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On convolution, convex, and starlike mappings

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Dedicated to the memory of Professor Gabriela Kohr

Abstract. Let C and S^* stand for the classes of convex and starlike mapping in $\mathbb{D}$, and let $\overline{\text{co}(C)}$, $\overline{\text{co}(S^*)}$ denote the closures of the respective convex hulls. We derive characterizations for when the convolution of mappings in $\overline{\text{co}(C)}$ is convex, as well as when the convolution of mappings in $\overline{\text{co}(S^*)}$ is starlike. Several characterizations in terms of convolution are given for convexity within $\overline{\text{co}(C)}$ and for starlikeness within $\overline{\text{co}(S^*)}$. We also obtain a correspondence via convolution between C and S^*, as well as correspondences between the subclasses of convex and starlike mappings that have n-fold symmetry.

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1. Introduction

The present paper is motivated by our interest in convex mappings of the unit disk, $\mathbb{D}$, in particular a representation formula for the pre-Schwarzian of such mappings that has been very useful in studying, for example, Schwarz-Christoffel mappings onto convex polygons. The famous Pólya-Schoenberg conjecture, resolved in 1973 by Ruscheweyh and Sheil-Small, [9], can be formulated in terms of this representation formula and leads to an open problem, stated in Section 4, regarding a certain product in the unit ball of $H^\infty(\mathbb{D})$.

We also revisit some classical themes related to convolution of holomorphic mappings in $\mathbb{D}$, with a particular focus in the classes C and S^* of convex and starlike mappings. The analysis will carry over naturally to the closures of the convex hulls $\overline{\text{co}(C)}$ and $\overline{\text{co}(S^*)}$. In Section 2, we will derive necessary and sufficient conditions for $f * g$ to be convex when $f, g \in \overline{\text{co}(C)}$, with two corollaries characterizing the mappings

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in $\overline{\text{co}(C)}$ that are convex. In the same vein, we will characterize when $f * g$ is starlike for $f, g \in \overline{\text{co}(S^*)}$, as well as give necessary and sufficient conditions for $f \in \overline{\text{co}(S^*)}$ to be starlike. Many more characterizations are probably possible.

In the Section 3 we will derive via convolution the classical theorem of Alexander for the correspondence between convex and starlike mappings, with the interesting special case of the correspondence between mappings onto convex polygons and starlike slit mappings. We also establish correspondences via convolution for the subclasses of C and S^* having n-fold symmetry.

We recall the definition of the convolution of two holomorphic functions. In terms of power series, if

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \quad \text{and} \quad g(z) = \sum_{n=0}^{\infty} b_n z^n$$

then their convolution is

$$(f * g)(z) = \sum_{n=0}^{\infty} a_n b_n z^n.$$

As well,

$$(f * g)(z) = \frac{1}{2\pi i} \int_{|\zeta|=\rho} f(\zeta) g(z\zeta^{-1}) \frac{d\zeta}{\zeta}, \quad |z| < \rho. \tag{1.1}$$

2. Convex Hulls

Let $\overline{\text{co}(C)}$ and $\overline{\text{co}(S^*)}$ stand, respectively, for the closures of the convex hulls of convex and starlike mappings of $\mathbb{D}$. It was shown in [1] that $f \in \overline{\text{co}(C)}$ if and only if there exists a probability measure μ on $\partial\mathbb{D}$ such that

$$f(z) = \int\limits_{|\zeta|=1} \frac{z}{1 - z\zeta} d\mu\,,$$

and that $f \in \overline{\text{co}(S^*)}$ if and only if there exists a probability measure ν on $\partial\mathbb{D}$ such that

$$f(z) = \int\limits_{|\zeta|=1} \frac{z}{(1 - z\zeta)^2} d\nu\,.$$

To make this actionable we will need a number of explicit convolutions.

