Stud. Univ. Babeş-Bolyai Math. 62(2017), No. 3, 295–307
DOI: 10.24193/subbmath.2017.3.03

Finite valuated groups as modules over their endomorphism ring

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Abstract. This paper discusses the structure of a finite valuated p-group when viewed as a module over its endomorphism ring. A category equivalence between full subcategories of the category of valuated p-groups and the category of right modules over the endomorphism ring of A is used to investigate the interaction between this module structure and homological properties of the underlying group. Examples are given throughout the paper.

Mathematics Subject Classification (2010): 20K30, 20K40, 20K10.

Keywords: Valuated p-group, endomorphism ring, Ulmer's theorem, projective module.

1. Introduction

Consider a prime p and a p-local Abelian group G. A *valuation* v on G assigns a value $v(g)$ to each $g \in G$ which is either an ordinal or ∞ subject to the rules

i) $v(px) > v(x)$ for all $x \in G$ where $\infty > \infty$,
ii) $v(x + y) \geq min\{v(x), v(y)\}$ for all $x, y \in G$, and
iii) $v(nx) = v(x)$ whenever n and p are relatively prime [11].

The third condition is redundant whenever G is a p-group. The valuated p-local groups are the objects of the category $\mathcal{V}_p$ studied extensively by Hunter, Richman and Walker (e.g. see [7], [8] and [11]). A group homomorphism $\alpha : (G, v) \to (H, w)$ is a $\mathcal{V}_p$*-morphism* if $w(\alpha(x)) \geq v(x)$ for all $x \in G$, and we write $\alpha \in \mathrm{Mor}(G, H)$ in this case. The category $\mathcal{V}_p$ is pre-Abelian, i.e. all maps have kernels and cokernels. While the kernel and cokernel of a $\mathcal{V}_p$-map $G \to H$ are its kernel and cokernel in the category $\mathcal{A}b$ of Abelian groups, their valuations are induced by those on G and H respectively. Consequently, monomorphisms and epimorphisms need not be kernels and cokernels; and $\mathcal{V}_p$ is not Abelian. Finally, the forgetful functor $\mathcal{F} : \mathcal{V}_p \to \mathcal{A}b$ strips a valuated group (G, v) of its valuation.

In this paper, all valuated groups are assumed to be finite valuated p-groups. Although the group structure of a finite valuated p-group is well understood, the addition of a valuation directly impacts its homological properties. In addition, Arnold discovered a surprising connection between finite valuated p-groups and torsion-free Abelian groups of finite rank in [3] by demonstrating that representation theory can be used to investigate finite rank Butler groups as well as finite valuated p-groups. Moreover, both classes of groups are equally difficult to describe.

This paper follows Arnold's approach by investigating valuated p-groups using tools which have traditionally been used in the discussion of torsion-free groups of finite rank. For instance, homological properties of Abelian groups A of finite torsion-free rank have been successfully studied by viewing A as a left module over its endomorphism ring. This paper extends this approach to finite valuated p-groups by considering such a group A as a module over its $\mathcal{V}_p$-endomorphism ring $R = \mathrm{Mor}(A, A)$ and by studying how this module structure affects the homological properties of A. Section 2 focuses on the case that A is projective as an R-module, while Section 3 considers the case that R has specific ring-theoretic properties.

2. Valuated p-Groups Projective as R-modules

A finite valuated p-group A-*free* if it is isomorphic to A^n for some $n < \omega$, and A-*projective* if it is a $\mathcal{V}_p$-direct summand of an A-free group. Since A is a left R-module, $H_A = \mathrm{Mor}(A, -)$ can be viewed as a functor from $\mathcal{V}_p$ to the category $\mathcal{M}_R$ of right R-modules, with the property that $H_A(P)$ is free (projective) if P is A-free (A-projective).

We begin our discussion with a few technical results. If α is a kernel in $\mathcal{V}_p$, then $\alpha = \ker(coker(\alpha))$ [12]; and a similar result holds for cokernels. However, composition of kernels (cokernels) in $\mathcal{V}_p$ need not be kernels (cokernels) [10]. Therefore, the usual homological constructions may not carry over from Abelian categories. Nevertheless, it is still possible to develop a homological algebra for pre-Abelian categories as Yakovlev showed in [14].

Lemma 2.1. *Let A, B and C be valuated p-groups. If $\alpha \in \mathrm{Mor}(A, B)$ is an epimorphism and $\beta \in Mor(B, C)$ such that $\beta\alpha$ is a cokernel of a $\mathcal{V}_p$-map δ, then β is a cokernel for $\alpha\delta$.*

Proof. Suppose that ϕ satisfies $\phi\alpha\delta = 0$. Since $\beta\alpha$ is a cokernel for δ, there is a map ψ such that $\psi\beta\alpha = \phi\alpha$. Because α is an epimorphism, $\phi = \psi\beta$. Since β is an epimorphism, ψ is unique with this property. □

A sequence $A \overset{\alpha}{\to} B \overset{\beta}{\to} C$ of valuated p-groups is is *left-exact* if α is a kernel for β, and *right-exact* if β is a cokernel for α. It is *exact in $\mathcal{V}_p$* if α is a kernel for β and β is a cokernel for α [11]. The functor $H_A : \mathcal{V}_p \to \mathcal{M}_R$ is left-exact since

$$0 \to H_A(U) \overset{H_A(\alpha)}{\longrightarrow} H_A(B) \overset{H_A(\beta)}{\longrightarrow} H_A(C) \quad (*)$$

is an exact sequence of right R-modules whenever

$$0 \to U \xrightarrow{\alpha} B \xrightarrow{\beta} C$$

is a left-exact sequence of valuated p-groups.

