Big\{(x,y) \in \mathbb{U}^n \times \mathbb{B}_Y : \|y\|_Y < \prod_{j=1}^n (1-|x_j|^2)^{1/(2n\alpha)}\Big\}. \tag{5.7}$$

In this case, we denote the operator $\mathfrak{F}_{n,\alpha}$ by $\Theta_{n,\alpha}$. Thus, we obtain the extension operator $\Theta_{n,\alpha} : \mathcal{L}S(\mathbb{U}^n) \to \mathcal{L}S(\Delta_\alpha)$ given by (cf. [14])

$$\Theta_{n,\alpha}(f)(z) = \Big(f(x), [J_f(x)]^{1/(2n\alpha)}y\Big), \quad z = (x,y) \in \Delta_\alpha. \tag{5.8}$$

5.1. Loewner chains and Pfaltzgraff-Suffridge type extension operators

We begin this section with the preservation of Loewner chains from the open unit ball $\mathbb{B}_X$ of an n-dimensional JB*-triple X into the domain $\mathbb{D}_\alpha$ given by (5.3) by the Pfaltzgraff-Suffridge type extension operator $\mathfrak{F}_{n,\alpha}$. This result is a generalization of [23, Theorem 2.1] (cf. [14, Theorem 2.1]).

Theorem 5.2. *Let $\mathbb{B}_X$ be the unit ball of an n-dimensional JB*-triple X, and let $\alpha \geq \frac{n}{2c(\mathbb{B}_X)}$. Also, let $\mathbb{D}_\alpha \subset Z = X \times Y$ be the domain given by* (5.3) *and $\mathfrak{F}_{n,\alpha}$ be the Pfaltzgraff-Suffridge type extension operator given by* (5.5). *Assume that $f \in S(\mathbb{B}_X)$ can be embedded as the first element of a Loewner chain on $\mathbb{B}_X$. Then $\mathfrak{F}_{n,\alpha}(f) \in S(\mathbb{D}_\alpha)$ can be embedded as the first element of a Loewner chain on $\mathbb{D}_\alpha$.*

As corollaries of Theorem 5.2, we obtain the following results (cf. [10], [14], [20], [21, Chapter 11]).

Corollary 5.3. *Let $\mathbb{B}_X$, $\mathbb{D}_\alpha$ and $\mathfrak{F}_{n,\alpha}$ be as in Theorem* 5.2. *If $\gamma \in (-\frac{\pi}{2}, \frac{\pi}{2})$ and $f \in \widehat{S}_\gamma(\mathbb{B}_X)$, then $\mathfrak{F}_{n,\alpha}(f) \in \widehat{S}_\gamma(\mathbb{D}_\alpha)$. In particular, if $f \in S^*(\mathbb{B}_X)$, then $\mathfrak{F}_{n,\alpha}(f) \in S^*(\mathbb{D}_\alpha)$.*

Let $\mathbb{B}_X = \mathbb{B}^n$ be the Euclidean unit ball in $\mathbb{C}^n$. Since $c(\mathbb{B}^n) = \frac{n+1}{2}$, in view of Theorem 5.2 and Corollary 5.3, we obtain the following consequence (cf. [14, Corollary 2.8], [23, Theorem 2.1]).

Corollary 5.4. *Let $\Gamma_{n,\alpha}$ be the extension operator given by (5.6), and let Ω_α be the domain given by (5.4), where $\alpha \geq \frac{n}{n+1}$. Then the following statements hold:*

(i) *If $f \in S(\mathbb{B}^n)$ can be embedded as the first element of a Loewner chain on $\mathbb{B}^n$, then $\Gamma_{n,\alpha}(f)$ can be embedded as the first element of a Loewner chain on Ω_α.*

(ii) *If $\gamma \in (-\frac{\pi}{2}, \frac{\pi}{2})$ and $f \in \widehat{S}_\gamma(\mathbb{B}^n)$, then $\Gamma_{n,\alpha}(f) \in \widehat{S}_\gamma(\Omega_\alpha)$. In particular, if $f \in S^*(\mathbb{B}^n)$, then $\Gamma_{n,\alpha}(f) \in S^*(\Omega_\alpha)$.*

(iii) *If $d \in [0,1)$ and $f \in S(\mathbb{B}^n)$ is an almost starlike mapping of order d on $\mathbb{B}^n$, then $\Gamma_{n,\alpha}(f)$ is almost starlike of order d on Ω_α.*

If $\mathbb{B}_X = \mathbb{U}^n$, then $c(\mathbb{U}^n) = n$, and we obtain the following result from Theorem 5.2 and Corollary 5.3.

