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# General decay rates of the solution energy in a viscoelastic wave equation with boundary feedback and a nonlinear source

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**Abstract.** In a bounded domain, we consider a viscoelastic equation

$$u_{tt} - \Delta u + \int_0^t g(t-\tau)\Delta u(\tau)d\tau = |u|^\gamma u$$

with a nonlinear feedback localized on a part of the boundary, where $\gamma > 0$ and the relaxation function $g$ satisfied $g'(t) \leq \xi(t)g^p(t),\ 1 \leq p < \frac{3}{2}$, and certain initial data. We establish an explicit and general decay rate result, using some properties of the convex functions. Our new results substantially improve several earlier related results in the literature.

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**Keywords:** General decay, nonlinear source, viscoelastic, wave equation, relaxation function.

## 1. Introduction

In this paper, we are concerned with the energy decay rate of the following viscoelastic problem with nonlinear boundary dissipation and a nonlinear source

$$\begin{cases} u_{tt} - \Delta u + \int_0^t g(t-\tau)\Delta u(\tau)d\tau = |u|^\gamma u, & in\ \ \Omega \times (0,\infty) \\ u = 0, & on\ \Gamma_0 \times (0,\infty) \\ \frac{\partial u}{\partial \nu} - \int_0^t g(t-\tau)\frac{\partial u}{\partial \nu}(\tau)d\tau + h(u_t) = 0, & on\ \Gamma_1 \times (0,\infty) \\ u(x,0) = u_0(x);\ u_t(x,0) = u_1(x), & x \in \Omega \end{cases} \tag{1.1}$$

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where $\Omega$ is a bounded domain of $\mathbb{R}^n$ with a smooth boundary $\partial\Omega = \Gamma_0 \cup \Gamma_1$. Here, $\Gamma_0$ and $\Gamma_1$ are closed and disjoint, with meas$(\Gamma_0) > 0$, $\nu$ is the unit outward normal to $\partial\Omega$, $\gamma > 0$, and $g$, $h$ are specific functions.

Let us mention some known results related to the viscoelastic problem with nonlinear boundary dissipation. In [7], Cavalcanti and al. considered the following problem

$$\begin{cases} u_{tt} - \triangle u + \int_0^t g\,(t-s) \triangle u\,(s)\,ds = 0, & \text{in } \Omega \times (0,\infty) \\ \frac{\partial u}{\partial \nu} - \int_0^t g\,(t-s)\,\frac{\partial u}{\partial \nu}\,(s)\,ds + h\,(u_t) = 0, & \text{on } \Gamma_1 \times (0,\infty) \\ u\,(x,t) = 0, & \text{on } \Gamma_0 \times (0,\infty) \\ u\,(x,0) = u_0, \;\; u_t\,(x,0) = u_1, & x \in \Omega. \end{cases} \tag{1.2}$$

The existence and uniform decay rate results were established under quite restrictive assumptions on damping term $h$ and the kernel function $g$. Later, Cavalcanti and al. [6] generalized this result without imposing a growth condition on $h$ and under a weaker assumption on $g$. Recently, Messaoudi and Mustafa [18] exploited some properties of convex functions [2] and the multiplier method to extend these results. They established an explicit and general decay rate result without imposing any restrictive growth assumption on the damping term $h$ and greatly weakened the assumption on $g$. Also, Li et al [11] have analyzed the global existence and decay estimates for nonlinear viscoelastic wave equation with boundary dissipation. They established uniform decay rate of the energy under suitable conditions on the initial data and the relaxation function g. Let us also mention other papers in connection with viscoelastic effects such as Dafermos [8] [9], Mustafa MI [22], Lagnese [10], Aassila et al. [1]. On considering the boundary dissipation,we refer the reader to related works Mohammad M. Al-Gharabli [3], [20], [21], [23] and the references therein.

In a situation in which a source term is competing with the viscoelastic dissipation, many authors have established stability results. For example, Messaoudi [16] looked at

$$\begin{cases} u_{tt} - \Delta u + \int_0^t g(t-\tau)\Delta u(\tau)d\tau = |u|^\gamma u, & in \;\; \Omega \times (0,\infty) \\ u = 0, & on\; \partial\Omega \times (0,\infty) \\ u(x,0) = u_0(x); \;\; u_t(x,0) = u_1(x), & x \in \Omega, \end{cases}$$

where $\Omega$ is a bounded domain in $\mathbb{R}^n$ $(n \geq 1)$ with a smooth boundary, $\gamma > 0$, and the relaxation function $g$ is a positive and uniformly decaying function satisfies a relation of the form

$$g'(t) \leq -\xi(t)g(t), \tag{1.3}$$

where $\xi$ is a nonincreasing differentiable function such that

$$\left|\frac{\xi'(t)}{\xi(t)}\right| \leq k, \quad \xi(t) > 0, \;\; \xi'(t) \leq 0, \;\; \forall t > 0, \qquad \int_0^\infty \xi(t)dt = +\infty.$$

He established a more general decay result, from which the usual exponential and polynomial decay rates are only special cases.