Lemma 2.1. *The following identities hold for functions of the variable $z \in \mathbb{D}$ and fixed parameters $\zeta, \xi \in \partial\mathbb{D}$.*

i) $\dfrac{z}{1 - z\zeta} * \dfrac{z}{1 - z\xi} = \dfrac{z}{1 - z\zeta\xi}$

ii) $\dfrac{z}{(1 - z\zeta)^2} * \dfrac{z}{1 - z\xi} = \dfrac{z}{(1 - z\zeta\xi)^2}$

iii) $\dfrac{z}{(1 - z\zeta)^2} * \dfrac{z\xi}{(1 - z\xi)^2} = \dfrac{z(1 + z\zeta\xi)}{(1 - z\zeta\xi)^3}$

iv) $\dfrac{z}{1-z\zeta} * \dfrac{1}{(1-z)^2} = \dfrac{z(2-z\zeta)}{(1-z\zeta)^3}$

v) $\dfrac{z}{1-z\zeta} * \dfrac{z^2}{(1-z)^2} = \dfrac{z^2\zeta}{(1-z\zeta)^2}$

vi) $\dfrac{z}{1-z\zeta} * \dfrac{z}{(1-z)^3} = \dfrac{z^2}{(1-z\zeta)^3}$

vii) $\dfrac{z}{1-z\zeta} * \dfrac{z^2}{(1-z)^3} = \dfrac{z^2\zeta}{(1-z\zeta)^3}$

Proof. The first identity follows directly from the power series of the functions convolved. For the remaining parts, we will use that $z(h_1 * h_2)' = h_1 * (zh_2')$.
ii) We have

$$\frac{z}{(1-z\zeta)^2} * \frac{z}{1-z\xi} = z\left(\frac{z}{1-z\zeta}\right)' * \frac{z}{1-z\xi} = z\left(\frac{z}{1-z\zeta} * \frac{z}{1-z\xi}\right)'$$
$$= z\left(\frac{z}{1-z\zeta\xi}\right)' = \frac{z}{(1-z\zeta\xi)^2}\,.$$

iii) Here we use that

$$\frac{z\xi}{(1-z\xi)^2} = z\left(\frac{z}{1-z\xi}\right)'.$$

iv) Since

$$\frac{1}{(1-z)^2} = 1 + \frac{z}{1-z} + \frac{z}{(1-z)^2}$$

the convolution is equal to

$$\frac{z}{1-z\zeta} + \frac{z}{(1-z\zeta)^2}\,,$$

which gives the result.

v) The identity follows at once from

$$\frac{z^2}{(1-z)^2} = 1 + \frac{2z}{(1-z)^2} - \frac{1}{(1-z)^2}\,.$$

vi) Here we write

$$\frac{z}{(1-z)^3} = \frac{z}{2}\left(\frac{1}{(1-z)^2}\right)'.$$

vii) The identity follows from

$$\frac{z^2}{(1-z)^3} = \frac{z}{(1-z)^3} - \frac{z}{(1-z)^2}\,.$$

□

We have a series of observations.

Theorem 2.2. *Let $f, g \in \overline{\mathrm{co}(C)}$ be represented by measures μ, τ, respectively. Then $f * g$ is convex if and only if*

$$\left| \iint\limits_{|\zeta|,|\xi|=1} \frac{z\zeta\xi}{(1-z\zeta\xi)^3} d\mu d\tau \right| \leq \left| \iint\limits_{|\zeta|,|\xi|=1} \frac{1}{(1-z\zeta\xi)^3} d\mu d\tau \right| .$$

Proof. Let $f, g \in \overline{\mathrm{co}(C)}$. Differentiating $z(f*g)' = f * (zg')$ gives

$$z(f*g)'' + (f*g)' = (f*(zg'))' = [(zf')*(zg')]/z \,.$$

Write

$$\psi := 1 + z\frac{(f*g)''}{(f*g)'} = \frac{(zf')*(zg')}{z(f*g)'} = \frac{(zf')*(zg')}{f*(zg')} \,.$$