Consider the functor $t_A : \mathcal{M}_R \to \mathcal{A}b$ defined by $t_A = - \otimes_R A$ for all $M \in \mathcal{M}_R$. If F is a free right R-module with basis $\{x_i \mid i \in I\}$, then

$$v(\Sigma_{i\in I} x_i \otimes a_i) = min\{v(a_i) \mid i \in I\}$$

defines a valuation on $t_A(F)$, and the resulting valuated group is denoted by $T_A(F)$ [1]. To define a valuation on $t_A(M)$ for an arbitrary right R-module M, we choose a free resolution

$$F_1 \xrightarrow{\alpha} F_0 \xrightarrow{\beta} M \to 0$$

of M. Applying t_A induces an exact sequence

$$T_A(F_1) \overset{t_A(\alpha)}{\longrightarrow} T_A(F_0) \overset{t_A(\beta)}{\longrightarrow} t_A(M) \to 0$$

where $t_A(\alpha)$ is a $\mathcal{V}_p$-map, which we denote as $T_A(\alpha)$, by [1]. Since $\mathcal{V}_p$ is pre-Abelian, there is a unique valuation v on $t_A(M)$ such that $t_A(\beta)$ becomes the $\mathcal{V}_p$-cokernel of $T_A(\alpha)$ [11]. We define $T_A(M) = (t_A(M), v)$, and observe $t_A = \mathcal{F}T_A$. The next result summarizes the basic properties of T_A which were established in [2, Section 2]:

Theorem 2.2. [2] *Let A be a finite valuated p-group.*

a) *$T_A : \mathcal{M}_R \to \mathcal{V}_p$ is a right exact functor.*
b) *The evaluation map $\theta_G : T_A H_A(G) \to G$ defined by $\theta_G(\alpha \otimes a) = \alpha(a)$ is a natural $\mathcal{V}_p$-map for all valuated p-groups G such that θ_P is an isomorphism for all A-projective groups P.*
c) *The natural map $\Phi_M : M \to \mathrm{Hom}(A, T_A(M))$ defined by $[\Phi_M(x)](a) = x \otimes a$ is a natural transformation such that $\theta_{T_A(M)} T_A(\Phi_M) = 1_{T_A(M)}$ for all right R-modules M. Moreover, Φ_P is an isomorphism for all finitely generated projective right R-modules P.*

An epimorphism $G \to H$ of valuated p-groups is *A-balanced* if the induced map $H_A(\alpha) : H_A(G) \to H_A(H)$ is onto. A valuated p-group G is *weakly A-generated* if we can find an A-balanced epimorphism

$$\oplus_I A \xrightarrow{\beta} G \to 0$$

for some index-set I. It is A-generated if β can be chosen to be a cokernel in $\mathcal{V}_p$. Although there is no need to distinguish between A-generated and weakly A-generated objects in an Abelian category, it is necessary to do this in the pre-Abelian case as was shown in [2].

A valuated p-group G is *A-presented* if there is an exact sequence

$$0 \to U \to F \to G \to 0$$

of valuated p-groups such that F is A-free and U is weakly A-generated. If this sequence can be chosen to be A-balanced, then G is called *A-solvable*. A valuated p-group G is A-presented if and only if $G \cong T_A(M)$ for some right R-module M.

Moreover, it is A-solvable if and only if θ_G is an isomorphism [2]. In particular, every A-projective group is A-solvable.

In a pre-Abelian category like $\mathcal{V}_p$, neither the 5-Lemma nor the Snake-Lemma need to hold [11]. The next result is frequently used in this paper as a substitute for the 5-Lemma throughout this paper:

Lemma 2.3. *Let A be a finite valuated p-groups. If $0 \to U \xrightarrow{\alpha} H \xrightarrow{\beta} G \to 0$ is a $\mathcal{V}_p$-exact sequence such that θ_H is an isomorphism, then there exists a commutative $\mathcal{V}_p$-diagram*

$$\begin{array}{ccccccccc} & & T_AH_A(U) & \xrightarrow{T_AH_A(\alpha)} & T_AH_A(H) & \xrightarrow{T_AH_A(\beta)} & T_A(M) & \longrightarrow & 0 \\ & & \downarrow \theta_U & & \wr \downarrow \theta_H & & \downarrow \theta & & \\ 0 & \longrightarrow & U & \xrightarrow{\alpha} & H & \xrightarrow{\beta} & G & \longrightarrow & 0 \end{array}$$

with $\mathcal{V}_p$-exact rows in which $M = imH_A(\beta) \subseteq H_A(G)$ and $\theta : T_A(M) \to G$ is the evaluation map. Moreover, θ is a cokernel, and $\theta = \theta_G T_A(\iota)$ where $\iota : M \to H_A(G)$ is the inclusion map.