In a situation in which a source term is competing with the viscoelastic dissipation and on considering the boundary dissipation, Shun and Hsueh [24] considered

$$
\begin{gathered}
u_{tt} - k_0 \triangle u(t) + \int_0^t g(t-s)\, div\,(a(x)\nabla u(s))\, ds + b(x)\, u_t = f(u)\,, \text{ in } \Omega \times (0,\infty)\,, \\
k_0 \frac{\partial u}{\partial \nu} - \int_0^t g(t-s)\,(a(x)\nabla u(s))\,.\nu ds + h(u_t) = 0, \text{on } \Gamma_1 \times (0,\infty)\,, \\
u(x,t) = 0, \text{ on } \Gamma_0 \times (0,\infty)\,, \\
u(x,0) = u_0, \quad u_t(x,0) = u_1, \; x \in \Omega,
\end{gathered}
$$

where $\Omega$ is a bounded domain in $\mathbb{R}^n$ $(n \geq 1)$ with a smooth boundary, the relaxation function $g$ is a positive and uniformly decaying function satisfying (1.3), and where $\xi$ is a nonincreasing differentiable positive function such that

$$
\int_0^\infty \xi(t)dt = +\infty.
$$

The authors established the general decay rate of the solution energy which is not necessarily of exponential or polynomial type. Another problems, in which in which a source term is competing with the viscoelastic dissipation and on considering the boundary dissipation, were discussed in [5], [14], [12] and [13], and the existence, uniform decay rate results were established.

In this article, we devote ourselves to the study of the problem (1.1). Motivated by previous work and by the idea of Messaoudi and Mustafa [17], which considers a wider class of relaxation functions $g$, we obtain a more general and explicit energy decay formula, to from which the exponential and the polynomial decay rates are only special cases of our result. In fact, our decay formulas extend and improve some results of the literature.

## 2. Preliminaries

In this section we prepare some material needed in the proof of our result. We have the imbedding: $H^1_{\Gamma_0} \hookrightarrow L^{2(\gamma+1)}(\Omega)$. Let $C_e > 0$ be the optimal constant of Sobolev imbedding which satisfies the following inequality:

$$
\|u\|_{2(\gamma+1)} \leq C_e \|\nabla u\|_2\,, \quad \forall u \in H^1_{\Gamma_0}, \tag{2.1}
$$

and we use the trace-Sobolev imbedding: $H^1_{\Gamma_0} \hookrightarrow L^k(\Gamma_1)\,,\ 1 \leq k < \frac{2(n-1)}{n-2}$. In this case, the imbedding constant is denoted by $B_1$, that is

$$
\|u\|_{k,\Gamma_1} \leq B_1 \|\nabla u\|_2. \tag{2.2}
$$

Next, we state the assumptions for problem (1.1) as follows.

For the relaxation function $g$ we assume the following:
$(G_1)$ $g : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ is a nonincreasing $C^1$ function satisfying

$$
g(0) > 0, \qquad 1 - \int_0^\infty g(s)ds = l > 0.
$$

$(G_2)$ There exists a nonincreasing differentiable function $\xi : \mathbb{R}_+ \to \mathbb{R}_+$, with $\xi(0) > 0$, and satisfying

$$g'(t) \le \xi(t) g^p(t), \quad 1 \le p < \frac{3}{2}, \quad t \ge 0.$$

$(G_3)$ For the nonlinear term, we assume

$$0 < \gamma \le \frac{2}{(n-2)}, \quad n \ge 3$$
$$\gamma > 0, \quad n = 1, 2.$$

$(G_4)$ $h : \mathbb{R} \longrightarrow \mathbb{R}$ is a nondecreasing $C^0$ function such that there exist a strictly increasing function $h_0 \in C^1([0, +\infty))$, with $h_0(0) = 0$, and positive constants $c_1$, $c_2$, $\epsilon$ such that

$$h_0(\mid s \mid) \le \mid h(s) \mid \le h_0^{-1}(\mid s \mid) \quad for\ all \ \mid s \mid \le \epsilon$$
$$c_1 \mid s \mid \le \mid h(s) \mid \le c_2 \mid s \mid \qquad for\ all \ \mid s \mid \ge \epsilon.$$

In addition, we assume that the function $H$, defined by $H(s) = \sqrt{s} h_0(\sqrt{s})$, is a strictly convex $C^2$ function on $(0, r^2]$, for some $r > 0$, when $h_0$ is nonlinear.

By using the Galerkin method and procedure similar to that of [11], and [23], we have the following local existence result for problem (1.1).

**Theorem 2.1.** *Let hypotheses* $(G1)$-$(G4)$ *hold and assume that* $u_0 \in H^1_{\Gamma_0} \cap H^2(\Omega)$, $u_1 \in H^1_{\Gamma_0}$. *Then there exists a strong solution* $u$ *of* (1.1) *satisfying*

$$\begin{aligned} u &\in L^\infty\left([0,T); H^1_{\Gamma_0} \cap H^2(\Omega)\right) \\ u_t &\in L^\infty\left([0,T); H^1_{\Gamma_0}\right) \\ u_{tt} &\in L^\infty\left([0,T); L^2(\Omega)\right), \end{aligned}$$

*for some* $T > 0$.