Because $\mathrm{Re}\{\psi\} \geq 0$ if and only if $|\psi - 1| \leq |\psi + 1|$, we have that $f*g$ is convex if and only if

$$|(zf'-f)*(zg')| \leq |(zf'+f)*(zg')| \,.$$

The correspondences in terms of the kernels are given by

$$zf' - f \longleftrightarrow \frac{z^2\zeta}{(1-z\zeta)^2}\,, \qquad zf'+f \longleftrightarrow \frac{z(2-z\zeta)}{(1-z\zeta)^2}\,,$$

hence

$$(zf'-f)*(zg') \longleftrightarrow \frac{2z\zeta\xi}{(1-z\zeta)^3}\,, \qquad (zf'+f)*(zg') \longleftrightarrow \frac{2}{(1-z\zeta\xi)^3}\,,$$

and the theorem follows. □

Corollary 2.3. *Let $f \in \overline{\mathrm{co}(C)}$ be represented by the measure μ. Then f is convex if and only if*

$$\left| \int\limits_{|\zeta|=1} \frac{z\zeta}{(1-z\zeta)^3} d\mu \right| \leq \left| \int\limits_{|\zeta|=1} \frac{1}{(1-z\zeta)^3} d\mu \right| .$$

Proof. The corollary follows by letting $g = z/(1-z)$. □

If we let $\mu = \sum_{k=1}^n \alpha_k \delta_{\zeta_k}$ be a finite sum of delta functions at points $\zeta_k \in \partial\mathbb{D}$, then f is convex if and only if

$$\left| \sum_{k=1}^n \frac{\alpha_k \zeta_k z}{(1-z\zeta_k)^3} \right| \leq \left| \sum_{k=1}^n \frac{\alpha_k}{(1-z\zeta_k)^3} \right| .$$

This inequality characterizes the finite convex combinations of rotations of a half-plane mapping that are convex.

Theorem 2.4. *The function $f \in \overline{\mathrm{co}(C)}$ is convex if and only if*

$$\left| f(z) * \frac{z^2}{(1-z)^3} \right| \le \left| f(z) * \frac{z}{(1-z)^3} \right| .$$

If $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$, then this holds if and only if

$$\left| \sum_{k=1}^{\infty} k(k+1) a_{k+1} z^k \right| \le \left| 2 + \sum_{k=1}^{\infty} (k+1)(k+2) a_{k+1} z^k \right| .$$

Proof. The first part of the theorem follows from parts vi) and vii) of Lemma 2.1, and the second follows directly from convolution. □

Theorem 2.5. *Let $f, g \in \overline{\mathrm{co}(S^*)}$ be represented by measures μ, τ, respectively. Then $f * g$ is starlike if and only if*

$$\left| \iint_{|\zeta|,|\xi|=1} \frac{z\zeta\xi}{(1-z\zeta\xi)^2} d\mu d\tau \right| \le \left| \iint_{|\zeta|,|\xi|=1} \frac{2 - z\zeta\xi}{(1-z\zeta\xi)^2} d\mu d\tau \right| .$$

Proof. For $f, g \in \overline{\mathrm{co}(S^*)}$ let

$$\phi = z\frac{(f*g)'}{f*g} = \frac{f*(zg')}{f*g} .$$

Then $\mathrm{Re}\{\phi\} \ge 0$ if and only if

$$|f*(zg' - g)| \le |f*(zg' + g)| ,$$

which proves the theorem from the representing kernels and Lemma 2.1. □

Corollary 2.6. *Let $f \in \overline{\mathrm{co}(S^*)}$ be represented by the measure μ. Then f is starlike if and only if*

$$\left| \int_{|\zeta|=1} \frac{z\zeta}{(1-z\zeta)^2} d\mu \right| \le \left| \int_{|\zeta|=1} \frac{2 - z\zeta}{(1-z\zeta)^2} d\mu \right| .$$

Proof. As before, we let $g(z) = z/(1-z)$. □

Theorem 2.7. *The function $f \in \overline{\mathrm{co}(S^*)}$ is starlike if and only if*

$$\left| f(z) * \frac{z^2}{(1-z)^2} \right| \le \left| f(z) * \frac{1}{(1-z)^2} \right| .$$

If $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$, then this holds if and only if

$$\left| \sum_{k=2}^{\infty} (k-1) a_k z^k \right| \le \left| z + \sum_{k=2}^{\infty} k a_k z^k \right| .$$

Proof. Parts iv) and v) of Lemma 1.1 give the first statement, which corresponds to the second inequality by convolving directly. □

3. Correspondences

Let

$$l(z) = \frac{z}{1-z}, \quad L_1(z) = \log\frac{1}{1-z}, \quad k_1(z) = \frac{z}{(1-z)^2}.$$

Note that $k_1 = zl'$ and $l = zL_1'$. We will use these functions together with convolution to recover Alexander's theorem relating convex and starlike mappings.