Proof. Since H_A is left-exact, every exact sequence

$$0 \to U \xrightarrow{\alpha} H \xrightarrow{\beta} G \to 0$$

of valuated groups induces an exact sequence

$$0 \to H_A(U) \xrightarrow{H_A(\alpha)} H_A(H) \xrightarrow{H_A(\beta)} M \to 0$$

of right R-modules where $M = im(H_A(\beta))$ is a submodule of $H_A(G)$. By Part a) of Theorem 2.2, the induced sequence

$$T_AH_A(U) \xrightarrow{T_AH_A(\alpha)} T_AH_A(H) \xrightarrow{T_AH_A(\beta)} T_A(M) \to 0$$

is right exact. Part b) of same result yields that θ_U and θ_G are $\mathcal{V}_p$-maps, and the commutativity of the diagram follows directly. Since $T_A(\iota)$ is a $\mathcal{V}_p$-map by another application of Theorem 2.2, the same holds for $\theta = T_A(\iota)\theta_G$. Using the fact that θ_H is a $\mathcal{V}_p$-isomorphism, we obtain $\theta[T_A(\beta)\theta_H^{-1}] = \beta$. Because $T_A(\beta)$ is a cokernel, θ is a cokernel by Lemma 2.1. □

Ulmer described the objects of an Abelian Groethendick category which are flat over their endomorphism ring [13]. When discussing the validity of Ulmer's result in $\mathcal{V}_p$, one immediately realizes that his original arguments need to be modified extensively because this category is only pre-Abelian. In particular, we want to remind the reader that a finite valuated p-group is flat as an R-module if and only if it is projective.

Theorem 2.4. *The following conditions are equivalent for a finite valuated p-group A:*

a) *A is projective as a left R-module.*
b) *Whenever $\phi \in \mathrm{Mor}(A^n, A)$ for some $n < \omega$, then* ker ϕ *is weakly A-generated.*
c) *Whenever $\phi \in \mathrm{Mor}(G, H)$ for A-solvable valuated p-groups G and H, then* ker ϕ *is weakly A-generated.*

Proof. *a*) $\Rightarrow$ *c*): For $K = \ker\ \phi$, consider the exact sequence

$$0 \to H_A(K) \longrightarrow H_A(G) \xrightarrow{\phi} M \to 0$$

of right R-modules in which $M = im(H_A(\phi))$ is a submodule of $H_A(H)$. Let ι denote embedding $M \subseteq H_A(H)$. By Proposition 2.3, we obtain a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & T_AH_A(K) & \longrightarrow & T_AH_A(G) & \xrightarrow{T_AH_A(\phi)} & T_A(M) & \longrightarrow & 0 \\ & & \downarrow \theta_K & & \wr\downarrow \theta_{\oplus_I G} & & \downarrow \theta & & \\ 0 & \longrightarrow & K & \longrightarrow & G & \xrightarrow{\phi} & H & & \end{array}$$

of $\mathcal{V}_p$-maps whose top-row is right exact in $\mathcal{V}_p$. Moreover, it is exact in $\mathcal{A}b$ since A is projective as a left R-module. Using the projectivity of A once more yields that $T_A(\iota)$ is a monomorphism, and the same holds for $\theta = \theta_H T_A(\iota)$ since H is A-solvable. Thus, θ is an isomorphism of Abelian groups. Because the 3-Lemma is valid in $\mathcal{A}b$, we obtain that θ_K is an epimorphism in $\mathcal{A}b$, and hence in $\mathcal{V}_p$.

Since *c*) $\Rightarrow$ *b*) is obvious, it remains to show *b*) $\Rightarrow$ *a*):

It suffices to establish that the inclusion map $\iota : I \to R$ induces a monomorphism $t_A(\iota) : t_A(I) \to t_A(R)$ of Abelian groups for all right ideals I of R. Since R is finite, $I = \{r_1, \ldots, r_n\}$. We define a map $\phi_1 : F = R^n \to I$ by $\phi_1(e_i) = r_i$ where $\{e_1, \ldots, e_n\}$ is an R-basis of F. Set $\phi = \iota\phi_1 : F \to R$. By b), the kernel K of the $\mathcal{V}_p$-map $T_A(\phi) : T_A(F) \to T_A(R)$ is weakly A-generated. Since A is finite, we can select a finite A-projective group P and an A-balanced epimorphism $\lambda : P \to K$. Because

$$0 \to K \to T_A(F) \overset{T_A(\phi)}{\longrightarrow} T_A(R)$$

is $\mathcal{V}_p$-exact, the induced sequence

$$0 \to H_A(K) \to H_AT_A(F) \overset{H_AT_A(\phi)}{\longrightarrow} H_AT_A(R)$$

is exact. Combining this sequence with $H_A(\lambda)$ yields that the top-row of the commutative diagram

$$\begin{array}{ccccc} H_A(P) & \xrightarrow{H_A(\lambda)} & H_AT_A(F) & \xrightarrow{H_AT_A(\phi)} & H_AT_A(R) \\ & & \wr\uparrow \Phi_F & & \wr\uparrow \Phi_R \\ & & F & \xrightarrow{\phi} & R \end{array}$$

of right R-modules is exact. In view of $\phi(F) = I$, the diagram gives us the exact sequence

$$(E)\ \ H_A(P) \overset{H_A(\lambda)}{\longrightarrow} H_AT_A(F) \overset{\phi_1\Phi_F^{-1}}{\longrightarrow} I \to 0$$

of right R-modules. Since $\theta_{T_A(M)}T_A(\Phi_M) = 1_{T_A(M)}$ for all right R-modules M, we obtain $\theta_{T_A(X)} = T_A(\Phi_X^{-1})$ for all finitely generated projective right R-modules X. Hence,

$$T_A(\phi)\theta_{T_A(F)} = T_A(\phi\Phi_F^{-1}) = T_A(\Phi_R^{-1}H_AT_A(\phi)) = \theta_{T_A(R)}T_AH_AT_A(\phi).$$