**Proposition 2.2.** *Suppose that* $(G1)$, $(G3)$ *and* $(G4)$ *hold. Let* $(u_0, u_1) \in V \times L^2(\Omega)$ *be given, satisfying* (2.7). *Then the solution* $u$ *of* (1.1) *is global and bounded.*

We introduce the following functionals

$$\begin{aligned} J(t) &= \frac{1}{2}\left(k_1 - \int_0^t g(s)\, ds\right) \|\nabla u\|_2^2 + \frac{1}{2}(g \circ \nabla u)(t) - \frac{1}{\gamma+2}\|u\|_{\gamma+2}^{\gamma+2} \\ E(t) &= J(u(t)) + \frac{1}{2}\|u_t\|_2^2, \quad \text{for } t \in [0,T) \qquad (2.3) \\ I(t) &= I(u(t)) = \left(k_1 - \int_0^t g(s)\, ds\right)\|\nabla u\|_2^2 + (g \circ \nabla u)(t) - \|u\|_{\gamma+2}^{\gamma+2}, \quad (2.4) \end{aligned}$$

where

$$(g \circ v)(t) = \int_0^t g(t-s)\|v(t) - v(s)\|_2^2\, ds, \tag{2.5}$$

and $E(t)$ is the energy functional.

A direct differentiation, using (1.1), leads to

$$E'(t) = \frac{1}{2}(g' \circ \nabla u)(t) - \frac{1}{2} g(t) \parallel \nabla u(t) \parallel_2^2 - \int_{\Gamma_1} u_t(t) h(u_t(t)) d\Gamma \le 0, \tag{2.6}$$

For completeness, using similar procedure in [15], we state the global existence result.

**Lemma 2.3.** *Suppose that* $(G1)$ *and* $(G3)$ *hold, and* $(u_0, u_1) \in V \times L^2(\Omega)$, *such that*

$$\beta = \frac{C_e^{\gamma+2}}{l} \left( \frac{2(\gamma+2)}{\gamma l} E(u_0, u_1) \right)^{\gamma/2} < 1 \tag{2.7}$$

$$I(u_0) > 0,$$

*then* $I(u(t)) > 0$, $\forall t > 0$.

**Proposition 2.4.** *Suppose that* $(G1)$, $(G3)$ *and* $(G4)$ *hold. Let* $(u_0, u_1) \in V \times L^2(\Omega)$ *be given, satisfying* (2.7). *Then the solution* $u$ *of* (1.1) *is global and bounded.*

Adopting the proof of [17], we have the following results which are crucial for the proof of our main result.

**Lemma 2.5.** *Assume that* $g$ *satisfies* $(G1)$ *and* $(G2)$ *then*

$$\int_0^{+\infty} \xi(t)\, g^{1-\sigma}(t)\, dt < +\infty, \quad \forall \sigma < 2 - p.$$

**Lemma 2.6.** *Assume that* $g$ *satisfies* $(G1)$ *and* $(G2)$, *and* $u$ *is the solution of* (1.1) *then, for* $0 < \delta < 1$, *we have*

$$(g \circ \nabla u)(t) \leq C \left[ \left( \int_0^{+\infty} g^{1-\sigma}(t)\, dt \right) E(0) \right]^{\frac{p-1}{p-1+\delta}} (g^p \circ \nabla u)^{\frac{\delta}{p-1+\delta}}(t).$$

*By taking* $\delta = \frac{1}{2}$, *we get*

$$(g \circ \nabla u)(t) \leq C \left[ \int_0^t g^{\frac{1}{2}}(s)\, ds \right]^{\frac{2p-2}{2p-1}} (g^p \circ \nabla u)^{\frac{1}{2p-1}}(t). \tag{2.8}$$

**Corollary 2.7.** *Assume that* $g$ *satisfies* $(G1)$ *and* $(G2)$, *and* $u$ *is the solution of* (1.1) *then*

$$\xi(t)(g \circ \nabla u)(t) \leq C \left[ -E'(t) \right]^{\frac{1}{2p-1}}. \tag{2.9}$$

If $G$ is a convex function on $[a, b]$, ($-G$ is convex), $f : \Omega \to [a, b]$ and $h$ are integrable functions on $\Omega$, with $h(x) \geq 0$ and $\int_\Omega h(x)\, dx = k > 0$, then Jensen's inequality states that

$$\frac{1}{k} \int_\Omega G[f(x)]\, h(x)\, dx \leq G\left[ \frac{1}{k} \int_\Omega f(x)\, h(x)\, dx \right]. \tag{2.10}$$

For the special case $G(y) = y^{\frac{1}{q}}$, $y \geq 0$, $p > 1$, we have

$$\frac{1}{k} \int_\Omega [f(x)]^{\frac{1}{q}}\, h(x)\, dx \leq \left[ \frac{1}{k} \int_\Omega f(x)\, h(x)\, dx \right]^{\frac{1}{q}}.$$

## 3. Decay of solutions

In this section we state and prove the main result of our work. For this purpose, we adopt the following result from [24] without proof.