Theorem 3.1. *If $f \in C$ then $f * k_1 \in S^*$. Conversely, if $g \in S^*$ then $g * L_1 \in C$.*

Proof. Let $f \in C$. Then $f * k_1 = f * (zl') = z(f * k_1)' = zf'$, hence

$$1 + z\frac{f''}{f'} = z\frac{(f * k_1)'}{f * k_1}, \tag{3.1}$$

which shows the first claim.

On the other hand, if $g \in S^*$ then $z(g * L_1)' = g * (zL_1') = g * l = g$, and thus

$$1 + z\frac{(g * L_1)''}{(g * L_1)'} = z\frac{g'}{g},$$

which establishes the second claim. □

The special case when f is a conformal mapping onto a convex polygon $\mathcal{P}$ is interesting. It follows from the Schwarz-Christoffel formula that

$$z\frac{f''}{f'} = -2\sum_{k=1}^{n} \frac{\beta_k z}{z - z_k},$$

where $z_k \in \partial\mathbb{D}$ are the pre-vertices, and $2\pi\beta_k$ are the exterior angles, satisfying $0 < \beta_k < 1$ with $\sum_{k=1}^{n} \beta_k = 1$. In [3] it was shown that

$$z\frac{f''}{f'} = \frac{2zB(z)}{1 - zB(z)}, \tag{3.2}$$

where $B(z)$ is a finite Blaschke product of degree $n-1$. Furthermore, the pre-vertices are the roots of the equation

$$zB(z) = 1.$$

Theorem 3.2. *Under the above convolutions, convex polygons correspond to slit mappings, with the number of vertices being equal to the number slits. If g is the slit mapping corresponding to f as above, then the pre-images ζ_k under g of the finite endpoints of the slits are given by the root of the equation*

$$zB(z) = -1,$$

while the pre-vertices z_k of the polygon are mapped under g to the point at infinity.

Proof. Let f map $\mathbb{D}$ onto a convex polygon $\mathcal{P}$. The correspondence of f with a starlike mapping g given by (3.1) is equivalent to

$$g = zf'.$$

On an open arc A_k between the pre-vertices z_k and z_{k+1} we have that $\arg\{zf'\}$ is constant, hence so is $\arg\{g\}$. We conclude that $g(A_k)$ lies on a slit. Since $f'(z) \to \infty$ as z approaches any pre-vertex, we see that $g = \infty$ at every pre-vertex z_k.

Let $\zeta_k \in A_k$ be the pre-image under g of the finite endpoint of the slit $g^{-1}(f(A_k))$. Then $g'(\zeta_k) = 0$, hence

$$1 + \zeta_k \frac{f''}{f'}(\zeta_k) = 0\,.$$

Using (3.2) we see that

$$\zeta_k B(\zeta_k) = -1\,,$$

as claimed. □

Corollary 3.3. *Let f be a conformal mapping onto a convex polygon, and let $B(z)$ be the associated Blaschke product in the representation (3.2). Then the roots of the equation*

$$zB(z) = -1$$

correspond to the points between consecutive pre-vertices where $|f'|$ attains the minimum value on that arc.