Because of this and Theorem 2.2, an application of T_A yields the commutative diagram

$$\begin{array}{ccccc} T_AH_A(P) & \xrightarrow{T_AH_A(\lambda)} & T_AH_AT_A(F) & \xrightarrow{T_AH_AT_A(\phi)} & T_AH_AT_A(R) \\ \wr\downarrow \theta_P & & \wr\downarrow \theta_{T_A(F)} & & \wr\downarrow \theta_{T_A(R)} \\ P & \xrightarrow{\lambda} & T_A(F) & \xrightarrow{T_A(\phi)} & T_A(R) \end{array}$$

of Abelian groups. Since it suffices to show that $t_A(\iota)$ is a monomorphism of Abelian groups, our computations are done from this point only in $\mathcal{A}b$ instead of in $\mathcal{V}_p$. In particular, we use the fact that the $\mathcal{V}_p$-kernel of a map is its kernel in $\mathcal{A}b$ with a valuation added. The symbols t_A and T_A can be used interchangeably when computing in $\mathcal{A}b$.

Observe that the bottom row of the last diagram is exact at $T_A(F)$ as a sequence of Abelian groups by the choice of P and λ. Since the vertical maps are isomorphisms, the top-row is exact at $T_AH_AT_A(F)$. Moreover, (E) induces the exact sequence

$$T_AH_A(P) \overset{T_AH_A(\lambda)}{\longrightarrow} T_AH_AT_A(F) \overset{T_A(\phi_1\Phi_F^{-1})}{\longrightarrow} T_A(I) \to 0$$

of Abelian groups. Therefore, the map $T_A(\phi_1\Phi_F^{-1})$ is a cokernel in $\mathcal{A}b$ for the left top-map $T_AH_A(\lambda)$. On the other hand, the projection

$$\pi : T_A(F) \to G = T_A(F)/K$$

is a cokernel of λ in $\mathcal{A}b$. Hence, there is an isomorphism $\sigma : T_A(I) = t_A(I) \to G$ of Abelian groups such that $\pi\theta_{T_A(F)} = \sigma T_A(\phi_1\Phi_F^{-1})$. Since the bottom row of the last diagram is exact at $T_A(F)$, there is a map $\tau : G \to T_A(R)$ with $\tau\pi = T_A(\phi)$ using the exactness of the bottom row of the last diagram once more. For $g \in \ker\, \tau$, select $x \in T_A(F)$ with $\pi(x) = g$. Then $0 = \tau\pi(x) = T_A(\phi)(x)$ yields $x = \lambda(y)$ for some $y \in P$. Hence, $g = \pi\lambda(y) = 0$, and τ is a monomorphism.

Because $H_ATA(\phi_1)\Phi_F = \Phi_I\phi_1$, we have

$$\begin{aligned} \theta_{T_A(R)}T_AH_AT_A(\iota)T_A(\Phi_I)T_A(\phi_1) &= \theta_{T_A(R)}T_AH_AT_A(\iota)T_AH_AT_A(\phi_1)T_A(\Phi_F) \\ &= \theta_{T_A(R)}T_AH_AT_A(\phi)T_A(\Phi_F) \\ &= T_A(\phi)\theta_{T_A(F)}T_A(\Phi_F) \\ &= \tau\pi\theta_{T_A(F)}T_A(\Phi_F) \\ &= \tau\sigma T_A(\phi_1\Phi_F^{-1})T_A(\Phi_F) \\ &= \tau\sigma T_A(\phi_1). \end{aligned}$$

Since $T_A(\phi_1)$ is an epimorphism, we obtain that

$$\theta_{T_A(R)}T_AH_AT_A(\iota)T_A(\Phi_I) = \tau\sigma$$

is a monomorphism since the maps on the right are monomorphisms, and the same holds for

$$T_A(\Phi_R)t_A(\iota) = T_AH_AT_A(\iota)T_A(\Phi_I)$$

using the fact that $T_A(R) \cong A$. Because $T_A(\Phi_R)$ is an isomorphism, $t_A(\iota)$ is one-to-one as desired. □

For a finite p-group G, let $e(A)$ denote the smallest $n < \omega$ such that $p^nG = 0$.