**Lemma 3.1.** *There exist positive constants $\varepsilon_1$, $\varepsilon_2$, $m$, $t_0$ such that the fun*

$$F(t) := E(t) + \varepsilon_1\psi_1(t) + \varepsilon_2\psi_2(t), \tag{3.1}$$

*is equivalent to $E$ and satisfies*

$$F'(t) \leq -mE(t) + c\int_{\Gamma_1} h^2(u_t)d\tau + c(g \circ \nabla u)(t). \tag{3.2}$$

*where*

$$\psi_1(t) := \int_\Omega uu_t dx, \tag{3.3}$$

$$\psi_2(t) := -\int_\Omega u_t \int_0^t g(t-\tau)(u(t)-u(\tau))d\tau dx.$$

**Lemma 3.2.** [19] *Under the assumptions $(G1)$, $(G2)$ and $(G4)$, the solution satisfies the estimates*

$$\int_{\Gamma_1} h^2(u_t)d\Gamma \leq \int_{\Gamma_1} u_t h(u_t)d\Gamma, \qquad \text{if } h_0 \text{ is linear} \tag{3.4}$$

$$\int_{\Gamma_1} h^2(u_t)d\Gamma \leq cH^{-1}(J(t)) - cE'(t), \qquad \text{if } h_0 \text{ is nonlinear} \tag{3.5}$$

*where*

$$J(t) = \frac{1}{\mid \Gamma_{12} \mid}\int_{\Gamma_{12}} u_t h(u_t)d\Gamma \leq E'(t),$$

*and*

$$\Gamma_{12} = \{x \in \Gamma_1 : |u_t| \leq \varepsilon_1\}.$$

*Proof.* **Case 1:** $h_0$ is linear, using $(G4)$ we have

$$c_1' \left|u_t\right| \leq \mid h(u_t) \mid \leq c_2' \left|u_t\right|,$$

and hence

$$h^2(u_t) \leq c_2' u_t h(u_t).$$

So, (3.4) is established.

**Case 2:** $h_0$ is nonlinear on $[0, \varepsilon]$:

First, we assume that $\max\{r, h_0(r)\} < \varepsilon$; otherwise we take $r$ smaller. Let $\varepsilon_0 = \min\{r, h_0(r)\}$; them for $\varepsilon_0 \leq\mid s \mid\leq \varepsilon$, using $(G4)$, we have

$$\mid h(s) \mid\leq \frac{h_0^{-1}(\mid s \mid)}{\mid s \mid} \mid s \mid\leq \frac{h_0^{-1}(\varepsilon)}{\varepsilon_0} \mid s \mid \quad \textit{and} \quad \mid h(s) \mid\geq \frac{h_0(\mid s \mid)}{\mid s \mid} \mid s \mid\geq \frac{h_0(\varepsilon_0)}{\varepsilon} \mid s \mid,$$

so, we conclude that

$$\begin{cases} h_0(\mid s \mid) \leq\mid h(s) \mid\leq h_0^{-1}(\mid s \mid) & \text{for all } \mid s \mid< \varepsilon_0 \\ c_1' \mid s \mid\leq\mid h(s) \mid\leq c_2' \mid s \mid & \text{for all } \mid s \mid\geq \varepsilon_0. \end{cases} \tag{3.6}$$

Since $H(s^2) = \mid s \mid h_0(\mid s \mid)$, then using (3.6), we obtain

$$H(h^2(s)) \leq sh(s) \quad for\ all \ \mid s \mid \leq \varepsilon_0,$$

which gives

$$h^2(s) \leq H^{-1}(sh(s)) \quad for\ all \ \mid s \mid \leq \varepsilon_0.$$

To estimate the last integral in (3.2), we consider the following partition of $\Gamma_1$:

$$\Gamma_{11} = \{x \in \Gamma_1 : |u_t| > \varepsilon_0\}, \qquad \Gamma_{12} = \{x \in \Gamma_1 : |u_t| \leq \varepsilon_0\}.$$

Recalling the definition of $\varepsilon_0$ and using (3.6), we obtain on $\Gamma_{12}$,

$$u_t h(u_t) \leq \varepsilon_0 h_0^{-1}(\varepsilon_0) \leq h_0(r)r = H(r^2) \tag{3.7}$$

and

$$u_t h(u_t) \leq \varepsilon_0 h_0^{-1}(\varepsilon_0) \leq r h_0^{-1} h_0(r) = r^2.$$

Jensen's inequality gives

$$H^{-1}(J(t)) \geq c \int_{\Gamma_{12}} H^{-1}(u_t h(u_t)) d\Gamma. \tag{3.8}$$

Thus, using (3.6) – (3.8), we get

$$\begin{aligned}
\int_{\Gamma_1} h^2(u_t) d\Gamma &= \int_{\Gamma_{12}} h^2(u_t) d\Gamma + \int_{\Gamma_{11}} h^2(u_t) d\Gamma \\
&\leq \int_{\Gamma_{12}} H^{-1}(u_t h(u_t)) d\Gamma + c \int_{\Gamma_{11}} u_t h(u_t) d\Gamma \\
&\leq c H^{-1}(J(t)) - cE'(t).
\end{aligned} \tag{3.9}$$