Proof. The mapping g in the previous theorem is starlike, therefore $|g(\zeta_k)|$ is the minimum value of $|g|$ on the arc A_k. The corollary follows because $g = zf'$. □

By appropriately modifying L_1, k_1, these correspondences carry through to subspaces of convex and starlike mappings with symmetries. For this, we begin with the functions L_2, k_2 defined by the conditions $L_2(0) = k_2(0) = 0$ and

$$L_2'(z) = \frac{1}{2}\left(L_1'(z) + L_1'(-z)\right) \quad , \quad k_2'(z) = \frac{1}{2}\left(k_1'(z) + k_1'(-z)\right)\,.$$

Then

$$z(f * L_2)' = \frac{1}{2} f * \left(L_1'(z) + L_1'(-z)\right) = \frac{1}{2}\left(f * \frac{z}{1-z} + \frac{z}{1+z}\right)\,.$$

If f is odd then

$$f * \frac{z}{1-z} = f * \frac{z}{1+z}\,,$$

and we are back in the case when $z(f * L_2)' = f$. Therefore, if $f \in \overline{\text{co}(S^*)}$ is odd then $f * L_2 \in \overline{\text{co}(C)}$, and is also odd because L_2 is odd. A similar analysis shows that if $g \in \overline{\text{co}(C)}$ is odd, then $g * k_2 \in \overline{\text{co}(S^*)}$ and is also odd.

For the general construction, we introduce the averaging operator A_n defined by

$$A_n(f)(z) = \frac{1}{n}\sum_{k=1}^{n} \omega^{-k} f(\omega^k z)\,,$$

where $\omega = e^{\frac{2\pi i}{n}}$. An equivalent definition is that $A_n(f)$ satisfies $A_n(f)(0) = 0$ and

$$A_n(f)'(z) = \frac{1}{n}\sum_{k=1}^{n} f'(\omega^k z)\,.$$

We thus see that $L_2 = A_2(L_1)$ and $k_2 = A_2(k_1)$.

A function f defined in $\mathbb{D}$ is said to be n-symmetric if $A_n(f) = f$. It is not difficult to see that f is n-symmetric if and only if $f(\omega z) = \omega f(z)$, which holds if and only if $f(z) = z\sum_{k=0}^{\infty} a_k z^{kn}$. Furthermore,

$$A_n(f) * g = f * A_n(g)\,,$$

a fact that follows immediately from the respective power series expansions. We can also see that $A_n(f) * g$ is always n-symmetric. We finally let $L_n = A_n(L_1)$ and $k_n = A_n(k_1)$ stand for the symmetrization of L_1 and k_1, and define C_n, S_n^* to be the set of convex and starlike mappings with n-fold symmetry. It is interesting to note that all the mappings L_n are univalent since $\mathrm{Re}\{L_n'\} > 0$ in $\mathbb{D}$. On the other hand, already the function k_2 is not even locally univalent in $\mathbb{D}$.

Theorem 3.4. *If $f \in C_n$ then $f * k_n \in S_n^*$. If $g \in S_n^*$ then $g * L_n \in C_n$.*

Proof. If $f \in C_n$ then $z(f * k_n') = (f * k_n)' = (A_n(f) * k_1)' = (f * k_1)'$, which allows us to conclude that $f * k_n \in S^*$ as argued in Theorem 2.2. Since it is also also symmetric, the first claim follows. The second claim is established in similar fashion. □

These results carry through to establish correspondences between the spaces $\overline{\mathrm{co}(C)}$ and $\overline{\mathrm{co}(S^*)}$, and the respective subspaces with n-fold symmetry $\overline{\mathrm{co}(C)}_n = \{A_n(f) : f \in \overline{\mathrm{co}(C)}\}$ and $\overline{\mathrm{co}(S^*)}_n = \{A_n(g) : g \in \overline{\mathrm{co}(S^*)}\}$. We have:

Theorem 3.5. *If $f \in \overline{\mathrm{co}(C)}$ then $f * k_1 \in \overline{\mathrm{co}(S^*)}$. Conversely, if $g \in \overline{\mathrm{co}(S^*)}$ then $g * L_1 \in \overline{\mathrm{co}(S^*)}$.*