Corollary 2.5. *Every finite valuated p-group A is a direct summand of a finite valuated p-group B such that* $e(A) = e(B)$ *and B is flat as a module over its endomorphism ring.*

Proof. Choose $n < \omega$ minimal with the property that $p^n A = 0$, and consider the group $B = \mathbb{Z}/p^n\mathbb{Z} \oplus A$ where $\mathbb{Z}/p^n$ carries the height valuation h. Since h is the smallest valuation on $\mathbb{Z}/p^n\mathbb{Z}$, and every B-generated group is bounded by p^n, the kernel of every map between any two B-generated groups is a $\mathcal{V}_p$-epimorphic image of $(\mathbb{Z}/p^n\mathbb{Z}, h)$. By Theorem 2.4, B is projective over its endomorphism ring. □

We continue our discussion by looking at simply presented groups. A *(p-)valuated tree* is a set X, on which a partial multiplication by p is defined, together with a function v assigning a value $v(x)$ to each $x \in X$ which is either an ordinal or ∞ subject to the rules

i) If $p^n x = x$ for some $0 < n < \omega$, then $px = x$, and there is exactly one element in X with this property, called the *root of* X.
ii) $v(px) > v(x)$ whenever px is defined.

Moreover, if $X_1, \ldots, X_n$ are rooted valuated trees, then the co-product $\cup_{i=1}^n X_i$ in the category of valuated p-tree is the tree that is obtained by joining $X_1, \ldots, X_n$ at their roots.

Associated with any rooted tree X is a *simply presented* valuated p-group $S(X)$ defined as F_X/R_X where F_X is a free $\mathbb{Z}_p$-module with basis $\{\langle x\rangle | x \in X\}$ and R_X is generated by the elements $p\langle x\rangle - \langle px\rangle$. If we set $\overline{x} = \langle x\rangle + R_X$, then every $g \in S(X)$ has a unique presentation $g = \Sigma_{x\in X} n_x \overline{x}$ with $0 \le n_x < p$, and the valuation on $S(X)$ is defined by

$$v(g) = min\{v(x) \mid n_x \neq 0\}.$$

Finally, a valuated cyclic p-group G of order p^n is of the form $G = S(X)$ for a valuated p-tree $X = \{x_0, \ldots, x_{n-1}\}$ such that $G = \langle x_0\rangle$ and $x_i = px_{i-1}$ for $i = 1, \ldots, n$.

A map $\psi : X \to Y$ between valuated trees is a *tree map* if $\psi(px) = p\psi(x)$ if px exists and $v(\psi(x)) \ge v(x)$. A tree map $r : X \to X$ is a *retraction* if $r^2 = r$. Hunter, Richman and Walker showed that there is an order preserving retraction from $S(X)$ onto X for all valuated trees [7]. Moreover, every tree map $\psi : X \to Y$ induces a $\mathcal{V}_p$-map $\overline{\psi} : S(X) \to S(Y)$.

Corollary 2.6. *The following conditions are equivalent for a finite valuated p-group A:*

a) *A is a cyclic group.*
b) *A is an indecomposable simply presented group which is projective as an R-module.*

Proof. It remains to show that an indecomposable simply presented group A is cyclic if it is projective as an R-module. Since A is indecomposable, R is a local ring. Therefore, all projective R-modules are free. Consequently, we can find $a \in A$ such that $A = Ra$, and $ra \neq 0$ for all non-zero $r \in R$.

Write $A = S(X)$ for some valuated tree X. Since A is indecomposable, X is irretractable and has a unique element y of order p. Let $x_1, \ldots, x_n$ be the elements of maximal order of X, and select $r_1, \ldots, r_n \in R$ such that $x_i = r_i a$ for $i = 1, \ldots, n$.

If $r_1, \ldots, r_n \in J(R)$, then $A = J(R)A$ because $x_1, \ldots, x_n$ generate A as an Abelian group, which is impossible by Nakayama's Lemma. Therefore, we may, without loss of generality, assume $r_1 \notin J(R)$. Thus, r_1 is a unit in R, and

$$A = Ra = Rr_1a = Rx_1.$$

Moreover, if $sx_1 = 0$, then

$$0 = sx_1 = sr_1(r_1^{-1}x_1) = sr_1a$$

from which we obtain $sr_1 = 0$. Then $s = 0$ since r_1 is a unit of R. Therefore, $\phi(x_1) \neq 0$ for all non-zero $\phi \in R$.

Suppose that $n > 1$, and define a map $r : X \to X$ by $r(x) = 0$ if $x \neq x_2$ and $r(x_2) = y$. Observe that $v(x_2) \leq v(y)$ by the choice of x_2 and y. For $x \neq x_2$, $px \neq x_2$ because x_2 is an element of maximal order. Thus, $r(px) = 0$. On the other hand $pr(x_2) = py = 0$ while $r(px_2) = 0$ since $px_2 \neq x_2$. Therefore, r is a map of valuated trees, and induces an endomorphism α of the valuated group A with $\alpha(x_1) = 0$ and $\alpha(x_2) = y \neq 0$, a contradiction. Consequently, X has only one element x_1 of maximal order, and $A = \langle x_1 \rangle$. □

However, Corollary 2.5 shows that a simply presented group which is flat as a module over its endomorphism ring need not be a direct sum of cyclic groups. Moreover, there are infinitely many isomorphism classes of indecomposable finite valuated p-groups G such that $p^4G = 0$ and $v(g) \leq 9$ for all $0 \neq g \in G$ [3, Example 8.2.5]. Furthermore, the category of indecomposable finite valuated p-groups G such that $p^5G = 0$ and $v(g) \leq 11$ for all $0 \neq g \in G$ has wild representation type [3, Example 8.2.6].