□

**Theorem 3.3.** *Let* $(u_0, u_1) \in \left(H^1_{\Gamma_0} \times L^2(\Omega)\right)$ *be given . Assume that* $(G1)$*-*$(G4)$ *are satisfied and* $h_0$**is linear**. *Then, for any* $t_0 > 0$*, there exist two positive constants* $K$*, and* $\lambda$ *such that the solution of* (1.1) *satisfies, for all* $t \geq t_0$*,*

$$E(t) \leq K e^{-\lambda \int_{t_0}^t \xi(s) ds}, \qquad if\ p = 1. \tag{3.10}$$

$$E(t) \leq K \left[\frac{1}{1 + \int_{t_0}^t \xi^{2p-1}(s)\, ds}\right]^{\frac{1}{2p-2}}, \quad 1 < p < \frac{3}{2}. \tag{3.11}$$

*Moreover, if*

$$\int_0^{+\infty} \left[\frac{1}{t\xi^{2p-1}(t) + 1}\right]^{\frac{1}{2p-2}} dt < +\infty, \quad 1 < p < \frac{3}{2}, \tag{3.12}$$

*then*

$$E(t) \leq K \left[\frac{1}{1 + \int_{t_0}^t \xi^{p}(s)\, ds}\right]^{\frac{1}{p-1}}, \qquad 1 < p < \frac{3}{2}. \tag{3.13}$$

*Proof.* Multiplying (3.2) by $\xi(t)$ and using Eqs. 3.4, we get

$$\begin{aligned}\xi(t)F'(t) &\leq -m\xi(t)E(t)+c\xi(t)(g\circ\nabla u)(t)+c\xi(t)\int_{\Gamma_1}h^2(u_t)d\tau\\ &\leq -m\xi(t)E(t)+c\xi(t)(g\circ\nabla u)(t)-c\xi(t)E'(t)\end{aligned}$$

which gives, as $\xi(t)$ is non-increasing,

$$(\xi\mathcal{F}+CE)'(t)\leq -m\xi(t)E(t)+c\xi(t)(g\circ\nabla u)(t),\quad \forall t\geq t_0. \tag{3.14}$$

Let $L(t):=\xi(t)\mathcal{F}(t)+CE(t)$, then clearly $L\sim E$ and we have, for some $m_1>0$,

$$L'(t)\leq -m_1\xi(t)L(t)+c\xi(t)(g\circ\nabla u)(t),\ \forall t\geq t_0.$$

Now, using the procedure similar to that of [17], we obtain the results of the theorem. □

**Theorem 3.4.** *Let* $(u_0,u_1)\in V\times L^2(\Omega)$ *be given, satisfying* (2.7). *Assume that* $(G1)-(G4)$ *hold and* $h_0$ **is nonlinear**. *Then there exist positive constants* $k_1$, $k_2$ *and* $k_3$*such that the solution of* (1.1) *satisfies , for all* $t\geq t_0$,

$$E(t)\leq k_3H_1^{-1}\left(k_1\int_{t_0}^{t}\xi(s)ds+k_2\right),\qquad p=1. \tag{3.15}$$

$$E(t)\leq k_3H_1^{-1}\left(k_1\int_{t_0}^{t}\xi^{2p-1}(s)ds+k_2\right),\quad 1<p<\frac{3}{2}. \tag{3.16}$$

Moreover, if

$$\int_0^{+\infty}H_1^{-1}\left(k_1t\xi^{2p-1}(t)+k_2\right)dt<+\infty,\quad 1<p<\frac{3}{2}, \tag{3.17}$$

then

$$E(t)\leq k_3H_2^{-1}\left(k_1\int_{t_0}^{t}\xi^{p}(s)ds+k_2\right),\qquad 1<p<\frac{3}{2}, \tag{3.18}$$

where $H_1(t)=\int_t^1\frac{1}{t^{2p-1}H'(\varepsilon_0t))}ds.$ and where $H_2(t)=\int_t^1\frac{1}{t^{2p-1}H'(\varepsilon_0t))}ds.$
Here, $H_1$ and $H_2$ are strictly decreasing and convex on $(0,1]$, with $\lim\limits_{t\longrightarrow 0}H_i(t)=+\infty$, $i=1,2$.