Corollary 5.5. *Let $\Theta_{n,\alpha}$ be the extension operator given by (5.8), and let Δ_α be the domain given by (5.7), where $\alpha \geq 1/2$. Then the following statements hold:*

(i) *If $f \in S(\mathbb{U}^n)$ can be embedded as the first element of a Loewner chain on $\mathbb{U}^n$, then $\Theta_{n,\alpha}(f)$ can be embedded as the first element of a Loewner chain on Δ_α.*

(ii) *If $\gamma \in (-\frac{\pi}{2}, \frac{\pi}{2})$ and $f \in \widehat{S}_\gamma(\mathbb{U}^n)$, then $\Theta_{n,\alpha}(f) \in \widehat{S}_\gamma(\Delta_\alpha)$. In particular, if $f \in S^*(\mathbb{U}^n)$, then $\Theta_{n,\alpha}(f) \in S^*(\Delta_\alpha)$.*

Next, we mention the following suggestive examples. If we combine Examples 5.6 and 5.7 with Theorem 5.2 and Corollary 5.3, we obtain concrete examples of starlike, spirallike of type γ, and mappings which can be embedded as the first elements of Loewner chains on the domain $\mathbb{D}_\alpha$, where $\alpha \geq \frac{n}{2c(\mathbb{B}_X)}$. If we combine Examples 5.6 and 5.7 with Corollary 5.4, we also obtain concrete examples of almost starlike mappings of order d on the domain Ω_α, where $\alpha \geq \frac{n}{n+1}$.

Example 5.6. Let $f \in \mathcal{L}S(\mathbb{U})$. Let $u \in X \setminus \{0\}$ be fixed and let $l_u \in T(u)$. Also, let $F_u \in H(\mathbb{B}_X)$ be given by

$$F_u(z) = \frac{f(l_u(z))}{l_u(z)} z, \quad z \in \mathbb{B}_X. \tag{5.9}$$

Then we have

$$[DF_u(z)]^{-1} F_u(z) = \frac{f(l_u(z))}{f'(l_u(z)) l_u(z)} z, \quad z \in \mathbb{B}_X.$$

Consequently, we deduce the following statements:

(i) $F_u \in S^*(\mathbb{B}_X)$ if and only if $f \in S^*$.

(ii) $F_u \in \widehat{S}_\gamma(\mathbb{B}_X)$, $\gamma \in (-\frac{\pi}{2}, \frac{\pi}{2})$ if and only if $f \in \widehat{S}_\gamma(\mathbb{U})$.

(iii) F_u is almost starlike of order $d \in [0,1)$ on $\mathbb{B}_X$ if and only if f is almost starlike of order d on $\mathbb{U}$.

We recall that a Loewner chain $(F_t)_{t\geq 0}$ on $\mathbb{B}_X$ is said to be normal if the family $\{e^{-t}F_t\}_{t\geq 0}$ is a normal family on $\mathbb{B}_X$.

Example 5.7. Let $f \in \mathcal{L}S(\mathbb{U})$. Let $u \in X \setminus \{0\}$ be fixed and let $l_u \in T(u)$. Also, let $F_u \in H(\mathbb{B}_X)$ be given by (5.9). Then F_u may be embedded in a normal Loewner chain on $\mathbb{B}_X$ if and only if $f \in S$.

5.2. Bloch mappings and Pfaltzgraff-Suffridge type extension operators

Next, we prove that locally univalent I-Bloch mappings on $\mathbb{B}_X$ of finite trace order are extended to R-Bloch mappings on Ω_α by the Pfaltzgraff-Suffridge type extension operator $\mathfrak{F}_{n,\alpha}$, for $\alpha \geq \frac{1}{2}$. In the case $n = 1$, $f \in \mathcal{LS}(\mathbb{U})$ is uniformly locally univalent on $\mathbb{U}$ if and only if f has a finite order (see [12, Theorem 2.1], [38]). Therefore, the following results are generalizations of Theorem 3.7.