Example 2.7. Let $A_1 = \langle a_1 \rangle$, $A_2 = \langle a_2 \rangle$ and $A_3 = \langle a_3 \rangle$ be cyclic groups of order p^3, and define a valuation on A_1 by $v(a_1) = 1$, $v(pa_1) = 4$ and $v(p^2a_1) = 5$ and on A_2 by $v(a_2) = 2$, $v(pa_2) = 3$ and $v(p^2a_2) = 5$. Finally, set $v(a_3) = \infty$.

To see that $A = A_1 \oplus A_2 \oplus A_3$ is not flat as an R-module, consider the map $\delta : A_1 \oplus A_2 \to A_3$ defined by $\delta((na_1, ma_2)) = (n - m)a_3$. It is easy to see that $K = \ker\ \delta =< (a_1, a_2) >$ and $v(a_1, a_2) = 1$, $v(pa_1, pa_2) = 3$, and $v(p^2a_1, p^2a_2) = 5$.

If $\phi \in \mathrm{Mor}(A_1, K)$, then $\phi(a_1) \in pK$ for otherwise

$$4 = v(pa_1) \leq v(\phi(pa_1)) = v(pa_1, pa_2) = 3.$$

Similarly, if $\psi \in \mathrm{Mor}(A_2, K)$, then $\psi(a_2) \in pK$ since otherwise

$$2 = v(a_2) \leq v(\psi(a_2)) = v(a_1, a_2) = 1.$$

Since $\mathrm{Mor}(A_3, A_1 \oplus A_2) = 0$, we have $im\ \theta_K \subseteq pK$, and K is not weakly A-generated. By Theorem 2.4, A is not projective as an R-module.

Example 2.8. If $A = \langle x \rangle$ is a cyclic group of order p^2 with the height valuation, then A is free as a module over its endomorphism ring $E = \mathbb{Z}/p^2\mathbb{Z}$. Moreover, $v(px) = 1$. On the other hand, $M = \mathbb{Z}/p\mathbb{Z}$ is a left E-module which fits into the exact sequence

$$E \xrightarrow{\alpha} E \xrightarrow{\beta} M \to 0$$

where $\alpha(1 + p^2\mathbb{Z}) = p + p^2\mathbb{Z}$ and $\beta(1 + p^2\mathbb{Z}) = 1 + p\mathbb{Z}$. Then $T_A(M) \cong \mathbb{Z}/p\mathbb{Z}$ and setting $v(1 + p\mathbb{Z}) = 0$ yields the cokernel valuation on $T_A(M)$. On the other hand, the map $\gamma : M \to E$ defined by $\gamma(1 + p\mathbb{Z}) = p + p^2\mathbb{Z}$ induces a monomorphism $T_A(\gamma) : T_A(M) \to A$ such that $im(T_A(\gamma)) = \langle px \rangle$. Since

$$0 = v(1 + p\mathbb{Z}) < v(px) = 1,$$

the map $T_A(\gamma)$ does not preserve valuations. If we consider the sequence

$$0 \to M \xrightarrow{\gamma} E \xrightarrow{\beta} M \to 0,$$

then $T_A(\gamma) : T_A(M) \to T_A(E)$ is not a kernel for $T_A(\beta)$.

Therefore, the class of A-solvable groups may behave quite different from the case that A is either a torsion-free or mixed Abelian group even if A is a finite valuated p-group which is projective over its endomorphism ring. For instance, the kernel of a map between two A-solvable groups need not be A-solvable, nor is a weakly A-generated subgroup U of an A-solvable group necessarily A-solvable.

Corollary 2.9. *Let A be a finite valuated p-group which is projective as an R-module. An A-generated subgroup U of an A-solvable group G is A-solvable.*

Proof. By Proposition 2.3, it remains to show that θ_U is an isomorphism in $\mathcal{V}_p$. Since A is projective as an R-module, one can argue as in the case of torsion-free groups that θ_U is an isomorphism of Abelian groups. Select an A-free group F and an A-balanced exact sequence $0 \to V \xrightarrow{\alpha} F \xrightarrow{\beta} U \to 0$. It induces the commutative diagram

$$\begin{array}{ccccccccc} & & & & T_AH_A(F) & \xrightarrow{T_AH_A(\beta)} & T_AH_A(U) & \longrightarrow & 0 \\ & & & & \wr\downarrow \theta_F & & \downarrow \theta_U & & \\ 0 & \longrightarrow & V & \xrightarrow{\alpha} & F & \xrightarrow{\beta} & U & \longrightarrow & 0. \end{array}$$