Simple calculations show that (3.16) and (3.17) yield

$$\int_{t_0}^{+\infty}E(t)\,dt<+\infty.$$

*Proof.* **Case of** $p=1$. Recalling $G(2)$ and (2.6), Multiplying (3.2) by $\xi(t)$, we obtain, for all $t\geq t_0$

$$\begin{aligned}\xi(t)\mathcal{F}'(t) &\leq -m\xi(t)E(t)+C\left(\xi(t)g\circ\nabla u\right)(t)+c\xi(t)\int_{\Gamma_1}h^2(u_t)d\tau \qquad (3.19)\\ &\leq -m\xi(t)E(t)-C\left(g'\circ\nabla u\right)(t)+c\xi(t)\int_{\Gamma_1}h^2(u_t)d\tau\\ &\leq -m\xi(t)E(t)-CE'(t)+c\xi(t)\int_{\Gamma_1}h^2(u_t)d\tau,\end{aligned}$$

which leads to

$$(\xi\mathcal{F}+CE)'(t) \leq -m\xi(t)E(t) + c\xi(t)\int_{\Gamma_1} h^2(u_t)d\tau, \quad \forall t \geq t_0. \tag{3.20}$$

Let $L(t) := \xi(t)\mathcal{F}(t) + CE(t)$, then clearly $L \sim E$ and we have, for some $m_1 > 0$,

$$L'(t) \leq -m_1\xi(t)L(t) + c\xi(t)\int_{\Gamma_1} h^2(u_t)d\tau, \ \forall t \geq t_0.$$

Now, using the procedure similar to that of [19], we obtain the results of the theorem.

**Case of** $1 < p < \dfrac{3}{2}$.

Multiplying (3.2) by $\xi(t)$ and we using 2.7, we obtain

$$\xi(t)F'(t) \leq -m\xi(t)E(t) + c\xi(t)\int_{\Gamma_1} h^2(u_t)d\tau + k(-E'^{\frac{1}{2p-1}}(t)),$$

multiplying by $\xi^{2p-2}(t)E^{2p-2}(t)$ and using Young's inequality

$$\begin{aligned}\xi^{2p-1}(t)E^{2p-2}(t)F'(t) &\leq -m\xi^{2p-1}(t)E^{2p-1}(t) + c\xi^{2p-1}(t)E^{2p-2}(t)\int_{\Gamma_1} h^2(u_t)d\tau \\ &\quad + k\left(-E'(t)\right)^{\frac{1}{2p-1}}(t)\xi^{2p-2}(t)E^{2p-2}(t) \\ &\leq -m\xi^{2p-1}(t)E^{2p-1}(t) + c\xi^{2p-1}(t)E^{2p-2}(t)\int_{\Gamma_1} h^2(u_t)d\tau \\ &\quad + k(-E'^{2p-1}(t)E^{2p-1}(t) \\ F_2'(t) &\leq k_1\xi^{2p-1}(t)E^{2p-1}(t) + c\xi^{2p-1}(t)E^{2p-2}(t)\int_{\Gamma_1} h^2(u_t)d\tau.\end{aligned} \tag{3.21}$$

With $F_2(t) = F(t)\xi^{2p-1}(t)E^{2p-2}(t) + kE(t); \quad F_0 \sim E$.
Therefore, using (3.5), (2.5) becomes

$$F_2'(t) \leq k_1\xi^{2p-1}(t)E^{2p-1}(t) + c\xi^{2p-1}(t)E^{2p-2}(t)\left(H^{-1}(\lambda(t)) - E'(t)\right)$$
$$F_2'(t) \leq k_1\xi^{2p-1}(t)E^{2p-1}(t) + c\xi^{2p-1}(t)E^{2p-2}(t)H^{-1}(\lambda(t)) - c\xi^{2p-1}(0)E^{2p-2}(0)E'(t)$$
$$F_3'(t) \leq k_1\xi^{2p-1}(t)E^{2p-1}(t) + c\xi^{2p-1}(t)E^{2p-2}(t)H^{-1}(\lambda(t))$$

with $F_3 = F_2 + CE$ then, $F_3 \sim E$.
Now, for $\epsilon_0 < r^2$ and $c_0 > 0$, using (3.9) and the fact that $E' \leq 0, \ H' \geq 0, \ H'' \geq 0$ on $(0, r^2]$, we find that the functional $F_2$ defined by

$$F_4(t) := H'\left(\epsilon_0\frac{E(t)}{E(0)}\right)F_2(t) + c_0E(t)$$

satisfies, for some $\alpha_1, \alpha_2 > 0$,

$$\alpha_1F_4(t) \leq E(t) \leq \alpha_2F_4(t) \tag{3.22}$$

and

$$F_4'(t) = \epsilon_0 \frac{E'(t)}{E(0)} H''\left(\epsilon_0 \frac{E(t)}{E(0)}\right) F_2(t) + H'\left(\epsilon_0 \frac{E(t)}{E(0)}\right) F_2'(t) + c_0 E'(t)$$
$$\leq -k\xi^p E^p(t) H'\left(\epsilon_0 \frac{E(t)}{E(0)}\right) + c\xi^p(t) E^{p-1}(t) H^{-1}(\lambda(t)) H'\left(\epsilon_0 \frac{E(t)}{E(0)}\right) + c_0 E'(t). \tag{3.23}$$

Let $H^*$ be the convex conjugate of $H$ in the sense of young (see [4] p. 61 – 64); then

$$H^*(s) = s\left(H'\right)^{-1}(s) - H\left[\left(H'\right)^{-1}(s)\right], \quad if\ s \in \left[0, H'^2)\right]$$

and $H^*$ satisfies the following Young's inequality:

$$AB \leq H^*(A) + H(B), \quad if\ A \in \left(0, H'^2)\right],\ B \in \left(0, r^2\right]. \tag{3.24}$$