Proof. The proof is very similar to that of Theorem 3.1, and we will give the details for the first claim. If $f \in \overline{\mathrm{co}(C)}$ then for some probability measure μ we have

$$f(z) = \int\limits_{|\zeta|=1} \frac{z}{1 - z\zeta}\,d\mu\,.$$

Hence

$$\begin{aligned}(f * k_1)(z) &= \int\limits_{|\zeta|=1} \frac{z}{1 - z\zeta} * k_1(z)d\mu \;=\; \int\limits_{|\zeta|=1} \frac{z}{1 - z\zeta} * (zl'(z))\,d\mu \\ &= \int\limits_{|\zeta|=1} z\left(\frac{z}{1 - z\zeta} * l(z)\right)' d\mu \;=\; \int\limits_{|\zeta|=1} z\left(\frac{z}{1 - z\zeta}\right)' d\mu \\ &= \int\limits_{|\zeta|=1} \frac{z}{(1 - z\zeta)^2}\,d\mu\,,\end{aligned}$$

showing that $f * k_1 \in \overline{\mathrm{co}(S^*)}$. □

We state without proof the last result in this section.

Theorem 3.6. *If $f \in \overline{\mathrm{co}(C)}_n$ then $f * k_n \in \overline{\mathrm{co}(S^*)}_n$. Conversely, if $g \in \overline{\mathrm{co}(S^*)}_n$ then $g * L_n \in \overline{\mathrm{co}(S^*)}_n$.*

4. On the Pólya-Schoenberg Conjecture

The Pólya-Schoenberg Conjecture (PSC) states that the convolution of convex mappings is again convex. Our point of departure for the discussion here is the representation formula for convex mappings f, namely

$$\frac{f''}{f'} = \frac{2z\phi}{1 - z\phi}\,, \tag{4.1}$$

for some holomorphic ϕ with $|\phi| \le 1$ in $\mathbb{D}$. For example, for ϕ a unimodular constant, the resulting f is a rotation of the half-plane mapping, while for $\phi = z$ we obtain the mapping onto a parallel strip. The representation formula can be derived from the classical characterization of convexity via positive real part, and Schwarz's lemma. Any choice of such a function ϕ will determine a unique (normalized) convex mapping.

Let now f, g be convex and represented as above by functions ϕ, ψ bounded by 1 in $\mathbb{D}$. By PSC $f * g$ is also convex, and thus must be represented by another such function χ, which can be thought of as determined by the functions ϕ and ψ. This dependence yields therefore a certain "product" in the unit ball of $H^\infty(\mathbb{D})$ inherited from convolution being associative and commutative. An independent proof that this product does indeed preserve the unit ball in $H^\infty(\mathbb{D})$ would provide an alternative proof of the PSC.

To be more precise, for f convex we define the operator

$$\Phi(f) = \frac{f''/f'}{2 + zf''/f'}\,,$$

which comes from expressing ϕ in terms of f in (4.1). As an example, we compute $\Phi(f * g)$ when f, g are Möbius transformations. If

$$f(z) = \frac{az + b}{cz + d}\,,\; ad - bc = 1\,,$$

then a simple calculation yields

$$\Phi(f) = -\frac{c}{d} = \frac{1}{f^{-1}(\infty)}\,.$$

If

$$g(z) = \frac{\alpha z + \beta}{\gamma z + \delta}\,,\; \alpha\delta - \beta\delta = 1\,,$$

then

$$\begin{aligned} f * g &= \frac{1}{\gamma\delta} f\left(-(\gamma/\delta)z\right) - \frac{\alpha}{\gamma} f(0) \\ &= \frac{1}{\gamma\delta}\frac{(a\gamma/\delta)z - b}{(c\gamma/\delta)z - d} - \frac{\alpha}{\gamma} f(0) \end{aligned}$$

is again a Möbius transformation with denominator $c\gamma^2 z - \gamma\delta d$. Therefore

$$\Phi(f * g) = \frac{1}{(f * g)^{-1}(\infty)} = \frac{c\gamma}{d\delta} = \Phi(f)\Phi(g)\,.$$

That the convolution corresponds to the actual product in $H^\infty(\mathbb{D})$ is exceptional and can be readily seen not to hold in general. But we find the problem of understanding and determining the properties of the Φ-operator in relation to convolution appealing.

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