Since θ_U is an isomorphism of Abelian groups, $T_AH_A(\beta)\theta_F^{-1}\alpha = 0$. There is a $\mathcal{V}_p$-map $\lambda : U \to T_AH_A(U)$ such that $T_AH_A(\beta)\theta_F^{-1} = \lambda\beta$ because β is a cokernel of α in $\mathcal{V}_p$. Then

$$\theta_U\lambda\beta = \theta_U T_AH_A(\beta)\theta_F^{-1} = \beta$$

yields $\theta_U\lambda = 1_U$. Thus, $\lambda\theta_U = 1_{T_AH_A(U)}$ since θ_U is an isomorphism of Abelian groups. Hence

$$v(x) = v(\lambda\theta_U(x)) \geq v(\theta_U(x)) \geq v(x)$$

for all $x \in T_AH_A(U)$. Thus, θ_U is a $\mathcal{V}_p$-isomorphism. □

Corollary 2.10. *The following conditions are equivalent for a finite valuated p-group A:*

a) *A is a progenerator for ${}_R\mathcal{M}$.*

b) i) *Whenever $\phi \in \mathrm{Mor}(G, H)$ for A-solvable valuated p-groups G and H, then $\ker\ \phi$ is weakly A-generated.*

   ii) *Whenever $\phi \in \mathrm{Mor}(G, H)$ is an epimorphism of A-solvable valuated p-groups G and H, then $H_A(\phi)$ is an epimorphism.*

Proof. *a)* $\Rightarrow$ *b)*: It remains to show that ii) holds. For this, consider the submodule $M = im\ H_A(\phi)$ of $H_A(H)$, and denote the inclusion map $M \to H_A(H)$ by ι. The evaluation map $\theta : T_A(M) \to H$ is a $\mathcal{V}_p$-map since it satisfies $\theta = \theta_H T_A(\iota)$. Moreover, it is one-to-one since A is a projective as a right R-module guarantees that $T_A(\iota)$ is a monomorphism of Abelian groups and θ_H is an isomorphism. On the other hand, it also fits into the commutative diagram

$$\begin{array}{ccccc} T_AH_A(G) & \xrightarrow{T_A(\phi)} & T_A(M) & \longrightarrow & 0 \\ \wr\downarrow\theta_G & & \downarrow\theta & & \\ G & \xrightarrow{\phi} & H & \longrightarrow & 0. \end{array}$$

Hence, θ is an isomorphism of Abelian groups, and the same holds for $T_A(\iota)$. However, the latter fits into the exact sequence

$$T_A(M) \xrightarrow{T_A(\iota)} T_AH_A(H) \to H_A(H)/M \to 0.$$

Therefore, $T_A(H_A(H)/M) = 0$. Since A is a projective generator, $M = H_A(H)$.

b) $\Rightarrow$ *a)*: By [9, Proposition 2.4], every faithful projective module is a generator. Since A is a projective left R-module by Theorem 2.4, it remains to show that it is faithful. Let M be a right R-module with $t_A(M) = 0$, and consider an exact sequence $P \to F \to M \to 0$ in which P and F are projective module. By Theorem 2.2, we obtain a right exact sequence $T_A(P) \to T_A(F) \to 0$ of valuated p-groups. By ii), the top sequence in the diagram

$$\begin{array}{ccccccc} H_AT_A(P) & \longrightarrow & H_AT_A(F) & \longrightarrow & 0 & & \\ \wr\uparrow\Phi_P & & \wr\uparrow\Phi_F & & & & \\ P & \longrightarrow & F & \longrightarrow & M & \longrightarrow & 0 \end{array}$$

is exact. Thus, $M = 0$. □

3. Hereditary and Quasi-Frobenius Endomorphism Rings

We conclude our discussion by considering finite valuated p-groups A whose endomorphism ring has specific ring-theoretic properties. We focus particularly on the cases that R is either hereditary or self-injective. We want to remind the reader that there is no need to deal with right/left conditions since R is finite [4].

A finite valuated p-group G is *A-torsion-less* if there is a monomorphism $G \to A^\ell$ for some $\ell < \omega$. We say that an exact sequence of valuated groups is *A-cobalanced* if A is injective with respect to it.

Theorem 3.1. *Let R be a finite valuated p-group A:*

a) *R is hereditary if and only if A is a direct sum of cyclic groups of order p.*
b) *R is (semi-)simple Artinian if and only if $A \cong B^m$ where B is a cyclic group of order p.*

c) *If R is a quasi-Frobenius ring, then every exact sequence $0 \to U \to G$ in which U is weakly A-generated and G is A-solvable is A-cobalanced. If A is a projective R-module, then the converse holds, and every A-presented group is A-torsionless.*

Proof. a) If R is hereditary, then so is eRe for any idempotent e of R [4]. If B is an indecomposable summand of A, then there is a primitive idempotent e of R such that eRe is the $\mathcal{V}_p$-endomorphism ring of B. Since eRe is a hereditary local ring, all right ideals of eRe are free eRe-modules. However, this means that eRe is a field since it is finite. Because, $pE(B)$ is a proper ideal of $E(B)$, we have $pB = 0$. By [8], B is a cyclic group. Hence, A is a direct sum of cyclic groups of order p.