With $A = H'\left(\epsilon_0 \frac{E(t)}{E(0)}\right)$ and $B = H^{-1}(\lambda(t))$, using (2.6), (3.7) and (3.23) – (3.24), we arrive at

$$\begin{aligned} F_4'(t) \quad \leq \quad & -k\xi^{2p-1}E^{2p-1}(t)H'\left(\epsilon_0 \frac{E(t)}{E(0)}\right) + c\xi(t)\lambda(t) \\ & +c\xi^{2p-1}(t)E^{2p-2}(t)H^*\left(H'\left(\epsilon_0 \frac{E(t)}{E(0)}\right)\right) + c_0 E'(t), \end{aligned}$$

that gives

$$\begin{aligned} F_4'(t) \quad \leq \quad & -k\xi^{2p-1}E^{2p-1}(t)H'\left(\epsilon_0 \frac{E(t)}{E(0)}\right) + c\epsilon_0\xi^{2p-1}(t)\frac{E^{2p-1}(t)}{E(0)} H'\left(\epsilon_0 \frac{E(t)}{E(0)}\right) \\ & -cE'(t) + c_0 E'(t). \end{aligned}$$

Consequently, with a suitable choice of $\epsilon_0$ and $k$, we obtain, for all $t \geq t_0$,

$$F_4'(t) \leq -k_1\xi^{2p-1}(t)\left(\frac{E(t)}{E(0)}\right)^{2p-1} H'\left(\epsilon_0 \frac{E(t)}{E(0)}\right) = -k_1\xi^{2p-1}(t) H_2\left(\frac{E(t)}{E(0)}\right), \tag{3.25}$$

where $H_2(t) = t^{2p-1}H'(\epsilon_0 t)$.
Since $h_0 \in C^1([0, +\infty])$, then it is evident that $H \in C^1([0, +\infty])$ and $H'(0) = h_0'(0)$. So, $H_2(0) = 0$ and since

$$H_2'(t) = (2p-1)t^{2p-2}H'(\epsilon_0 t) + \epsilon_0 t^{2p-1}H''(\epsilon_0 t)$$

then, using the strict convexity of $H$ on $(0, r^2]$, we find that $H_2'(t), H_2(t) > 0$ on $[0, 1]$. Thus, with $R(t) = \frac{\alpha_1 F_4(t)}{E(0)}$, and using (3.22) and (3.25), we have $R \sim E$ and, for some $k_1 > 0$,

$$R'(t) \leq -k_1\xi^{2p-1}(t)H_2(R(t)), \quad \forall\ t \geq t_0.$$

Then, a simple integration gives, for some $k_2 > 0$,

$$R(t) \leq H_1^{-1}\left(k_1 \int_{t_0}^{t} \xi^{2p-1}(s)ds + k_2\right), \quad \forall t > t_0,$$

where $H_1(t) = \int_t^1 \dfrac{1}{H_2(s)} ds$.
To establish (3.18) Multiplying(3.2) by $\xi(t)$ and recall Remark 3. So, we have

$$\begin{aligned}\xi(t)\mathcal{F}'(t) &\le -m\xi(t)E(t) + C\xi(t)(g\circ\nabla u)(t) + c\xi(t)\int_{\Gamma_1} h^2(u_t)d\tau \\ &= -m\xi(t)E(t) + C\frac{\eta(t)}{\eta(t)}\int_0^t [\xi^p(s)g^p(s)]^{\frac{1}{p}} \|\nabla u(t) - \nabla u(t-s)\|_2^2 \\ &+ c\xi(t)\int_{\Gamma_1} h^2(u_t)d\tau, \end{aligned} \tag{3.26}$$

where

$$\begin{aligned}\eta(t) &= \int_0^t \|\nabla u(t) - \nabla u(t-s)\|_2^2 ds \le C\int_0^t \|\nabla u(t)\|_2^2 + \|\nabla u(t-s)\|_2^2 ds \\ &\le C\int_0^t [E(t) + E(t-s)] ds \le 2C\int_0^t E(t-s) ds \\ &= 2C\int_0^t E(s) ds < 2C\int_0^{+\infty} E(s) ds < +\infty.\end{aligned}$$

Applying Jensens's inequality (2.10) for the second term on the right hand side of (3.26), with

$$G(y) = y^{\frac{1}{p}},\ y > 0,\ f(s) = \xi^p(s)g^p(s)$$

and

$$h(s) = \|\nabla u(t) - \nabla u(t-s)\|_2^2,$$

to get

$$\begin{aligned}\xi(t)F'(t) &\le -m\xi(t)E(t) + c\xi(t)\int_{\Gamma_1} h^2(u_t)d\tau, \\ &\quad + C\eta(t)\left[\frac{1}{\eta(t)}\int_0^t \xi^p(s)g^p(s)\|\nabla u(t) - \nabla u(t-s)\|_2^2 ds\right]^{\frac{1}{p}}\end{aligned}$$