Theorem 5.8. *Let $\mathbb{B}_X$ be the open unit ball of an n-dimensional JB*-triple X. Let $\mathfrak{F}_{n,\alpha}$ be the Pfaltzgraff-Suffridge type extension operator given by* (5.5)*, and let Ω_α be the domain given by* (5.4)*, where $\alpha \geq \frac{1}{2}$. If $f \in \mathcal{LS}(\mathbb{B}_X)$ is an I-Bloch mapping on $\mathbb{B}_X$ which has finite trace order, then $F = \mathfrak{F}_{n,\alpha}(f) \in \mathcal{LS}(\Omega_\alpha)$ is an R-Bloch mapping on Ω_α.*

Next, we obtain the following consequences of Theorem 5.8.

Corollary 5.9. *Let $\mathbb{B}_X$, $\mathfrak{F}_{n,\alpha}$ and Ω_α be as in Theorem* 5.8. *If $f \in \mathcal{LS}(\mathbb{B}_X)$ is a bounded mapping on $\mathbb{B}_X$ which has finite trace order, then $F = \mathfrak{F}_{n,\alpha}(f) \in \mathcal{LS}(\Omega_\alpha)$ is an R-Bloch mapping on Ω_α.*

Corollary 5.10. *Let $\mathbb{B}_X$, $\mathfrak{F}_{n,\alpha}$ and Ω_α be as in Theorem* 5.8. *Then the following statements hold:*

(i) *If $f \in K(\mathbb{B}_X)$ is an I-Bloch mapping on $\mathbb{B}_X$, then $F = \mathfrak{F}_{n,\alpha}(f) \in S(\Omega_\alpha)$ is an R-Bloch mapping on Ω_α.*

(ii) *If $f \in K(\mathbb{B}_X)$ is a bounded mapping on $\mathbb{B}_X$, then $F = \mathfrak{F}_{n,\alpha}(f) \in S(\Omega_\alpha)$ is an R-Bloch mapping on Ω_α.*

As a corollary of Theorem 5.8, we obtain that the Pfaltzgraff-Suffridge type extension operator $\Gamma_{n,1}$ given by (5.6) maps locally univalent Bloch mappings of finite trace order from the Euclidean unit ball $\mathbb{B}^n$ into locally univalent Bloch mappings on the unit ball $\mathbb{B}_H$ of a complex Hilbert space H with $\dim H \geq n+1$. Note that $\mathbb{B}_H$ can be regarded as the domain

$$\Omega_1 = \{(x,y) \in \mathbb{C}^n \times H_1 : \|x\|_e^2 + \|y\|_{H_1}^2 < 1\},$$

where H_1 is a complex Hilbert space with $\dim H_1 \geq 1$.

Corollary 5.11. *Let $\mathbb{B}_H$ be the unit ball of a complex Hilbert space H with $\dim H \geq n+1$. Then the following statements hold:*

(i) *If $f \in \mathcal{LS}(\mathbb{B}^n)$ is a Bloch mapping, which has finite trace order, then $F = \Gamma_{n,1}(f) \in \mathcal{LS}(\mathbb{B}_H)$ is a Bloch mapping on $\mathbb{B}_H$.*

(ii) *If $f \in K(\mathbb{B}^n)$ is a bounded mapping on $\mathbb{B}^n$, then $F = \Gamma_{n,1}(f) \in S(\mathbb{B}_H)$ is a Bloch mapping on $\mathbb{B}_H$.*

In view of Corollary 5.11, we obtain the following result related to the preservation of normalized locally univalent Bloch functions under the Roper-Suffridge extension operator (cf. Theorem 3.7). This result is an improvement of [20, Theorem 2.6].

Corollary 5.12. *Let $\mathbb{B}_H$ be the unit ball of a complex Hilbert space H with* $\dim H \geq 2$, *and let $f \in \mathcal{LS}(\mathbb{U})$. Then the following statements hold:*

(i) *If f is a uniformly locally univalent Bloch function on $\mathbb{U}$, then $F = \Gamma_{1,1}(f) \in \mathcal{LS}(\mathbb{B}_H)$ is a Bloch mapping on $\mathbb{B}_H$.*

(ii) *If f is a bounded convex function on $\mathbb{U}$, then $F = \Gamma_{1,1}(f) \in S(\mathbb{B}_H)$ is a Bloch mapping on $\mathbb{B}_H$.*

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