Conversely, if A has the described form, then $A = A_1 \oplus \ldots \oplus A_n$ where $A_i \cong B_i^{\ell_i}$ and each B_i is a cyclic group of order p. If $B_i = \langle b_i \rangle$, then no generality is lost if we assume $v(b_i) < v(b_j)$ for $i < j$ and $v(b_i) \neq \infty$ for $i < n$. Then $\mathrm{Mor}(B_i, B_j) \cong \mathbb{Z}/p\mathbb{Z}$ if $i \leq j$, and $\mathrm{Mor}(B_i, B_j) = 0$ otherwise. Therefore, R is Morita-equivalent to a lower triangular matrix ring over $\mathbb{Z}/p\mathbb{Z}$. By [5], R is hereditary.

b) We continue using the notation from a). If $A = A_1 \oplus \ldots \oplus A_n$ and $n > 1$, then $\mathrm{Mor}(A_i, A_j) = 0$ for $i > j$, but $\mathrm{Mor}(A_i, A_j) \neq 0$ for $i < j$. In particular, $N(R) \neq 0$. b) now follows immediately.

c) If R is quasi-Frobenius, then we consider an exact sequence $0 \to U \xrightarrow{\alpha} G$ in which U is an epimorphic image of an A-projective group and G is A-solvable. For $\phi \in \mathrm{Mor}(U, A)$, we can find a map $\psi : H_A(G) \to R$ such that $\psi H_A(\alpha) = \phi$. Since both, α and ϕ, fit into the commutative diagram

$$\begin{array}{ccc} T_A H_A(U) & \xrightarrow{T_A H_A(.)} & T_A H_A(G) \\ \Big\downarrow \theta_U & & \wr \Big\downarrow \theta_G \\ U & \xrightarrow{\;\cdot\;} & G, \end{array}$$

we obtain

$$T_A(\psi)\theta_G^{-1}\alpha\theta_U = \theta_A T_A(\psi) T_A H_A(\alpha) = \theta_A T_A H_A(\phi) = \phi\theta_U.$$

Because θ_U is a $\mathcal{V}_p$-epimorphism, $T_A(\psi)\theta_G^{-1}\alpha = \phi$.

Conversely, let

$$0 \to I \xrightarrow{\alpha} R$$

be an exact sequence and $\phi \in \mathrm{Hom}_R(I, R)$. Because A is a flat R-module,

$$0 \to T_A(I) \xrightarrow{T_A(\alpha)} T_A(R)$$

is a $\mathcal{V}_p$-exact sequence. Since $T_A(I)$ is an image of an A-projective group, there is a map $\psi \in \mathrm{Mor}(T_A(R), T_A(R))$ such that $\psi T_A(\alpha) = T_A(\phi)$. We consider commutative diagrams of the form

$$\begin{array}{ccccc} 0 & \longrightarrow & H_A T_A(I) & \xrightarrow{H_A T_A(.)} & H_A T_A(R) \\ & & \Big\uparrow \Phi_I & & \wr \Big\uparrow \Phi_R \\ 0 & \longrightarrow & I & \xrightarrow{\;\cdot\;} & R \end{array}$$

to obtain

$$\begin{aligned} \Phi_R^{-1} H_A(\psi)\Phi_R \alpha &= \Phi_R^{-1} H_A(\psi) H_A T_A(\alpha)\Phi_I \\ &= \Phi_R^{-1} H_A T_A(\phi)\Phi_I \\ &= \Phi_R^{-1}\Phi_R \phi = \phi. \end{aligned}$$

Finally, if G is an A-presented group, then $G \cong T_A(M)$ for some finitely generated right R-module M by [2] as mentioned before. Let E be an injective hull of M. Since R is quasi-Frobenius, E is projective. Thus, M can be embedded into a free R-module F, which can be chosen to be finite since M is finite. Then $T_A(M)$ is isomorphic to a submodule of $T_A(F)$ since A is projective. □

Corollary 3.2. *Let A be a finite valuated p-group whose endomorphism ring is self-injective. Every exact sequence*

$$0 \to P \xrightarrow{\alpha} G$$

such that P is A-projective and G is A-solvable splits.

□

We conclude with two examples that show that the endomorphism ring of a direct sum of cyclic valuated p-groups may or may not be quasi-Frobenius:

Example 3.3. a) Let A_1 be a cyclic group of order p^n, and A_2 a cyclic valuated group of order p^n whose generator x satisfies $v(p^{n-1}x) > n$. Then, the endomorphism ring of $A = A_1 \oplus A_2$ is the lower triangular matrix ring over $\mathbb{Z}/p^n\mathbb{Z}$, which is not self-injective.

b) By [6, Example 1], the ring

$$R = \begin{bmatrix} \mathbb{Z}/p^3\mathbb{Z} & p\mathbb{Z}/p^3\mathbb{Z} \\ p\mathbb{Z}/p^3\mathbb{Z} & \mathbb{Z}/p^3\mathbb{Z} \end{bmatrix}$$

is quasi-Frobenius. Consider two cyclic valuated groups $A_1 = (\langle x_1 \rangle, v_1)$ and $A_2 = (\langle x_2 \rangle, v_2)$ of order p^3 such that $v_1(x_1) = 1$, $v_1(px_1) = 4$, $v_2(x_2) = 2$, $v_2(px_2) = 3$ and $v_1(p^2x_1) = v_2(p^2x_2) \geq 5$. In view of the fact that $\mathrm{Mor}(A_i, A_j) \cong \mathbb{Z}/p^2\mathbb{Z}$ for $i \neq j$, we obtain that $A = A_1 \oplus A_2$ has R as its $\mathcal{V}_p$-endomorphism ring.

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