where we assume that $\eta(t) > 0$.
Therefore, we obtain

$$\begin{aligned}\xi(t)F'(t) &\le -m\xi(t)E(t) + c\xi(t)\int_{\Gamma_1} h^2(u_t)d\tau \\ &\quad + C\eta^{\frac{p-1}{p}}(t)\left[\xi^{p-1}(0)\int_0^t \xi(s)g^p(s)\|\nabla u(t) - \nabla u(t-s)\|_2^2 ds\right]^{\frac{1}{p}} \\ &\le -m\xi(t)E(t) + C(-g'\circ\nabla u)^{\frac{1}{p}}(t) + c\xi(t)\int_{\Gamma_1} h^2(u_t)d\tau \\ &\le -m\xi(t)E(t) + C(-E'(t))^{\frac{1}{p}} + c\xi(t)\int_{\Gamma_1} h^2(u_t)d\tau.\end{aligned}$$

Multiplying by $\xi^p(t)E^p(t)$, and repeating the same computations as in above, we arrive at

$$E(t) \leq k_3 H_2^{-1}\left(k_1 \int_{t_0}^{t} \xi^p(s)ds + k_2\right), \qquad 1 < p < \frac{3}{2},$$

where $H_2(t) = \int_t^1 \dfrac{1}{t^p H'(\varepsilon_0 t))} ds$.

**Remark 3.5.** In the case where $\| \nabla u(t) - \nabla u(t-s) \| = 0$ and hence from (3.2) we have

$$\mathcal{F}'(t) \leq -mE(t) + c\int_{\Gamma_1} h^2(u_t)d\tau,$$

using the procedure similar to that of [19], we obtain

Case $h_0$ linear

$$E(t) \leq Ce^{-mt}.$$

Case $h_0$ nonlinear

$$E(t) \leq H_1^{-1}(k_1 t + k_2), \quad \forall t > t_0$$

This completes the proof of our main result. □

**Example 3.6.** As in [17], we give an example to illustrate the existence of relaxation function $g$ and $\xi$ satisfying $(G2)$:

**If** $p = 1$**:**

Let $g(t) = ae^{-b(1+t)}$, where $b > 0 < \nu \leq 1$ and $a > 0$ is chosen so that $\int_0^{+\infty} g(t)\,dt < 1$. Then $g'(t) = -\xi(t)g(t)$ where $\xi(t) = b$.

**If** $1 < p < \dfrac{3}{2}$**:**

Let $g(t) = \frac{a}{(1+t)^\nu}$, $\nu > 2$, where $a > 0$ is a constant so that $\int_0^{+\infty} g(t)\,dt < 1$. We have

$$g'(t) = -\frac{a\nu}{(1+t)^{\nu+1}} = -b\left(\frac{a}{(1+t)^\nu}\right)^{\frac{\nu+1}{\nu}} = -bg^p(t), \quad p = \frac{\nu+1}{\nu} < \frac{3}{2}, \quad b > 0.$$

with $\xi(t) = b$.

**Example 3.7.** As in [2, 6], we give an example to illustrate the energy decay rates given by **Theorem** (3.3) and **Theorem** (3.4).
If $h$ satisfies

$$c_1 \min\{| s |, | s |^q\} \leq | h(s) | \leq c_2 \max\left\{| s |, | s |^{1/q}\right\},$$

for some $c_1$, $c_2 > 0$ and $q \geq 1$. Then $h_0(s) = cs^q$ and $\bar{H}(s) = \sqrt{s}h_0(\sqrt{s}) = cs^{\frac{q+1}{2}}$ is a strictly convex $C^2$ function on $(0, \infty)$, then $H_1^{-1}(t) = (ct + c_1)^{\frac{-2}{4p+q-5}}$, and the relaxation function $g$ and $\xi$ given in Example 3.6.

Then, we obtain for some constants $c,\ c',\ c'' > 0$:
If $p = 1$ and $q = 1$ **($h_0$ is linear)**, by **Theorem** (3.3) we arrive at

$$E(t) \le ce^{-c'\int_0^t \xi(s)ds} = ce^{-c'bt}.$$

If $1 < p < \frac{3}{2}$ and $q = 1$ **($h_0$ is linear)**, by **Theorem** (3.3) we arrive at

$$E(t) \le c\left(c'\int_0^t \xi^{2p-1}(s)ds + c''\right)^{-\frac{1}{2p-2}} = c\left(c'bt + c''\right)^{-\frac{1}{2p-2}}.$$

If $p = 1$ and $q > 1$ **($h_0$ is nonlinear)**, by **Theorem** (3.4) we arrive at

$$E(t) \le c\left(c'\int_0^t \xi(s)ds + c''\right)^{-\frac{2}{q-1}} = c\left(c'bt + c''\right)^{-\frac{2}{q-1}}.$$

If $1 < p < \frac{3}{2}$ and $q > 1$ **($h_0$ is nonlinear)**, by **Theorem** (3.4) we arrive at

$$E(t) \le c\left(c'\int_0^t \xi^{2p-1}(s)ds + c''\right)^{-\frac{2}{4p+q-5}} = c\left(c'bt + c''\right)^{-\frac{2}{4p+q-5}}.$$

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